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SOME NATURAL AXIOM SYSTEM OF THE PLANE EUCLIDEAN GEOMETRY

Abstract. The paper contains some system of axioms of the plane Euclidean geometry concerning the notions of points, segments and congruence of segments.

Introduction. The notion of segment is primary with regard to the notion of line in the school teaching. In this paper we attempt to modify Hilbert's axioms of the plane Euclidean geometry by changing the concept "line" and axioms relating to incidence and betweenness for a "segment" and axioms relating to this notion. The remaining axioms we retain as in [1] or [2] with only small changes making it possible to formulate these axioms in the new terminology.

In Section 1 we recall Hilbert's axioms modified as in [1] and [2]. In Section 2 we formulate the proposed axiomatics and in Section 3 the equivalence of these systems of axioms is proved.

1. Hilbert's axiomatic system. A structure $\mathscr{E} = (E, L, |, \mu, \equiv)$, where E is a set of points, L—a set of lines, $| \subset E \times L$—a relation of incidence, $\mu \subset E^3$—a relation of betweenness and $\equiv \subset E^4$—a relation of an equal distance, is called a *Euclidean plane* if and only if $\mathscr{E}$ satisfies the following axioms:

H1 $\forall_{a,\, b \in E}\ \exists_{A \in L}\ (a,\ b | A)$,

H2 $\forall_{a,\, b \in E}\ \forall_{A,\, B \in L}\ (a,\ b | A,\ B \wedge a \neq b \Rightarrow A = B)$,

H3 $\forall_{A \in L}\ \exists_{a,\, b \in E}\ (a,\ b | A \wedge a \neq b)$,

H4 $\exists_{a,\, b,\, c \in E}\ \forall_{A \in L}\ \sim(a,\ b,\ c | A)$,

H5 $\forall_{a,\, b,\, c \in E}\ (b \mu ac \Rightarrow a \neq b \neq c \neq a \wedge \exists_{A \in L}\ (a,\ b,\ c | A))$,

H6 $\forall_{a,\, b,\, c \in E}\ (b \mu ac \Rightarrow\ \sim a \mu bc \wedge \sim c \mu ab)$,

H7 $\forall_{a,\, b \in E}\ \forall_{A \in L}\ (a \neq b \wedge a,\ b | A \Rightarrow \exists_{c \in E} (c | A \wedge b \mu ac))$,

H8 (the axiom of Pasch)

$$\forall_{a,\, b,\, c,\, d \in E}\ \forall_{A \in L}\ (d | A \wedge d \mu ab \wedge \sim a | A \wedge \sim b | A \wedge \sim c | A \wedge \forall_{B \in L} \sim (a,\ b,\ c | B) \Rightarrow \exists_{e \in E}\ (e | A \wedge (e \mu ac \vee e \mu bc))),$$

H9 $\forall_{a,\, b \in E}\ (ab \equiv ba)$,

and axioms:

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H10 transitivity of the relation $\equiv$,
H11 adding of segments,
H12 marking off a segment on a half-line,
H13 marking of a triangle,
H14 about five segments,
H15 Archimedean property,
H16 Cantor's continuity axiom
and
H17 (Euclid's parallel axiom)

$$\forall_{A,B,C\in L}\ \forall_{a\in E}\ (a|A,\ B \wedge \sim a|C \wedge \forall_{b\in E}\ (b|c \Rightarrow \sim b|A \wedge \sim b|B) \Rightarrow A = B).$$

Since corresponding axioms A10—A16 formulated in Section 2 differ from H10—H16 only in the form of notation (the notion of segment instead of the betweenness relation), only their names are specified above.

2. A *S*-axiom system. Given a set M, whose elements are called points, and a function $[\,]$: $M \times M \rightarrow 2^M$ satisfying the following conditions (axioms):
A1 $\forall_{a,b\in M}\ (\{a,\ b\} \subset [ab])$,
A2 $\forall_{a,b\in M}\ ([ab] \subset \{a,\ b\} \Leftrightarrow a = b)$,
A3 $\forall_{a,b,c,d\in M}\ ([ab] = [cd] \Leftrightarrow \{a,\ b\} = \{c,\ d\})$,
A4 $\forall_{a,b,c\in M}\ (c \in [ab] \Leftrightarrow [ac] \cup [cb] = [ab])$,
A5 $\forall_{a,b,c\in M}\ (c \in [ab] \Rightarrow [ac] \cap [cb] = \{c\})$,
A6 $\forall_{a,b,c,d,k,l\in M}\ \exists_{e,f,g,h\in M}\ (k,\ l \in [ab] \cap [cd] \wedge k \neq l$
$\Rightarrow [ab] \cup [cd] = [ef] \wedge [ab] \cap [cd] = [gh] \wedge \{a,\ b,\ c,\ d\} = \{e,\ f,\ g,\ h\})$,
A7 $\exists_{a,b,c\in M}\ (a \notin [bc] \wedge b \notin [ac] \wedge c \notin [ab])$.

The value of the function $[\,]$ at a pair of points a, b is called a *segment* with terminal points a, b or more shortly a segment $[ab]$. The set of all segments is denoted here by S, i.e.

(2.1) $$S := \{s \in 2^M\colon\ \exists_{a,b\in M}\ (s = [ab])\}.$$

We say that the points a, b, c are *collinear* and we write $w(a,\ b,\ c)$ iff one of these points belongs to a segment whose terminal points are the remaining two points, i.e.

(2.2) $$w(a,\ b,\ c) \Leftrightarrow (a \in [bc] \vee b \in [ac] \vee c \in [ab]).$$

For three arbitrary noncollinear points a, b, c a set

(2.3) $$[abc] := \{p \in M\colon\ \exists_{q\in[ab]}\ (p \in [qc])\}$$

is called a *triangle with vertices a, b, c* or more shortly a triangle $[abc]$. We admit that this set does not depend on the order of the vertices, i.e. we admit the axiom
A8 $\forall_{a,b,c\in M}\ (\sim w(a,\ b,\ c) \Leftrightarrow [abc] = [bca])$.

In the set S we define a 2-ary relation $\|$ of parallelity of segments by

(2.4) $$s_1 \| s_2 :\Leftrightarrow ((\exists_{s_3\in S}\ (s_1 \cup s_2 \subset s_3)) \vee \forall_{s_4,s_5\in S}\ (s_1 \subset s_4 \wedge s_2 \subset s_5 \Rightarrow s_4 \cap s_5 = \varnothing))$$

and we assume that this relation is transitive, i.e.
A9 $\forall_{s_1,s_2,s_3\in S}\ (s_1 \| s_2 \wedge s_2 \| s_3 \Rightarrow s_1 \| s_3)$.

A structure $\mathscr{M} = (M, S, [\,], \equiv_s)$, where $\equiv_s \subset S^2$, is called an *S-Euclidean plane* or more shortly an *S-structure* iff $\mathscr{M}$ satisfies A1—A9 and the following axioms:

A10 (transitivity of the relation $\equiv_s$)

$\forall_{s_1, s_2, s_3 \in S}\ (s_2 \equiv_s s_1 \wedge s_3 \equiv_s s_1 \Rightarrow s_2 \equiv_s s_3)$,

A11 (adding of segments)

$$\forall_{a, b, c, d, e, f \in M}\ (b \in [ac] \wedge e \in [df] \wedge [ab] \equiv_s [de] \wedge [bc] \equiv_s [ef] \Rightarrow [ac] \equiv_s [df]),$$

A12 (marking off a segment on a half-line)

$\forall_{a, b, c, d \in M}\ (a \neq b \Rightarrow \exists^1_{e \in M}\ (b \in [ae] \wedge [be] \equiv_s [cd]))$,

A13 (marking off a triangle)

$$\forall_{a, b, c, d, e \in M}\ (\sim w(a, b, c) \wedge [ab] \equiv_s [de] \Rightarrow \exists^2 f \in M\ ([ac] \equiv_s [df] \wedge [bc] \equiv_s [ef])),$$

A14 (about five segments)

$$\forall_{a, b, c, d, e, f, g, h \in M}\ (c \in [ab] \wedge g \in [ef] \wedge \sim w(a, b, d) \wedge [ac] \equiv_s [eg] \wedge [cb] \equiv_s [gf] \wedge [ad] \equiv_s [eh] \wedge [cd] \equiv_s [gh] \Rightarrow [bd] \equiv_s [fh]),$$

A15 (the Archimedean property)

$$\forall_{a, b, c, d \in M}\ (c \neq d \Rightarrow \exists_{n \in \mathbf{N}} \exists_{p_0, \ldots, p_n \in M}\ (a = p_0 \wedge p_1 \in [p_0 p_2] \wedge \ldots \wedge p_{n-1} \in [p_{n-2} p_n] \wedge [p_0 p_1] \equiv_s [p_1 p_2] \equiv_s \ldots \equiv_s [p_{n-1} p_n] \equiv_s [cd] \wedge b \in [p_{n-1} p_n])),$$

A16 (Cantor's continuity property)

$\forall (s_n)_{n \in \mathbf{N}} (\forall_{n \in \mathbf{N}}\ (s_n \in S \wedge s_{n+1} \subset s_n) \Rightarrow \exists_{p \in M} \forall_{n \in \mathbf{N}} (p \in s_n))$.

The relation $\equiv_s$ is called a *congruence of segments.*

3. Equivalence of the given axiomatic systems. If $\mathscr{E} = (E, L, |, \mu, \equiv)$ is the Euclidean plane, then putting

$$[ab] := \{p \in E \colon p = a \vee p\mu ab \vee p = b\}, \tag{3.1}$$

$$[ab] \equiv_s [cd] \Leftrightarrow ab \equiv cd \tag{3.2}$$

we easily find that $(E, S, [\,], \equiv_s)$ is an S-structure.

Now we assume that $\mathscr{M} = (M, S, [\,], \equiv_s)$ is the S-Euclidean plane and for arbitrary points $a, b \in M$, $a \neq b$ we define a line $a \cdot b$ as a set

$$a \cdot b := \{p \in M \colon \exists_{s \in S}\ (a, b, p \in s)\}. \tag{3.3}$$

Putting

$$L := \{A \in 2^M \colon \exists_{a, b \in M}\ (a \neq b \wedge A = a \cdot b)\}, \tag{3.4}$$

$$(a | A :\Leftrightarrow a \in A) \text{ for } a \in M \text{ and } A \in L, \tag{3.5}$$

by (2.1), A1—A3 and A7, we find that the conditions H1, H3 and H4 are satisfied.

LEMMA *If* $a, b, c \in M$, $b \neq a \neq c$ *and* $c \in a \cdot b$ *then* $a \cdot b = a \cdot c$.

Proof. It follows form the assumptions and (3.3) that there exists a segment $s_1 \in S$ such that $a, b, c \in s_1$. Hence, by (3.3), $b \in a \cdot c$ and from symmetry of assumptions, it suffices to prove an inclusion $a\cdot b\subset a\cdot c$. If $p\in a\cdot b$ then there exists a segment $s_2\in S$ such that a, b, $p\in s_2$, and from A6, $s_1\cup s_2\in S$ because a, $b\in s_1\cap s_2$ and $a\neq b$. Let $s=s_1\cup s_2$. Since $a,b,c,p\in s$, then $p\in a\cdot c$. Hence $a\cdot b\subset a\cdot c$.

It follows immediately from the lemma that for arbitrary a, $b\in M$, $a\neq b$ and $A\in L$ the condition a, $b\in A$ implies an equality $A=a\cdot b$. This proves H2.

Now we define a parallelism of lines by

$$A\|B\Leftrightarrow(A=B\vee A\cap B=\emptyset) \tag{3.6}$$

for $A, B\in L$. From (2.4), (3.4) and A9 we derive a condition

$$\forall_{A,\,B,\,C\in L}\ (A\|B\wedge B\|C\Rightarrow A\|C),$$

which proves H17.

Putting

$$a\mu bc\Leftrightarrow(a\neq b\neq c\neq a\wedge a\in[bc]) \tag{3.7}$$

for $a,b,c\in M$, by A3—A5, we obtain H5 and H6.

To prove the axiom of Pasch (H8) let us assume that points a, b, c are not collinear and a line A cuts the segment $[ab]$ at a point d different from a and b, i.e. $d\in A$, $d\mu ab$, a, b, $c\notin A$ and $\sim w(a, b, c)$. According to Euclid's parallel axiom H17 a line A meets at least one of the lines $a\cdot c$, $b\cdot c$ and it suffices to consider one of these cases. Let A meets $b\cdot c$ at a point p. Since $p\in b\cdot c$ then either $p\mu bc$ or $b\mu cp$ or $c\mu bp$. Hence it suffices to consider the alternative $b\mu cp\vee c\mu bp$ then $\sim w(a, c, p)$ and from A4, A5, (2.3) and A8 we obtain the equality $[acp]=[abc]\cup[abp]$. Since $d\in[acp]$ then there exists a point $e\in[ac]$ such that $d\in[pe]$. Moreover $a\neq e\neq c$. Hence $e|A$ and $e\mu ac$. Next, let us assume that $c\mu bp$. Analogically to the above we obtain $\sim w(a, b, p)$, $[abp]=[abc]\cup[acp]$ and $[abc]=[adc]\cup[dbc]$. The axiom A3 implies the existence of a point $q\in M$ such that $q\mu dp$, Now, if $q\in[acp]$, then from (2.3) there exists a point e such that $e\in[ac]$ and $q\in[ep]$, i.e. $e\mu ac$ and $e|A$. If $q\in[abc]$ then $q\in[adc]=[acd]$ and, according to (2.3), there exists a point $e\in[ac]$ such that $q\in[de]$, i.e. $e\mu ac$ and $e|A$.

Hence $p\mu bc$ or there exists a point $e\in A$ such that $e\mu ac$. This proves H8.

Finally, using (3.2) as a definition of the relation $\equiv\subset M^4$, from A3, A10—A16 we obtain the conditions H9—H16. The condition H7 follows from A12.

This reasoning proves the following theorem.

THEOREM. *The Euclidean plane $\mathscr{E}$ and the S-Euclidean plane $\mathscr{M}$ are definitionally equivalent.*

This paper is based on the author's work for his master's degree. It was inspirated by doc. dr E. Siwek and completed under the supervision of prof. dr hab. L. Dubikajtis at the Silesian University in Katowice.

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