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ON A PROBLEM OF Z. DARÓCZY

Abstract. Some conditions on $F: X \to \mathbf{R}$ ($X = (0, \infty)$ or $X = \mathbf{N}$) which guarantee that all solutions of the equation

$$F(x) = F(x+1) + F(x(x+1))$$

have to be of the form $F(x) = \frac{F(1)}{x}$ are given.

During the Twenty-fourth International Symposium on Functional Equations in South Hadley Z. Daróczy posed the following problem (see [3]): Let $X = \mathbf{N}$ or $X = (0, \infty)$. Find some weak conditions on solutions $F: X \to \mathbf{R}$ of the equation

$$F(x) = F(x+1) + F(x(x+1)) \tag{1}$$

under which the function F is of the form

$$F(x) = \frac{F(1)}{x}, \quad x \in X. \tag{2}$$

In the case $X = \mathbf{N}$ he conjectured that the positivity of the solution is such a condition. This, however, has been recently disproved by M. Laczkovich and R. Redheffer (see [4, Corollary 2], also [6, Problem 19 and Remarks 27 and 28]). Some answer to Daróczy's question was given by K. Baron [1] who proved that the solution $F: X \to \mathbf{R}$ has form (2) provided there exists a finite limit

$$\lim_{x \to \infty} xF(x). \tag{3}$$

Recently he has strengthened this result showing that if $F: X \to \mathbf{R}$ is a solution of equation (1) and the limit (3) exists then it is necessarily finite (see [2, Theorem]).

Below we provide a few conditions satisfying the request of Z. Daróczy. As K. Baron observed (cf. [2, Remark]), making use of a result of M. Kuczma, one can derive some of them from the theorem presented in [2]. This concerns Theorem 1 as well as the "continuous" parts of Theorems 2 and 3. Nevertheless,

Manuscript received November 24, 1987, and in final form June 30, 1989.
AMS (1991) subject classification: 39 B 22.
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we present them here for their independent and immediate proofs. Moreover, Theorems 2 and 3 deal also with discrete versions of the proposed conditions. Theorem 4 yields another answer to Daróczy's question.

Let us start with a simple observation showing that equation (1) may be reduced to the Cauchy equation on the graph of a function. Namely we have the following fact.

REMARK 1. *A function* $F:(0, \infty)\to\mathbf{R}$ *is a solution of equation* (1) *if and only if the function* $(0, \infty)\ni x\mapsto F(1/x)$ *satisfies the equation*

$$\varphi(x+f(x)) = \varphi(x)+\varphi(f(x)) \tag{4}$$

where $f:(0, \infty)\to\mathbf{R}$ *is given by*

$$f(x) = \frac{2}{1+\sqrt{1+4/x}}.$$

Indeed, putting $\varphi(x) = F(1/x)$, $x\in(0, \infty)$, we see that the condition

$$F(x) = F(x+1)+F(x(x+1)), \qquad x\in(0, \infty),$$

holds if and only if

$$F\left(\frac{1}{y}\right) = F\left(\frac{y+1}{y}\right)+F\left(\frac{y+1}{y^2}\right), \quad y\in(0, \infty),$$

which can be rewritten as

$$\varphi(y) = \varphi\left(\frac{y}{y+1}\right)+\varphi\left(\frac{y^2}{y+1}\right), \quad y\in(0, \infty),$$

or equivalently, putting $x = \dfrac{y^2}{y+1},$

$$\varphi(x+f(x)) = \varphi(x)+\varphi(f(x)), \quad x\in(0, \infty).$$

This remark explains the situation a little. It is known that usually no continuity or regularity condition guarantees the uniqueness of solutions of the Cauchy equation on the graph of a given function. In the case of equation (1) (so consequently for equation (4) also) this was observed by Z. Moszner. He proved in [5] that if $X = (0, \infty)$ then the general solution as well as continuous solutions of (1) depend on an arbitrary function. So it seems that conditions giving the uniqueness in the problem of Z. Daróczy have to be of a "geometric" or "asymptotic" type like these from Baron's results or Theorems 1—4 below. On the other hand it should not be expected that the reduction of equation (1) to the Cauchy equation (4) will produce some new interesting results. As one can easily check applying a result of M. C. Zdun (cf. [7, Theorem 1]) to equation (4) we obtain another proof of Baron's theorem presented in [1]. But it seems that none of known results concerning equations of type (4) is helpful in proving his theorem in the more general setting from [2].

Before stating results we formulate and prove a useful fact.

LEMMA. *Let* x_0 *be a positive number and let* $F_0:(x_0, \infty)\cap X\to\mathbf{R}$ *be a function satisfying equation* (1). *Then there is exactly one solution* $F:X\to\mathbf{R}$ *of equation* (1) *such that* $F|_{(x_0,\infty)\cap X} = F_0$.

Proof. Choose an $m \in \mathbf{N}_0$ in such a manner that

$$m < x_0 \leqslant m+1.$$

First assume for a moment that $m \geqslant 1$ and for every $n = 1, \ldots, m$ define the function $F_n : (x_0 - n, \infty) \cap X \to \mathbf{R}$ by

$$F_n(x) = F_{n-1}(x+1) + F_{n-1}(x(x+1)). \tag{5}$$

Clearly F_m is a (unique) extension of the function F_0 to a solution of equation (1) that is defined on the set $(x_0 - m, \infty) \cap X$. This procedure shows that we can assume in fact the relation $0 < x_0 \leqslant 1$.

Now put

$$x_n = \frac{\sqrt{4x_{n-1}+1}-1}{2}, \quad n \in \mathbf{N},$$

and observe that

$$0 < x_n < x_{n-1}, \quad n \in \mathbf{N},$$
$$\lim_{n \to \infty} x_n = 0,$$

and, for any $n \in \mathbf{N}$,

$$\text{if } x \geqslant x_n \text{ then } x+1,\ x(x+1) \geqslant x_{n-1}.$$

Thus the formula

$$F(x) = F_n(x), \quad x \in (x_n, \infty) \cap X,\ n \in \mathbf{N},$$

where the functions $F_n : (x_n, \infty) \cap X \to \mathbf{R}$ are given by (5), correctly defines the function F that is a (unique) extension of F_0 to a solution of equation (1) defined on the set X.

THEOREM 1. *Let $F : X \in \mathbf{R}$ be a solution of equation* (1). *Assume that, for a non-negative number a, the function $(a, \infty) \cap X \ni x \mapsto xF(x)$ is monotonic. Then the function F is of form* (2).

Proof. Putting

$$\Phi(x) = xF(x), \quad x \in X,$$

and making a simple computation we get

$$\Phi(x) = \frac{x}{x+1}\Phi(x+1) + \frac{1}{x+1}\Phi(x(x+1)), \quad x \in X. \tag{6}$$

Assume for instance that the function $\Phi|_{(a, \infty) \cap X}$ is increasing. Let us put $x_0 = \max\{1, a\}$ and fix an $x \in (x_0, \infty) \cap X$. Then

$$x(x+1) > x+1 > x > a$$

whence, by (6),

$$\Phi(x) \geqslant \frac{x}{x+1}\Phi(x+1) + \frac{1}{x+1}\Phi(x+1) = \Phi(x+1) \geqslant \Phi(x),$$

and, consequently, $\Phi(x+1) = \Phi(x)$. Thus the function $\Phi|_{(x_0, \infty) \cap X}$ is constant which, due to Lemma, gives our assertion.

In the sequel, in the case $X = \mathbf{N}$, convexity of the function F will mean that

$$F(n+1) \leqslant \frac{F(n) + F(n+2)}{2}, \quad n \in \mathbf{N}.$$

In can be easily shown that every such a function is the restriction of a convex function defined on the half-line $(0, \infty)$. (Any continuous function $F^*:(0, \infty)\to\mathbf{R}$ coinciding with F on $\mathbf{N}$, linear on each interval $(n-1, n]$, $n\in\mathbf{N}$, and such that

$$F^*(0+)\geqslant 2F(1)-F(2)$$

satisfies the requirement.) Consequently, given a convex function $F:X\to\mathbf{R}$ we can assert that

$$F(\lambda x+(1-\lambda)y)\leqslant \lambda F(x)+(1-\lambda)F(y)$$

for any x, $y\in X$ and $\lambda\in[0,1]$ such that $\lambda x+(1-\lambda)y\in X$ and that the function F is monotonic in a vicinity of infinity.

As an immediate consequence of Theorem 1 we have the following fact.

COROLLARY *Let $F:X\to\mathbf{R}$ be a solution of equation* (1). *Assume that, for a non-negative number a, the function* $(a, \infty)\cap X\ni x\mapsto xF(x)$ *is convex* (*concave*). *Then the function F is of form* (2).

THEOREM 2. *Let $F:X\to\mathbf{R}$ be a solution of equation* (1). *Assume that, for a non-negative number a, the function* $(a, \infty)\cap X\ni x\mapsto x^2F(x)$ *is convex* (*concave*). *Then the function F is of form* (2).

Proof. Fix an integer $n>\max\{1, a\}$. Then

$$n+1 = \left(1-\frac{1}{n^2}\right)n+\frac{1}{n^2}n(n+1).$$

Thus

$$(n+1)^2F(n+1)\leqslant\left(1-\frac{1}{n^2}\right)n^2F(n)+\frac{1}{n^2}n^2(n+1)^2F(n(n+1)),$$

and, making use of equality (1),

$$(n+1)^2F(n+1)\leqslant(n^2-1)F(n)+(n+1)^2(F(n)-F(n+1)),$$

i.e.

$$2(n+1)^2F(n+1)\leqslant 2n(n+1)F(n).$$

This means that

$$(n+1)F(n+1)\leqslant nF(n), \quad n\in\mathbf{N},\ n>\max\{1, a\},$$

whence, in virtue of Theorem 1,

$$n^2F(n) = F(1)n, \quad n\in\mathbf{N}.$$

Thus the function $\mathbf{N}\ni x\mapsto x^2F(x)$ is linear. So, due to the convexity of the function $(a, \infty)\cap X\ni x\mapsto x^2F(x)$,

$$x^2F(x) = F(1)x, \quad x\in(\max\{1, a\}, \infty)\cap X.$$

Making use of Lemma we complete the proof.

THEOREM 3. *Let $F:X\to\mathbf{R}$ be a solution of equation* (1). *Assume that, for a non-negative number a, either*

$$F(x)>0, \quad x\in(a, \infty)\cap X,$$

or

$$F(x)<0, \quad x\in(a, \infty)\cap X,$$

and the function $(a, \infty)\cap X \ni x \mapsto 1/F(x)$ *is convex (concave). Then the function* F *is of form* (2).

Proof. Without loss of generality we may assume that

$$F(x)>0, \quad x\in(a, \infty)\cap X, \tag{7}$$

which, by equality (1), gives the relation

$$F(x)-F(x+1)>0, \quad x\in(a, \infty)\cap X. \tag{8}$$

Fix an integer $n>\max\{1, a\}$. Then we have

$$n+1 = \left(1-\frac{1}{n^2}\right)n+\frac{1}{n^2}n(n+1),$$

whence

$$\frac{1}{F(n+1)}\leqslant\left(1-\frac{1}{n^2}\right)\frac{1}{F(n)}+\frac{1}{n^2}\frac{1}{F(n(n+1))},$$

and, using equality (1),

$$\frac{1}{F(n+1)}\leqslant\left(1-\frac{1}{n^2}\right)\frac{1}{F(n)}+\frac{1}{n^2}\frac{1}{F(n)-F(n+1)}.$$

Therefore

$$\frac{1}{F(n+1)}-\frac{1}{F(n)}\leqslant\frac{1}{n^2}\frac{F(n)-(F(n)-F(n+1))}{F(n)(F(n)-F(n+1))},$$

i.e.

$$\frac{F(n)-F(n+1)}{F(n)F(n+1)}\leqslant\frac{1}{n^2}\frac{F(n+1)}{F(n)(F(n)-F(n+1))}.$$

Hence, according to inequalites (7) and (8), we get

$$(F(n)-F(n+1))^2\leqslant\frac{1}{n^2}F(n+1)^2$$

and, using (7) and (8) again, we infer that

$$F(n)-F(n+1) \leqslant \frac{1}{n}F(n+1).$$

Thus

$$(n+1)F(n+1)\geqslant nF(n), \quad n\in\mathbf{N},\ n>\max\{1, a\},$$

whence, due to Theorem 1,

$$\frac{1}{F(n)}=\frac{1}{F(1)}n, \quad n\in\mathbf{N}.$$

In other words, by the convexity of the function $(a, \infty)\cap X\ni x\mapsto 1/F(x)$,

$$\frac{1}{F(x)}=\frac{1}{F(1)}x, \quad x\in(\max\{1, a\}, \infty)\cap X.$$

Now to complete the proof it is enough to apply Lemma.

THEOREM 4. *Let* $F:(0,\infty)\to\mathbf{R}$ *be a solution of equation* (1). *Assume that, for a positive number* a, *the function* $(0,a)\ni x\mapsto F(1/x)$ *is convex (concave). Then the function* F *is of form* (2).

Proof. Fix an integer $n>\max\{1, 1/a\}$. Then

$$\frac{1}{n(n+1)}<\frac{1}{n+1}<\frac{1}{n}<a$$

and

$$\frac{1}{n+1}=\left(1-\frac{1}{n}\right)\frac{1}{n}+\frac{1}{n}\,\frac{1}{n(n+1)}.$$

Hence, because of the convexity of the function $(0,a)\ni x\mapsto F(1/x)$,

$$F(n+1)\leqslant\left(1-\frac{1}{n}\right)F(n)+\frac{1}{n}\,F(n(n+1)),$$

i.e., taking into account equality (1),

$$F(n+1)\leqslant\left(1-\frac{1}{n}\right)F(n)+\frac{1}{n}(F(n)-F(n+1)).$$

Consequently

$$(n+1)F(n+1)\leqslant nF(n),\quad n\in\mathbf{N},\ n>\max\{1, 1/a\},$$

whence, in virtue of Theorem 1,

$$F(n)=\frac{F(1)}{n},\quad n\in\mathbf{N}.$$

Since the function $(0,a)\ni x\mapsto F(1/x)$ is convex it follows that

$$F\left(\frac{1}{x}\right)=F(1)x,\quad x\in(0,\min\{1, a\}).$$

In other words

$$F(x)=\frac{F(1)}{x},\quad x\in(\max\{1, 1/a\},\infty).$$

Recalling Lemma we end the proof.

The results given above suggest to ask about the uniqueness of solutions of equation (1) first of all in the class of convex functions. However, as follows from the quoted result of Laczkovich and Redheffer and from the following simple remark, in the case $X=\mathbf{N}$ the answer is negative.

REMARK 2. *Let* $F:\mathbf{N}\to\mathbf{R}$ *be a non-negative solution of equation* (1). *Then the function* F *is decreasing and convex.*

Indeed, in view of equality (1), we have

$$F(n)-F(n+1)=F(n(n+1))\geqslant 0,\quad n\in\mathbf{N},$$

i.e. the function F is decreasing. Moreover, for every $n\in\mathbf{N}$,

$$\begin{aligned}F(n)-2F(n+1)+F(n+2)&=(F(n)-F(n+1))-(F(n+1)-F(n+2))\\&=F(n(n+1))-F((n+1)(n+2))\geqslant 0\end{aligned}$$

whence

$$2F(n+1) \leqslant F(n)+F(n+2)$$

which means that the function F is convex.

In the case $X = (0, \infty)$ the problem of the uniqueness of convex solutions seems to be still open. If the answer was negative it would be interesting to study the problem in smaller classes, for instance for completely monotonic functions or logarithmically convex functions.

Regarding the case of the "continuous" variable we have only the following simple observation.

REMARK 3. *Let* $F:(0, \infty) \to \mathbf{R}$ *be a convex solution of equation* (1). *Then* F *is the zero function or the function* F *is positive and strictly decreasing.*

To prove this assume at first that $F(x_0) = 0$ for an $x_0 \in (0, \infty)$. If $x_0 \in (0, 1)$ then, by (1), $F(x_1) = 0$ for an $x_1 \in [x_0(x_0+1), x_0+1]$. In general, if the function F vanishes at a point $x_n \in (0, 1)$, $n \in \mathbf{N}$, then $F(x_{n+1}) = 0$ for an $x_{n+1} \in [x_n(x_n+1), x_n+1]$. Since $x_{n+1} > x_n$ and

$$x_{n+1} - x_n \geqslant x_n(x_n+1) - x_n = x_n^2$$

it follows that $x_n \in [1, \infty)$ for an $n \in \mathbf{N}$. Thus we may assume without loss of generality that $x_0 \in [1, \infty)$.

We shall define by induction a sequence $(x_n : n \in \mathbf{N})$ of numbers greater than 1 such that

$$F(x_n) = 0, \quad n \in \mathbf{N}, \tag{9}$$

and

$$\lim_{n\to\infty} x_n = \infty. \tag{10}$$

If the number $x_n \in [1, \infty)$ is defined it follows from (9) that the function F vanishes at a point $x_{n+1} \in [x_n+1, x_n(x_n+1)]$. Since $x_{n+1} \geqslant x_n + 1$ the sequence $(x_n : n \in \mathbf{N})$ has evidently property (10). From (9) and (10), because of the convexity of F, we get

$$F(x) = 0, \quad x \in [x_0, \infty),$$

which, on account of Lemma, means that

$$F(x) = 0, \quad x \in (0, \infty).$$

Now suppose that F takes a negative value at a point. Then, according to the first part of the proof, $F(x) \neq 0$ for every positive x, whence, by the continuity of F,

$$F(x) < 0, \quad x \in (0, \infty). \tag{11}$$

Since F is convex there exists an $a \in (0, \infty)$ such that the function $F|_{(a, \infty)}$ is monotonic. If it were increasing then, due to (11) and the convexity of F, the function $F|_{(a, \infty)}$ would be costant and negative which, by (1), is impossible. If the function $F|_{(a, \infty)}$ were decreasing then, using (1) and (11), we would have

$$F(x) = F(x+1) + F(x(x+1)) < F(x+1) \leqslant F(x)$$

for every $x \in (a, \infty)$ which is also impossible.

Therefore we can assume that

$$F(x) > 0, \quad x \in (0, \infty). \tag{12}$$

If the function F were not decreasing then, because of its convexity, there would exist an $a \in (0, \infty)$ such that the function $F|_{(a, \infty)}$ is increasing. But then, by (1) and (12), we would have

$$F(x) = F(x+1) + F(x(x+1)) > F(x+1) \geqslant F(x)$$

for every $x \in (a, \infty)$. It is also clear that the function F is in fact strictly decreasing. Otherwise, since F is decreasing and convex, there would exist an $a \in (0, \infty)$ such that the function $F|_{(a, \infty)}$ is constant and (cf. (12)) positive. But such a function does not satisfy equation (1).

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