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ON PERIODIC BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL EQUATION OF n-TH ORDER

Abstract. The problem (1) is investigated. New sufficient conditions are derived for the existence of at least one solution in the space $\mathscr{C}^{(n-1)}_{[0,w]}$. The proof is based on the topological degree method in the Banach space of solutions.

Let us consider the following problem

$$(1)\qquad \begin{cases} (Lx)(t)-P(t,x(t),\dots,x^{(n-2)}(t)). \quad x(t) = Q(t,x(t),\dots,x^{(n-2)}(t)), \\ x^{(i)}(0) = x^{(i)}(w), \quad i = 0,1,\dots,n-1,\ t\in[0,\ w],\ n\geqslant 2, \end{cases}$$

in which

$$Lx(t) = \sum_{i=0}^{n} a_i x^{(n-i)}(t), \quad a_i = \text{const},$$

P: $[0,\ w]\times \mathbf{R}^{n-1}\to\mathbf{R}^1$ is continuous, $|P(\cdot)|\leqslant M = \text{const}$,
Q: $[0,\ w]\times \mathbf{R}^{n-1}\to\mathbf{R}^1$ is continuous and

$$\lim_{r\to\infty} r^{-1}\int_0^w \sup_{|x|\leqslant r}|Q(t,\ x)|\,\mathrm{d}t = 0 \quad \text{for} \quad x = (x_1,\dots,x_n)\in\mathbf{R}^{n-1},\ |x|\leqslant r.$$

Our aim is to demonstrate a sufficient condition for the existence of a solution of problem (1).

The solution x is here a function x:$[0,\ w]\to\mathbf{R}^1$, $x\in\mathscr{C}^{(n-1)}_{[0,w]}$, satisfying problem (1).

Let us take the following definitions and assumptions:

$\mathscr{C}^{(k)}_{[0,w]}$ denotes the space of k times continuously differentiable real functions in [0, w] with the norm

$$|x| = \sum_{i=0}^{k} \max_{[0,w]}|x^{(i)}(t)|$$

or

$$(2)\qquad |x| = \max_{i=0,\dots k}\ (\max_{[0,w]}|x^{(i)}(t)|),$$

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or

(3) $$|x| = \max(|x(0)|,\ |x'(0)|,\dots,|x^{(n-1)}(0)|,\ \max_{[0,\,w]}|x^{(n)}(t)|),$$

respectively.

REMARK. *When P, Q are w-periodic in $t \in \mathbf{R}^1$, then the solutions of problem (1) are w-periodic in t.*

Let us now consider the differential operator

$$(Lx)(t) = a_0 x^{(n)}(t) + \dots + a_{n-1}x(t) + a_n, \quad a_0 \neq 0,\ t \in [0,\ w],\ n \geqslant 1,$$

$$x^{(i)}(0) = x^{(i)}(w), \quad i = 0, 1, \dots, n-1,$$

and assume that the characteristic equation

$$W(\lambda) = a_0\lambda^n + \dots + a_n$$

has nonzero and distinct characteristic roots $\lambda_1, \dots, \lambda_n$.

LEMMA 1. *For every characteristic polynomial $W(\lambda)$ there exists a real number μ that the characteristic equation*

$$W_\mu(\lambda) = a_0\lambda^n + \dots + (a_{n-1} + \mu)\lambda + a_n$$

has distinct and nonzero characteristic roots.

This fact is a consequence of the continuous dependence of the characteristic roots of $W_\mu(\lambda)$ on its coefficients. $W_\mu(\lambda)$ is related to the differential operator

$$(L_\mu x)(t) = a_0 x^{(n)}(t) + \dots + a_{n-2}x''(t) + (a_{n-1} + \mu)x'(t) + a_n x(t).$$

REMARK. It can happen that problem (1) has many solutions, e.g.

$$x'' + 4x = \sin 2t,\ x(0) = x(\pi),\ x'(0) = x'(\pi),$$

has a general solution of the form

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + 1/16 \sin 2t - 1/4 \cos 2t$$

and for every (C_1, C_2) we get a solution of the problem. In this case

$$\exp(\lambda_j w) = \exp(\pm 2\pi i) = 1.$$

But if we consider the problem

$$x'' + 4x = \sin t,\ x(0) = x(\pi),\ x'(0) = x'(\pi),$$

then the general solution is of the form

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + 1/3 \sin^3 t \cos 2t - 1/3 \cos^3 t \sin 2t + 1/2 \cos t \sin 2t,$$

and it contains no solution of the problem.

LEMMA 2. *The problem*

(4) $$(Lx)(t) = \varphi(t),\ \varphi \text{ is given},\ x^{(i)}(0) = x^{(i)}(w),\ i = 0, 1, \dots, n-1,$$

has exactly one solution if and only if the problem

(5) $$(Lx)(t) = 0, \quad x^{(i)}(0) = x^{(i)}(w), \quad i = 0, 1, 2, \ldots, n-1,$$

has only the zero solution.

LEMMA 3. *If the characteristic roots of the differential operator* (5) *are nonzero, distinct and* $\exp(\lambda_j w) \neq 1$ *for* $j = 1, 2, \ldots, n$, *then the solutions of* (4) *are of the form*

(6) $$x(t) = \int_0^w G(t, s)\varphi(s)\,ds,$$

where

$$G(t, s) = \begin{cases} \sum_{j=1}^{n} b_j \exp(\lambda_j(t-s)) & \text{for } 0 \leqslant s \leqslant t \leqslant w, \\ \sum_{j=1}^{n} b_j \exp(\lambda_j(t-s-w)) & \text{for } 0 \leqslant t < s \leqslant w, \end{cases}$$

and

$$b_j = W'(\lambda_j)/(1 - \exp(\lambda_j w)), \quad j = 1, 2, \ldots, n.$$

Together with (6) let us consider the vector field

(7) $$(\Phi x)(t) = x(t) - (Fx)(t),$$

or in the explicit form

(8) $$(\Phi x)(t) = x(t) + \int_0^w G_\mu(t, s)([P(s, x(s), \ldots, x^{(n-2)}(s)) + \mu]x(s) + Q(s, x(s), \ldots, x^{(n-2)}(s)))\,ds; \quad x \in \mathscr{C}^{(n-1)}_{[0, w]}.$$

We can now show, that the operation

$$(\Phi x)(t) - x(t) = \int_0^w G_\mu(t, s)([P(s, x(s), \ldots, x^{(n-2)}(s)) + \mu]x(s)) + Q(s, x(s), \ldots, x^{(n-2)}(s)))\,ds$$

is completely continuous in the space $\mathscr{C}^{(n-1)}_{[0, w]}$.

Let $\{x_m\}_{m=1,2,\ldots} \subset \mathscr{C}^{(n-1)}_{[0, w]}$ and $|x_m| \leqslant N$, where $|\cdot|$ denotes the $\mathscr{C}^{(n-1)}_{[0, w]}$ norm. From the last inequality it follows that

$$|x_m(t)| \leqslant N, \quad |x'_m(t)| \leqslant N, \ \ldots, \ |x_m^{(n-1)}(t)| \leqslant N, \quad m = 1, 2, \ldots$$

and $t \in [0, w]$. So we can take a subsequence $\{x_{m_k}(t)\}$ such that

$$x_{m_k}(t) \rightrightarrows x(t), \quad x'_{m_k}(t) \rightrightarrows x'(t), \quad x_{m_k}^{(n-1)}(t) \rightrightarrows x^{(n-1)}(t), \quad m_k \to \infty, \quad t \in [0, w].$$

For the derivatives $x_{m_k}^{(i)}(t)$, $i = 1, 2, \ldots, n-2$, we have the formulae

$$(9) \qquad ((F x_{m_k})(t))^{(i)} = \int_0^w \frac{\partial^i G_\mu(t,\ s)}{\partial t^i}\Big([P(s,\dots,x^{(n-2)}(s)+\mu]x(s) + Q(s, x(s),\dots,x^{(n-2)}(s))\Big)\,ds.$$

From our assumptions for the functions P, Q and (9) it follows that

$$(10) \qquad ((F x_{m_k})(t))^{(i)} \rightrightarrows ((F x)(t))^{(i)}, \quad i = 0, 1, \dots, n-2.$$

For $i = n-1$ we have $|x_{m_k}^{(n-1)}(t)| \leqslant N$, $x_{m_k}(t) \rightrightarrows x(t)$, so

$$(11) \qquad ((F x_{m_k})(t))^{(n-1)} = \int_0^w \frac{\partial^{n-1} G_\mu(t,\ s)}{\partial t^{n-1}}\big[(P(s, x(s),\dots,x_{m_k}^{(n-2)}(s)) + \mu)\, x_{m_k}(t) + Q(s,\dots,x_{m_k}^{(n-2)}(s))\big]\,ds + \frac{P(t, x_{m_k}(t),\dots,x_{m_k}^{(n-2)}(t)+\mu}{a_0}\, x_{m_k}(t).$$

The right hand side terms of (11) converge uniformly as $m_k \to \infty$, so the same is true for

$$(12) \qquad ((F x_{m_k})(t))^{(n-1)}, \quad m_k \to \infty \ \text{ and } \ t \in [0,\ w].$$

Relations (10), (11) and (12) guarantee that the operation F defined in (7) and (8) is completely continuous in the space $\mathscr{C}^{(n-1)}_{[0,\ w]}$. Together with the vector field (8) let us consider

$$(13) \qquad (\psi x)(t) = x(t) - \int_0^w G_\mu(t,\ s)\,[P(s, x(s),\dots,x^{(n-2)}(s)) + \mu]\, x(s)\,ds, \quad x \in \mathscr{C}^{(n-1)}_{[0,\ w]}.$$

Zero vectors of (7) and (8) and (13) with $\mu = 0$ are solutions of problem (1) or

$$\begin{cases} (Lx(t) - [P(t, x(t),\dots,x^{(n-2)}(t)) + \mu]\, x(t) = 0, \\ x^{(i)}(0) = x^{(i)}(w), \quad i = 0, 1, \dots, n-1, \end{cases}$$

respectively.

Now we can formulate the following theorem.

THEOREM. *If we assume that there exists a real number μ such that the problem*

$$(L_\mu x)(t) = 0, \quad x^{(i)}(0) = x^{(i)}(w), \quad i = 0, 1, \dots, n-1,$$

is invertible, and the Green function $G_\mu(t,\ s)$ satisfies the condition

$$(14) \qquad \sum_{j=0}^{n-1} \max_{[0,\ w]} \int_0^w \left| \frac{\partial^j G_\mu(t,\ s)}{\partial t^j} \right| ds < \frac{1}{M + |\mu|},$$

where $|P(t,\ x(t),\ \dots,\ x^{(n-2)}(t))| \leqslant M$ for $x \in \mathscr{C}^{(n-1)}_{[0,\ w]}$, then problem (1) *has at least one solution in the space $\mathscr{C}^{(n-1)}_{[0,\ w]}$.*

REMARK. *If we take the space* $\mathscr{C}^{(n-1)}_{[0,w]}$ *with norm* (2) *or* (3) *respectively, then condition* (14) *has the form*

$$\max_{0\leqslant i\leqslant n-1}\ \max_{[0,w]}\left(\int_0^w \left|G_\mu^{(i)}(t,\ s)\right| \mathrm{d}s\right)<\frac{1}{M+|\mu|} \tag{15}$$

or

$$\max\left(\max_{0\leqslant i\leqslant n-1}\left|\int_0^w G_\mu^{(i)}(t,\ s)\,\mathrm{d}s\right|;\ \max_{[0,w]}\left|\int_0^w G_\mu^{(n-1)}(t,\ s)\,\mathrm{d}s\right|\right)<\frac{1}{M+|\mu|}. \tag{16}$$

Proof. First we discuss the vector field (13). It is completely continuous and we shall prove that it is non singular on the sphere $S_R = \{x\colon\ x\in\mathscr{C}^{(n-1)}_{[0,w]},\ |x| = R\}$. We have the following inequality

$$\left|x(t)-\int_0^w G_\mu(t,s)(P(s,x(s),\dots,x^{(n-2)}(s))+\mu)\,x(s)\,\mathrm{d}s\right|$$
$$\geqslant |x(t)|-\left|\int_0^w G_\mu(t,\ s)(P(s,x(s),\dots,x^{(n-2)}(s))+\mu)\,x(s)\,\mathrm{d}s\right|.$$

But

$$\left|\int_0^w G_\mu(t,\ s)(P+\mu)\,x(s)\,\mathrm{d}s\right|$$
$$=\sum_{j=0}^{n}\max_{[0,w]}\left|\left[\int_0^w G_\mu(t,\ s)(P(s,\dots,x^{(n-2)}(s))+\mu)\,x(s)\,\mathrm{d}s\right]^{(j)}\right|$$
$$=\sum_{j=0}^{n-1}\max_{[0,w]}\left|\int_0^w G_\mu^{(j)}(t,\ s)(P(s,\dots,x^{(n-2)}(s))+\mu)\,x(s)\,\mathrm{d}s\right|$$
$$\leqslant\sum_{j=0}^{n-1}\max\left|G_\mu^{(j)}(t,\ s)\right|(|P(s,\dots,x^{(n-2)}(s))|+|\mu|)|x(s)|\,\mathrm{d}s$$
$$\leqslant\sum_{j=0}^{n-1}\max_{[0,w]}\int_0^w |G_\mu^{(j)}(t,\ s)|(M+|\mu|)\,R\,\mathrm{d}s.$$

So we have

$$\left|x(t)-\int_0^w G_\mu(t,\ s)(P(s,\dots,x^{(n-2)}(s))+\mu)x(s)\,\mathrm{d}s\right|$$
$$\geqslant R-\left|\int_0^w G_\mu(t,\ s)(P(s,\dots,x^{(n-2)}(s))+\mu)x(s)\,\mathrm{d}s\right|$$
$$\geqslant R(1-(M+|\mu|))\left(\sum_{j=0}^{n-1}\max_{[0,w]}\int_0^w |G_\mu(t,\ s)|\,\mathrm{d}s\right)>0.$$

From the last inequality it follows that $\psi(x)\neq 0$ for $x\in S_R$.

REMARK. It follows form the nonsingularity of ψ on S_R for arbitrary $R>0$ that the problem

$$\begin{cases}(L_\mu x)(t) = (P(t,x(t),\dots,x^{(n-2)}(t)+\mu)\,x(t), & t\in[0,\ w],\\ x^{(j)}(0) = x^{(j)}(w),\ j = 0,1,\dots,n-1,\end{cases}$$

has only the zero solution.

Now we can see that

$$\gamma(\psi, S_R) \neq 0,$$

where $\gamma(\psi, S_R)$ denotes the rotation of the vector field ψ on S_R. Let us consider the difference

$$\Lambda(x) = \psi(x) - \lambda\psi(-x) \qquad \text{for } x \in S_R \text{ and } \lambda > 0.$$

Easy calculations lead to

$$\Lambda(x)(t) = (1+\lambda)\left[x(t) - \int_0^w G_\mu(t, s)\left[\mu + \frac{P(s, x(s), \ldots, x^{(n-2)}(s))}{1+\lambda} + \frac{P(s, -x(s), \ldots, -x^{(n-2)}(s))}{1+\lambda}\right]x(s)\,ds\right].$$

From the last inequality we have

$$|\psi(x)(t) - \lambda\psi(-x)(t)| \geqslant (1+\lambda)\left(|x(t)| - \left|\int_0^w G_\mu(t, s)\left[\mu + \frac{P(s, x(s), \ldots, x^{(n-2)}(s)) + \lambda P(s, -x(s), \ldots, -x^{(n-2)}(s))}{1+\lambda}\right]x(s)\,ds\right|\right)$$

$$\geqslant (1+\lambda)R\left[1 - \sum_{j=0}^{n-1} \max_{[0, w]} \int_0^w |G_\mu^{(j)}(t, s)|(M+|\mu|)\,ds\right] > 0.$$

From the last inequality it follows that the vectors $\psi(x)$ and $\psi(-x)$ are not equiparallel in the antipodal points x and $-x$ on the sphere S_R and

$$\gamma(\psi, S_R) \neq 0.$$

Now we can demonstrate that the vector field Φ defined by (8) is nonsingular on S_R and

$$\gamma(\Phi, S_R) \neq 0.$$

Let us denote

$$\delta = (M+|\mu|) \sum_{j=0}^{n-1} \max_{[0, w]} \int_0^w |G_\mu^{(j)}(t, s)|\,ds.$$

Then $\delta \in [0, 1)$ and

$$|(\psi x)(t)| \geqslant R(1-\delta) > 0 \quad \text{for } x \in S_R.$$

Let us take $R = R_0$, so that

$$(17) \qquad R_0(1-\delta) > \sup_{x \in \mathscr{C}^{(n-1)}_{[0, w]}} \left|\int_0^w G_\mu(t, s)\,Q(s, \ldots, x^{(n-2)}(s))\,ds\right| = \text{const} = c.$$

Then we have

$$|\Phi(x) - \psi(x)| \leqslant c < R_0(1-\delta) \leqslant |\psi(x)|, \quad x \in S_{R_0},$$

and finally

$$|\Phi(x) - \psi(x)| < |\psi(x)|, \quad x \in S_{R_0}.$$

If we assume that

$$(18) \qquad \lim_{r\to\infty} 1/r \int_0^w \sup_{|x| \leqslant r} |Q(t, x)|\,dt = 0 \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbf{R}^n,$$

then the last part of our demonstration must be changed. Beginning with the formula (17) it is to see, that for sufficiently large R_0 we have

$$(19)\qquad R_0(1-\delta)>\int_0^w \sup_{|x|\leqslant r}|G_\mu(t,\ s)Q(t,\ x)|\,dt,\quad x\in\mathbf{R}^n.$$

In formula (19) the growth of the left hand side is linear in R_0 and that of the right hand side is sublinear. This is, in fact the essence of assumption (18).

As an example of (1) let us investigate

$$(20)\qquad x^{(n)}(t)-P(t,x(t),\ldots,x^{(n-2)}(t))x(t) = Q(t,\ldots,x^{(n-2)}(t))$$

$$x^{(j)}(0) = x^{(j)}(1),\qquad i = 0,1,\ldots,n-1.$$

Here $(Lx)(t) = x^{(n)}(t)$ and we replace $(Lx)(t)$ by $(L_\mu x)(t)$. If we take $\mu = \varepsilon^n$, $0<\varepsilon\ll 1$, then simple calculations lead to the condition

$$\int_0^1 G_\mu^{(i)}(t,\ s)\,ds = \begin{cases}\displaystyle\sum_{j=0}^{n-1}\frac{-e^{\lambda_j t}}{n\lambda_j^{n-i-1}e^{\lambda_j}} & \text{for } 0\leqslant s\leqslant t\leqslant 1,\\[2ex] \displaystyle\sum_{j=0}^{n-1}\frac{e^{\lambda_j t}}{n\lambda_j^{n-i-1}} & \text{for } 0\leqslant t<p\leqslant 1,\end{cases}$$

where λ_j are the characteristic roots of $\lambda^n-\varepsilon^n = 0$. Condition (14) takes the form

$$(21)\qquad \max_{0\leqslant i\leqslant n-1}\left(\max_{[0,\,1]}\left|\sum_{j=0}^{n-1}\frac{-e^{\lambda_j(t-1)}}{n\lambda_j^{n-i-1}}\right|,\ \max_{[0,\,1]}\left|\sum_{j=0}^{n-1}\frac{e^{\lambda_j t}}{n\lambda_j^{n-i-1}}\right|\right)<\frac{1}{M+|\mu|}.$$

and we can formulate the following remark: If the function P is such that $|P|\ll 1$, then conditions (14), (15), (16) and (21) are satisfied provided M is sufficiently small. In other words the problem (20) always has solutions if the number M is sufficiently small.

REMARK. Various forms of problem (1) have been investigated by many authors (see [1] — a survey article and [2] — [7]). In the book [6] the linear form of (1) was discovered by an algebraic method. In [5] the Banach fixed point theorem was used to derive a sufficient condition for the existence of solutions. In [7] the Schauder fixed point theorem was applied and the author generalized certain results in [4].

In our article the existence of solutions of (1) has been proved using the topological degree method. This appears to be the first attempt to use this method for solving problem (1).

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