MAREK PIĘTKA*

UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR THE PARTIAL DIFFERENTIAL EQUATIONS AND THE CONVOLUTION EQUATIONS

Abstract. The method of construction of classes of uniqueness of solutions for differential and convolutional equations (containing the classical partial difierential equations) is presented in this paper. It tries to explain the anomaly of uniqueness of solutions for the Laplace and wave equations and non-uniqueness for the equation of heat.

Introduction. This paper present the method of the construction of classes of the uniqueness of the Cauchy problem for the linear partial differential equations of order n with constant coefficients and equations with convolution. The method takes the advantage of the operational calculus given by J. Mikusiński in [1]. Some preliminary notes connected to the operational calculus will be given nearer.

The operational function in the Mikusiński's sense is the function defined on the interval $J \times \mathbf{R}$ with values in the field $\mathcal{Y}$ of Mikusiński's operators [1]. In this paper we will consider the special case of the operational functions of the form

$$x(\lambda) = q\{y(\lambda, t)\},$$

where $q \in \mathcal{Y}$ and $y(\cdot, \cdot)$ is the function defined on $J \times \mathbf{R}^+$ (J — an interval included in $\mathbf{R}$) with values in $\mathbf{R}$ or $\mathbf{C}$. If the function $y(\cdot, \cdot)$ belongs to the class $\mathscr{C}^{(n)}$ on $J \times R^+$ we say that $x(\cdot)$ belongs to $\mathscr{C}^{(n)}$ in the operational sense.

The derivative of such a function $x(\cdot)$ we define by the formula

$$Dx(\lambda) = q\left\{\frac{\partial}{\partial \lambda} y(\lambda, t)\right\}.$$

The definition of the derivative fulfils the fundamental condition: if $x'(\lambda) = 0$ on the interval J then $x(\lambda) = \text{const}$ on the interval.

Similarly we may define the n-th derivative of the function $x(\cdot)$.

In the case when $q = 1$ the function $x(\cdot)$ is called the parametric function.

It is the one-to-one transformation of parametric functions of the class $\mathscr{C}^{(n)}$ (in the operational sense) to the set of real (complex) functions of two variables of the class $\mathscr{C}^{(n)}$.

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* Instytut Matematyki Uniwersytetu Śląskiego, ul. Bankowa 14, 40-007 Katowice, Poland.

It is given in [1] that there is the similarity between linear differential equations considered in the set of operational functions and linear partial differential equations with solutions in the class $\mathscr{C}^{(n)}$ by applying the differential operator $s \in \mathscr{Y}$. For example the equation

$$\frac{\partial^2 x(\lambda, t)}{\partial \lambda^2} = a \frac{\partial^2 x(\lambda, t)}{\partial t^2}$$

with the initial conditions:

$$x(\lambda, 0) = 0,$$

$$\frac{\partial}{\partial t} x(\lambda, 0) = 0$$

transforms to the operational form

$$x''(\lambda) - as^2 x(\lambda) = 0.$$

Similarly the equation of heat

$$\frac{\partial^2 x(\lambda, t)}{\partial \lambda^2} = a \frac{\partial x(\lambda, t)}{\partial t}$$

with the initial condition

$$x(\lambda, 0) = 0$$

transforms to the operational equation

$$x''(\lambda) - asx(\lambda) = 0.$$

Problems of the uniqueness of the solution for the last equation (in the halfplane) were considered by A. Tychonoff in [3]. The similar results were given by J. Mikusiński in [1] taking the advantage of the estimation of operational functions of the exponential type [2]. The solution of the linear operational differential equation is formed by the sum of exponential functions with polynomial coefficients. Applying the estimation of such functions he gave the sufficient conditions which formed the class of uniqueness of the Cauchy problem in the halfplane for the heat equation.

Preliminary theorems. The operational equation has the form

(1) $$\sum_{l=0}^{m} A_l D^l x(\lambda) = f(\lambda).$$

Methods used in solving the operational equation are similar to ones applied in the ordinary differential equation with constant coefficients and the solution has the similar from

(2) $$u(\lambda) = \sum_{i=1}^{k} \sum_{j=0}^{r_i - 1} c_{ij} \lambda^j e^{w_i \lambda} + h(\lambda),$$

where w_i are certain roots of the characteristic equation

(3) $$\sum_{l=0}^{m} A_l w^l = 0$$

in the field of operators $\mathscr{Y}$. r_i means the mulitiplicity of the root w_i and c_{ij} are operators belonging to $\mathscr{Y}$. $h(\cdot)$ is a certain solution of the equation (1). Each root of the equation (3) has the form

$$w = \sum_{l=0}^{\infty} b_l s^{p-lq}, \tag{4}$$

where $p, q \in \mathbf{Q}$ (the set of rational numbers), $q \geqslant 0$ and $b_l \in \mathbf{C}$ (the set of complex numbers).

The root w generates the exponential function $e^{w\lambda}$ if and only if $p<1$ or in the case $p=1$ and $b_0 \in \mathbf{R}$.

LEMMA 1.1. *If $p \leqslant -1$ then w given by form (4) is a classical continuous function.*

The proof is given in [1].

LEMMA 1.2. *If $f \in \mathscr{C}_{[0,\infty)}$ and $\lambda>0$ then there exists a continuous function $H(\cdot,\cdot)$ defined on $\mathbf{R}\times\mathbf{R}^+$ such that the exponential function $e^{\lambda f}$ has the form*

$$e^{\lambda f} = 1+\{H(\lambda, t)\}.$$

The proof is given in [1].

LEMMA 1.3. *Let $T>0$ be fixed and $H(\cdot,\cdot)$ be the function given by Lemma 1.2. Then for $\lambda \in \mathbf{R}^+$*

$$\max_{t\in[0,T]} |H(\lambda, t)| \leqslant \lambda M_f(T) \exp[T\lambda M_f(T)],$$

where

$$M_f(T) = \max_{t\in[0,T]} |f(t)|.$$

Proof. The function $H(\cdot,\cdot)$ has more precise characterization ([1])

$$H(\lambda, t) = \sum_{k=1}^{\infty} \frac{f^k \lambda^k}{k!},$$

where f^k denotes the convolution of k functions f.

By the method of the mathematical induction we get that

$$\max_{t\in[0,T]} |f^k(t)| \leqslant T^{k-1}[M_f(T)]^k.$$

Hence by the form of the function H we get the estimation.

The method used by J. Mikusiński for the heat equation will be applied to the differential-convolutional equation of the order n. In the construction of the class of uniqueness of the solution for such equations we will apply the generalization of the theorem given in [2].

The generalization reads as follows.

THEOREM 1.1. *Let the operational function*

$$x(\lambda) = e^{\left(\sum_{l=1}^{n} b_l s^{p_l}\right)\lambda} \tag{6}$$

be defined on $\mathbf{R}^+$, where for $l \in \{2, \ldots, n\}$

$$p_1 > p_l, \quad p_l \in (-1, 1) \quad \text{and} \quad p_1 \in (0, 1),$$

$$b_l \in \mathbf{C}, \quad b_l = d_l + \mathrm{i}e_l, \quad d_l,\ e_l \in \mathbf{R} \quad \textit{and} \quad d_1 < 0$$

and

(7) $$|e_1| < |d_1|(1 - 2^{1/p_1})\,\mathrm{ctg}[p_1(\pi/2)].$$

Then the function $x(\lambda) = \{F(\lambda, t)\}$, *i.e.* $x(\cdot)$ *is the parametric function for* $\lambda > 0$. *Moreover, there exist positive constants* C_1 *and* C_2 *such that if* λ, $t > 0$ *and* $\lambda/t > C_1$ *and* $\lambda/(t^{p_1}) > C_2$ *then there exist positive constants* G_1 *and* G_2 *such that the inequality*

(8) $$|F(\lambda, t)| < G_1\left(\frac{\lambda}{t}\right)^{\frac{1}{1-p_1}} \exp\left[-G_2\left(\frac{\lambda}{t^{p_1}}\right)^{\frac{1}{1-p_1}}\right]$$

holds.

The proof is given in [4].

LEMMA 1.4. *Let the function* $x(\cdot)$ *fulfil the assumptions given in Theorem 1.1. Let* $f(\cdot)$ *and* $g(\cdot)$ *belong to the set of the continuous functions defined on* $\mathbf{R}^+$. *Then the function*

(9) $$Y(\lambda) = e^{\left(\sum_{l=1}^{n} b_l s^{p_l}\right)\lambda}\{g(t)\}\, e^{f\lambda} = \{N(\lambda, t)\},$$

i.e. $Y(\cdot)$ *is the parametric function for* $\lambda > 0$. *Moreover, for each* $T > 0$ *there exist positive constants* C_3, G_3 *and* G_4 *such that if* $\lambda > C_3$ *and* $0 \leqslant t \leqslant T$ *then*

(10) $$|N(\lambda, t)| < \max_{t\in[0,T]} |g(t)|\, G_4 \exp[-G_3\,\lambda^{1/(1-p_1)}].$$

Proof. Following Lemmas 1.1, 1.2, 1.3 and Theorem 1.1 we have that $Y(\lambda)$ is the parametric function. Moreover, we get the estimation

$$\max_{t\in[0,T]} |N(\lambda, t) \leqslant \max_{t\in[0,T]} |g(t)|\{1 + T\lambda M_f(T)\exp[T\lambda M_f(T)]\}\int_0^T |F(\lambda, t)|\,dt.$$

For $\lambda > \max\{C_1 T,\ C_2 T^{p_1}\}$ we have

$$\int_0^T |F(\lambda, t)|\,dt < G_1 \lambda^{\frac{1}{1-p_1}} \int_0^T t^{-\frac{1}{1-p_1}} \exp\left[-G_2\left(\frac{\lambda}{t^{p_1}}\right)^{\frac{1}{1-p_1}}\right] dt.$$

Thus, changing the variable

$$w = G_1 \lambda^{\frac{1}{1-p_1}} t^{-\frac{1}{1-p_1}},$$

we get the estimation

$$\int_0^T |F(\lambda, t)|\,dt < \frac{G_1}{G_2}\exp\left[-G_2 T^{-\frac{p_1}{1-p_1}} \lambda^{\frac{1}{1-p_1}}\right].$$

Hence and following the fact that $\lambda \exp[TM_f(T)\lambda] > 1$ for $\lambda > \max\{1,\ C_1 T,\ C_2 T^{p_1}\}$ we get

$$\max_{t\in[0,T]} |N(\lambda, t)| <$$

$$< \max_{t\in[0,T]} |g(t)| \frac{G_1}{G_2}[1 + TM_f(T)]\cdot\lambda\cdot\exp[TM_f(T)\lambda - G_2 T^{-\frac{p_1}{1-p_1}} \lambda^{\frac{1}{1-p_1}}].$$

Let us denote

$$G_3 = \frac{G_1}{G_2}[1+TM_f(T)].$$

Following the inequalities

$$1/(1-p_1)>1 \quad \text{and} \quad G_2 T^{-[p_1/(1-p_1)]}>0$$

we get that there exists a constant $G_0(T)$ such that for $\lambda > G_0(T)$

$$\lambda \exp[TM_f(T)\lambda - (1/2) G_2 T^{-\frac{p_1}{1-p_1}} \lambda^{\frac{1}{1-p_1}}]>1.$$

Hence, putting

$$G_4 = (1/2)G_2 T^{-\frac{p_1}{1-p_1}},$$

we get condition (10) for $\lambda > \max\{1, G_0(T), C_1 T, C_2 T^{p_1}\} = C_3$.

Theorem 1.2 will show that if one of roots of the characteristic equations is generated by $p = 1$ and $b_0 \in \mathbf{R}$ then solution (2) does not generate the parametric function.

THEOREM 1.2. *Let* $p = 1$ *and* $0 \neq b_0 \in \mathbf{R}$ *and* $0 \neq c \in \mathscr{Y}$ *then* $ce^{w\lambda}$ *(*w *has form* (4)) *is not the parametric function defined on the whole real axis.*

Proof. Let w has form (4), i.e.

$$w = \sum_{l=0}^{\infty} b_l s^{1-lq}.$$

Let us denote by w_0 the power series

$$w_0 = b_1 s^{1-q} + b_2 s^{1-2q} + \dots$$

Following [1] we obtain that there exists a function $0 \neq g \in \mathscr{C}_{\mathbf{R}}^{(\infty)}$ (in the classical sense) such that $ge^{w_0\lambda}$ is the parametric function on $\mathbf{R}$.

Let us suppose that there exists $0 \neq c \in \mathscr{Y}$ such that $ce^{(b_0 s + w_0)\lambda}$ is the parametric function on $\mathbf{R}$ (it is a contradiction to the thesis of the theorem). We do not loose the generality if we suppose that $c \in \mathscr{C}_{\mathbf{R}}$. We will make the additional assumption that $b_0 > 0$. The proof in the case $b_0 < 0$ is similar. The function $ce^{(b_0 s + w_0)\lambda}$ being the parametric one on $\mathbf{R}$ is, of course, the parametric function for $\lambda > 0$. Let us put $\{M(\lambda, t)\} = ce^{(b_0 s + w_0)\lambda}$ ($M(\cdot, \cdot)$ means a continuous complex function defined on $\mathbf{R} \times \mathbf{R}^+$). By a simple calculation we have

$$gc = e^{-s(b_0\lambda)}[\{M(\lambda, t)\} g e^{-w_0\lambda}],$$

where the function g is that one for which $ge^{-w_0\lambda}$ is the parametric function on $\mathbf{R}$.

The values of the operational function $e^{-s(b_0\lambda)}$ are translation operators ([1]). Denoting by $Q(f)$ the support number of the function f, i.e.

$$Q(f) = \sup\{r\colon f(x) = 0 \text{ if } x \leqslant r\},$$

we get

$$Q(gc) = Q\{e^{-s(b_0\lambda)}[\{M(\lambda, t)\} g e^{-w_0\lambda}]\} \geqslant b_0\lambda.$$

The number λ may be chosen arbitrarily and then $cg = 0$ on $\mathbf{R}^+$. Hence $c = 0$ by Titchmarsh Theorem and we get a contradiction to the hypothesis.

Following the theorem we get that the solution of (1) being the parametric function may only be expected in the case $p<1$.

The operational function (6) is the parametric one if and only if the each part of the sum is the parametric function. Hence coefficients p_i are less than one.

Main theorem. Let us consider the differential-convolutional equation

$$\sum_{l=0}^{m_1}\sum_{r=0}^{n_1} a_{lr}\frac{\partial^{l+r}}{\partial\lambda^l\partial t^r}u(\lambda,\ t)+\sum_{l=0}^{m_2}\sum_{r=0}^{n_2} b_{lr}\int_0^t(t-y)^{c_r}\frac{\partial^l}{\partial\lambda^l}u(\lambda,\ y)\,\mathrm{d}y = f(\lambda,\ t) \tag{11}$$

with the initial condition

$$\frac{\partial^r}{\partial t^r}u(\lambda,\ 0) = 0 \quad \text{for} \quad r = 0,1,\ldots,n_1-1 \tag{12}$$

defined on the region $\mathbf{R}\times\mathbf{R}^+$. Constants a_{lr}, b_{lr} are complex numbers and $c_r\in\mathbf{Q}$ and $c_r>-1$. The function $f(\cdot,\ \cdot)\in\mathscr{C}_{\mathbf{R}\times\mathbf{R}^+}$.

A solution of (11), (12) is a smooth function on $\mathbf{R}\times\mathbf{R}^+$, fulfiling equation (11) and initial conditions (12). J. Mikusiński presented in [1] the connection between equation (11), (12) and the operational equation

$$\sum_{l=0}^{m_3} A_l D^l u(\lambda) = f(\lambda), \tag{13}$$

where $m_3 = \max\{m_1, m_2\}$,

$$A_l = \sum_{r=0}^{n_1} a_{lr}s^{r+c}+\sum_{r=0}^{n_2} b_{lr}\Gamma(c_r+1)s^{d_r}, \tag{14}$$

$c = \max\limits_{r=0,1,2,\ldots,n_2}\{c_r+1\}$, $d_r = c-c_r-1$ for $r\in\{0,1,2,\ldots,n_2\}$, and

$$f(\lambda) = s^c\{f(\lambda,\ t)\}. \tag{15}$$

Let us denote by $\mathscr{U}_\alpha$ the set of smooth complex functions $u(\cdot,\ \cdot)$ defined on $\mathbf{R}\times\mathbf{R}^+$ such that for each fixed $T>0$ there exist constants $\varepsilon>0$ and $M_{\varepsilon T}>0$ that the inequality

$$|u(\lambda,\ t)|<M_{\varepsilon T}\exp[|\lambda|^{1/(1-\alpha)-\varepsilon}] \tag{16}$$

holds for $(\lambda,\ t)\in\mathbf{R}\times[0,\ T]$.

Theorem 2.1 will present the sufficient condition for the class $\mathscr{U}_\alpha$ in which equation (11) — (12) has at most one solution.

THEOREM 2.1. *Let equation* (13) *be the equation connected to equation* (11) — — (12) *and let its general solution has form* (2), *i.e.*

$$u(\lambda) = \sum_{i=1}^{k}\sum_{j=0}^{r_i-1} c_{ij}\lambda^j e^{w_i\lambda}+h(\lambda)$$

and all w_i *have the expansions*

$$w_i = \sum_{l=0}^{\infty} b_{li}s^{p_i-lq_i} \quad \textit{for} \quad i\in\{1,\ldots,k\}. \tag{17}$$

Let

$$p_0 = \min_{i\in\{1,\dots,k\}} \{p_i\} \quad \textit{and} \quad p_0 \in (0, 1).$$

Let for $j \in \{1,\dots,k\}$ $b_{0j} = d_{0j} + \mathrm{i}e_{0j}$, *where* $d_{0j}, e_{0j} \in \mathbf{R}$ *and*

$$(18) \qquad |e_{0j}| < [1 - 1/(2^{p_j})]\,|d_{0j}|\operatorname{ctg}[p_j(\pi/2)]$$

and in the case when $p_j = p_k$

$$(19) \qquad |e_{0j} - e_{0k}| < [1 - 1/2^{p_j}]\,|d_{0j} - d_{0k}|\operatorname{ctg}[p_j(\pi/2)].$$

Then equation (11) — (12) *has at most one solution in the class* $\mathcal{U}_{p_0}$.

Proof. The previous remarks lead to the conclusion that the only interesting case of the proof is if all $p_i \in (0, 1)$. In the opposite case the solution of (13) (if it exists) is not a parametric function on $\mathbf{R}\times\mathbf{R}^+$.

From the form of the general solution of (13) we may deduce that it is sufficient to prove that all coefficients $c_{ij} \in \mathcal{Y}$ vanish (the only solution of the homogeneous equation is the trivial one).

Let us assume, without loosing the generality, that w_i are in the decreasing order considering their coefficients p_i, i.e. w_j follows w_i if $p_i > p_j$. Numbers p_i may be equal for different w_i and in this case w_{i_k} follows w_{i_j} if $|d_{0i_j}| > |d_{0i_k}|$. The last case is when $p_{i_k} = p_{i_j}$ and $|d_{0i_j}| = |d_{0i_k}|$ then w_{i_j} follows w_{i_k} if $d_{0i_k} > 0$.

Let us assume that a function u, being a solution of (13) and fulfilling previous assumptions belongs to $\mathcal{U}_{p_0}$. Following (2) and (17) it has the form

$$(20) \qquad \{u(\lambda,\ t)\} = \sum_{i=1}^{k}\sum_{j=0}^{r_i-1} c_{ij}\lambda^j \mathrm{e}^{\left(\sum_{l=0}^{\infty} b_{li}s^{p_i - lq_i}\right)\lambda}$$

for $(\lambda,\ t) \in \mathbf{R}\times\mathbf{R}^+$. Without loosing the generality we may assume that $c_{ij} \in \mathscr{C}_{[0,\,\infty)}$. It is enough to prove that for each (i, j) $c_{ij} = 0$ and then it follows that u vanishes on $\mathbf{R}$.

We will show that for each $T > 0$ and (i, j)

$$\max_{t\in[0,\,T]} |c_{ij}(t)| = 0.$$

Taking the advantage of ordering of the parts of the sum (20) at first we will estimate c_{1r_1-1}, then c_{1r_1-2} and so on up to c_{10}, after that we will estimate c_{2r_2-1} up to c_{20} and then we will follow the estimation up to c_{k0}. The method of estimation will be shown in the case of the function c_{1r_1-1}.

At first we will calculate c_{1r_1-1} using the equality (20). We have

$$(21) \qquad \{c_{1r_1-1}(t)\} = \{u(\lambda,\ t)\}\lambda^{-r_1+1}\mathrm{e}^{-\left(\sum_{l=0}^{\infty} b_{l1}s^{p_1 - lq_1}\right)\lambda} - \sum_{j=0}^{r_1-2}\{c_{1j}(t)\}\lambda^{j-r_1+1}$$

$$- \sum_{i=2}^{k}\sum_{j=0}^{r_i-1}\{c_{ij}(t)\}\lambda^{j-r_1+1}\mathrm{e}^{\left(\sum_{l=0}^{\infty} b_{li}s^{p_i - lq_i} - \sum_{l=0}^{\infty} b_{l1}s^{p_1 - lq_1}\right)\lambda}.$$

Without loosing the generality we may assume that $d_{01}>0$. We will show that for $\lambda>0$ each part of the sum on the right side of the last equality is the parametric function. In the case $d_{01}<0$ we can prove it in the similar way for $\lambda<0$.

Let us consider the function

$$e^{-\left(\sum_{l=0}^{\infty} b_{l1}s^{p_1-lq_1}\right)\lambda} = e^{-\left(\sum_{l=0}^{\mu} b_{l1}s^{p_1-lq_1}\right)\lambda} \cdot e^{-\left(\sum_{l=\mu+1}^{\infty} b_{l1}s^{p_1-lq_1}\right)\lambda}$$

where μ fulfils the conditions $p_1-\mu\cdot q_1>-1$ and $p_1-(\mu+1)q_1\leqslant -1$. If follows from Theorem 1.1 that the function

$$e^{\left(\sum_{l=0}^{\mu} b_{l1}s^{p_1-lq_1}\right)\lambda} \text{ is parametric for } \lambda>0.$$

Taking into account the conditions given on μ and p_1 and Lemma 1.1 we notice that $-\left(\sum_{l=\mu+1}^{\infty} b_{l1}s^{p_1-lq_1}\right)$ is the classical continuous function and then

$$e^{-\left(\sum_{l=\mu+1}^{\infty} b_{l1}s^{p_1-lq_1}\right)\lambda} = e^{f\lambda} \quad \text{and} \quad f\in\mathscr{C}_{[0,\infty)}.$$

By virtue of Lemma 1.2 there exists a continuous function $H(\cdot,\cdot)$ such that $e^{f\lambda}=1+\{H(\lambda,t)\}$. Hence the first part of sum (21), i.e.

$$\{u(\lambda,t)\}\lambda^{-r_1+1}e^{-\left(\sum_{l=0}^{\infty} b_{l1}s^{p_1-lq_1}\right)\lambda}$$

is the parametric function. The second part of sum (21), i.e.

$$\sum_{j=0}^{r_1-2}\{c_{1j}(t)\}\lambda^{j-r_1+1}$$

is the parametric function by the definition of parametric functions. The third part of sum (21) equals

$$\sum_{i=2}^{k}\sum_{j=0}^{r_1-1}\{c_{ij}(t)\}\lambda^{j-r_1+1}e^{\left(\sum_{l=0}^{\infty} b_{li}s^{p_i-lq_i}-\sum_{l=0}^{\infty} b_{l1}s^{p_1-lq_1}\right)\lambda}.$$

The function

(22) $$e^{\left(\sum_{l=0}^{\infty} b_{li}s^{p_i-lq_i}-\sum_{l=0}^{\infty} b_{l1}s^{p_1-lq_1}\right)\lambda}$$

is of the form

$$e^{\left(\sum_{l=0}^{\infty} b_l s^{p_l}\right)\lambda}$$

and it may be distributed into two parts

(23) $$e^{\left(\sum_{l=0}^{\infty} b_l s^{p_l}\right)\lambda} = e^{\left(\sum_{l=0}^{\mu} b_l s^{p_l}\right)\lambda}\cdot e^{\left(\sum_{l=\mu+1}^{\infty} b_l s^{p_l}\right)\lambda}$$

where μ fulfils conditions $p_\mu>-1$ and $p_{\mu+1}\leqslant -1$. On account of Lemma 1.2 the second exponential function on the right side of equality (23) has the form $e^{f\lambda}=1+\{H(\lambda,t)\}$. Taking into account the ordering of w_i ($d_{01}>0$) and the fact that $p_0\in(0,1)$ we see that the assumptions of Theorem 1.1 are fulfilled and the first exponential function on the right side of equality (23) is the parametric function. Hence the third part of sum (21) is the parametric function.

Now we will estimate each part of sum (21). Let us fix $T>0$. By the assumptions of Theorem 1.2 we notice that the function

$$\{c_{ij}(t)\}\, e^{\left(\sum_{l=0}^{\infty} b_{lis}p_i - lq_i - \sum_{l=0}^{\infty} b_{l1s}p_1 - lq_1\right)\lambda}$$

fulfils the assumptions of Lemma 1.4. Hence there exist constans C_3^{ij}, G_3^{ij}, and G_4^{ij} *such that for* $\lambda > C_3^{ij}$ and $t \in [0, T]$

$$\left|\{c_{ij}(t)\}\, e^{\left(\sum_{l=0}^{\infty} b_{lis}p_i - lq_i - \sum_{l=0}^{\infty} b_{l1s}p_1 - lq_1\right)\lambda}\right| < \max_{t\in[0,T]} |c_{ij}(t)|\, G_3^{ij} \exp[-G_4^{ij} \lambda^{\frac{1}{1-p_1}}].$$

There also exist constans C_3^0, G_3^0 and G_4^0 such that for $\lambda > C_3^0$

$$\left|\{u(\lambda, t)\}\, \lambda^{-r_1+1} e^{-\left(\sum_{l=0}^{\infty} b_{l1s}p_1 - lq_1\right)\lambda}\right| < \max_{t\in[0,T]} |u(\lambda, t)|\, \lambda^{-r_1+1} G_3^0 \exp[-G_4^0 \lambda^{1/(1-p_1)}].$$

Putting

$$C_3 = \max_{(i,j)} \{C_3^{ij}, C_3^0\},\quad G_3 = \max_{(i,j)} \{G_3^{ij}, G_3^0\},\quad G_4 = \max_{(i,j)} \{G_4^{ij}, G_4^0\}$$

we get for $\lambda > C_3$

$$\begin{aligned}\max_{t\in[0,T]} |c_{1r_1-1}(t)| \leqslant{}& \max_{t\in[0,T]} |u(\lambda, t)|\, \lambda^{-r_1+1} G_3 \exp[-G_4 \lambda^{1/(1-p_1)}]\\ &+ \sum_{j=0}^{r_1-2} \max_{t\in[0,T]} |c_{1j}(t)|\, \lambda^{j-r_1+1}\\ &+ G_3\left(\sum_{i=2}^{k}\sum_{j=0}^{r_i-1} \max_{t\in[0,T]} |c_{ij}(t)|\, \lambda^{j-r_1+1}\right)\\ &\cdot \exp[-G_4 \lambda^{1/(1-p_1)}].\end{aligned}$$

The function $u(\cdot, \cdot) \in \mathscr{U}_{p_0}$. Thus there exist $\varepsilon > 0$ and $M_{\varepsilon T} > 0$ such that for sufficiently large λ

$$\max_{t\in[0,T]} |u(\lambda, t)| < M_{\varepsilon T} \exp[\lambda^{\frac{1}{1-p_0} - \varepsilon}].$$

Following the fact that $p_0 \leqslant p_1$ and then $1/(1-p_0) - \varepsilon < 1/(1-p_1)$ we get

$$\lim_{\lambda\to\infty} \{G_3 \max_{t\in[0,T]} |u(\lambda, t)|\, \lambda^{-r_1+1} \exp[-G_4 \lambda^{1/(1-p_1)}]\} = 0.$$

For each $j \in \{0, 1, \ldots, r_1-2\}$ we have $j - r_1 + 1 < 0$ whence

$$\lim_{\lambda\to\infty} \left\{\sum_{j=0}^{r_1-2} \max_{t\in[0,T]} |c_{ij}(t)|\, \lambda^{j-r_1+1}\right\} = 0.$$

Finally, due to the fact that $-G_4 \lambda^{1/(1-p_1)} < 0$, we obtain

$$\lim_{\lambda\to\infty} \left\{G_3\left(\sum_{i=2}^{k}\sum_{j=0}^{r_i-1} \max_{t\in[0,T]} |c_{ij}(t)|\, \lambda^{j-r_1+1}\right) \exp[-G_4 \lambda^{1/(1-p_1)}]\right\} = 0$$

and hence $\max_{t\in[0,T]} |c_{1r_1-1}(t)| = 0$. Then $c_{1r_1-1}(t) = 0$ on $[0, T]$ and, consequently, $c_{1r_1-1} = 0$ on $\mathbf{R}^+$, since the constant T was chosen arbitrarily. This means that in fact the function $c_{1r_1-1}\lambda^{r_1-1}e^{w_1\lambda}$ does not exist in sum (20).

Repeating the applied method to the each coefficient c_{ij} we get finally that each of them vanishes and this fact finishes the proof.

REMARKS. The result obtained in Theorem 2.1 leads to the explanation of a certain anomaly lying in different behaviour of the uniqueness of the solution of the wave equation and the heat equation. The wave equation has only the trivial solution in the class of smooth functions defined on the halfplane but the uniqueness for the heat equation depends on the degree of increasing of the solution.

The following example will try to explain this anomaly.

EXAMPLE. Let us consider the family of equations

$$u(\lambda, t) = \frac{1}{\Gamma(c_n+1)}\int_0^t (t-r)^{c_n}\frac{\partial^2}{\partial\lambda^2}u(\lambda, r)\,dr, \tag{24}$$

where $c_n = (n-1)/(n+1)$. This family of equations is conneted to the family of operational equations

$$D^2u(\lambda) - s^{c_n+1}u(\lambda) = 0. \tag{25}$$

Let us notice that for $n = 1$ equation (23) has the form

$$D^2u(\lambda) - su(\lambda) = 0 \tag{26}$$

and operational equation (26) is connected to the heat equation with the null initial condition. When n tends to infinity equations (24) transforms to the form

$$D^2u(\lambda) - s^2u(\lambda) = 0. \tag{27}$$

Equation (27) is connected to the wave equation with null initial conditions. The general solution of (24) is

$$u(\lambda) = c_1e^{s^{\frac{n}{n+1}}\lambda} + c_2e^{s^{-\frac{n}{n+1}}\lambda} \tag{28}$$

and the sets $\mathscr{U}_\alpha$ consist of the smooth functions fulfilling the condition

$$|u(\lambda, t)| < \exp[|\lambda|^{n+1-\varepsilon}], \tag{29}$$

where $\varepsilon > 0$ is arbitrarily small. It follows from (29) for $n = 1$ that $\mathscr{U}_\alpha = \mathscr{U}_1$ and it is the same class of functions for which Tichonoff Theorem of uniqueness holds. In the case $n \to \infty$, by (29), we get no restrictions, agreely to the d'Alembert Theorem of the existence and uniqueness of the solution of the wave equation.

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