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THE STABILITY OF THE PEXIDER EQUATION

Abstract. A stability theorem for the Pexider functional equation (1) is proved. It is an analogue of the classical theorem of Hyers [5] relating to the stability of the Cauchy functional equation.

In 1941 D. H. Hyers proved that *if* $f:E\to E'$ *maps a Banach space* E *into a Banach space* E' *and satisfies for some* $\varepsilon>0$ *the condition*

$$\|f(s+t)-f(s)-f(t)\|\leqslant\varepsilon,\quad s,\ t\in E,$$

then there exists exactly one mapping $l:E\to E'$ *such that* $l(s+t)=l(s)+l(t)$ *and* $\|f(s)-l(s)\|\leqslant\varepsilon$ *for all* $s,\ t\in E$. It was an affirmative answer to a question of S. Ulam [8] concerning the stability of the linear functional equation. Since that time many papers appeared concerning the stability of other functional equations as well as generalizing the Hyers theorem (see, for instance, [1]—[4], [6], [7]). In the present note we shall give another result of such a type; namely, we shall prove the following stability theorem for the Pexider functional equation

$$f(s+t)=g(s)+h(t) \tag{1}$$

with three unknown functions f, g and h.

THEOREM. *Let* $(S,\ +)$ *be an abelian semigroup with zero and let* Y *be a sequentially complete topological vector space over the field* $\mathbf{Q}$ *of all rational numbers. Assume that* V *is a non-empty,* $\mathbf{Q}$*-convex symmetric and bounded subset of* Y. *If functions* $f:S\to Y$, $g:S\to Y$ *and* $h:S\to Y$ *satisfy the condition*

$$f(s+t)-g(s)-h(t)\in V,\quad s,\ t\in V, \tag{2}$$

then there exist functions $f_1:S\to Y$, $g_1:S\to Y$ *and* $h_1:S\to Y$ *satisfying the equation* (1) *for all* $s,t\in S$ *and such that* $f_1(s)-f(s)\in 3\,\mathrm{seqcl}\,V$, $g_1(s)-g(s)\in 4\,\mathrm{seqcl}\,V$ *and* $h_1(s)-h(s)\in 4\,\mathrm{seqcl}\,V$ *for all* $s\in S$.

In the proof of this theorem a basic role is played by a lemma on the existence of additive selections of subadditive multifunctions due to Z. Gajda and R. Ger. A multifunction $F:S\to 2^Y$ is said to be *subadditive* iff

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$$F(s+t)\subset F(s)+F(t), \quad s,t\in S,$$

where the symbol "+" on the right side means the algebraic sum of sets. Given a bounded set $A\subset Y$ and a neighbourhood U of zero we define the relative diameter of A with respect to U as $\operatorname{diam}_U A := \inf\{q\in\mathbf{Q}\cap(0,\infty): A-A\subset qU\}$. By $\operatorname{seqcl} A$ we denote the sequential closure of A.

LEMMA (Gajda and Ger [4]). *Let $(S, +)$ be an abelian semigroup and let Y be a sequentially complete topological vector space over $\mathbf{Q}$. Assume that $F:S\to 2^Y\setminus\{\varnothing\}$ is a subadditive multifunction such that $F(s)$ is $\mathbf{Q}$-convex for all $s\in S$ and $\sup\{\operatorname{diam}_U F(s):\ s\in S\}<\infty$ for every $\mathbf{Q}$-balanced neighbourhood U of zero. Then there exists an additive function $\varphi:S\to Y$ such that $\varphi(s)\in\operatorname{seqcl} F(s)$ for all $s\in S$.*

Proof of Theorem. Let $a:=g(0)$, $b:=h(0)$ and $f_0:=f-a-b$. Then, setting in (1) $t=0$ and $s=0$ in succesion, we obtain

$$f_0(s)+a-g(s) = f(s)-b-g(s)\in V, \quad s\in S, \tag{3}$$

and

$$f_0(t)+b-h(t) = f(t)-a-h(t)\in V, \quad t\in S. \tag{4}$$

Consider the multifunction $F_0:S\to 2^Y$ defined by $F_0(s):=f_0(s)+3V$, $s\in S$. This multifunction is subadditive because, in view of (2), (3), (4) and the symmetricity of V, we have

$$\begin{aligned}F_0(s+t) &= f_0(s+t)+3V = f(s+t)-a-b+3V\subset g(s)+h(t)-a-b+4V\\ &\subset f_0(s)-V+f_0(t)-V+4V = F_0(s)+F_0(t).\end{aligned}$$

Moreover, for an arbitrary neighbourhood U of zero we have

$$\sup\{\operatorname{diam}_U F(s):\ s\in S\} = \inf\{q\in\mathbf{Q}\cap(0,\ \infty): 3V-3V\subset qU\}<\infty,$$

since the set $3V-3V$ is bounded. Therefore, in virtue of the lemma of Gajda and Ger, there exists an additive function $\varphi:S\to Y$ such that

$$\varphi(s)\in\operatorname{seqcl} F_0(s), \quad s\in S.$$

Put $f_1:=\varphi+a+b$, $g_1:=\varphi+a$ and $h_1:=\varphi+b$. It is easily seen that these functions satisfy the equation (1). Moreover, for all $s\in S$ we have

$$f_1(s)-f(s) = \varphi(s)+a+b-f_0(s)-a-b\in\operatorname{seqcl} F_0(s)-f_0(s) = 3\operatorname{seqcl} V.$$

On the other hand, using (3) we get

$$\begin{aligned}g_1(s)-g(s) &= \varphi(s)+a-g(s)\in\operatorname{seqcl} F_0(s)+a-g(s)\\ &= f_0(s)+\operatorname{seqcl} 3V+a-g(s)\subset 3\operatorname{seqcl} V+V\subset 4\operatorname{seqcl} V,\end{aligned}$$

and similarly, by means of (4), we obtain

$$h_1(s)-h(s)\in 4\operatorname{seqcl} V$$

for all $s\in S$. This completes our proof.

As an immediate consequence of this theorem we obtain the following.

COROLLARY. *Assume that* $(S, +)$ *is an abelian semigroup with zero and* $(Y, \|\cdot\|)$ *is a Banach space. If functions* $f: S \to Y$, $g: S \to Y$ *and* $h: S \to Y$ *satisfy the inequality*

$$\|f(s+t)-g(s)-h(t)\| \leqslant \varepsilon \tag{5}$$

for all $s, t \in S$ *and some* $\varepsilon > 0$, *then there exist functions* $f_1: S \to Y$, $g_1: S \to Y$ *and* $h_1: S \to Y$ *which solve the equation* (1) *and satisfy for all* $s \in S$ *the conditions*

$$\|f_1(s)-f(s)\| \leqslant 3\varepsilon, \quad \|g_1(s)-g(s)\| \leqslant 4\varepsilon \quad \text{and} \quad \|h_1(s)-h(s)\| \leqslant 4\varepsilon. \tag{6}$$

REMARK. The functions f_1, g_1 and h_1 occuring in the assertion of our theorem need not be unique. Assume, for example, that $f: S \to Y$, $g: S \to Y$ and $h: S \to Y$ (Y is a Banach space) are a solution of the Pexider equation (1). Then the assumption (5) is fulfilled trivially with an arbitrary $\varepsilon > 0$. On the other hand, for every $a, b \in Y$ such that $\|a\| \leqslant 4\varepsilon$, $\|b\| \leqslant 4\varepsilon$ and $\|a+b\| \leqslant 3\varepsilon$ the functions $f_1 := f+a+b$, $g_1 := g+a$ and $h_1 := h+b$ satisfy the equation (1) as well as conditions (6).

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