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ON THE EXISTENCE OF SOLUTIONS OF A NICOLETTI PROBLEM FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH ADVANCED ARGUMENT

Abstract. Our purpose is to prove the existence of solutions of a Nicoletti problem for an integro-differential equation with advanced argument of the form

(1) $$x'(t) = \int_0^{h(t)} f(t, x(t+s))\, d_s r(t,s),\ t \in \boldsymbol{R}^+,$$

(2) $$Nx = \eta.$$

The problem (1)—(2) is called the generalized Nicoletti problem (see [4], [7]). The field of our study is a Banach space, and we apply the theory of measure of noncompactness, and the fixed point theorem of Darbo (see [1]). The literature on the differential equations with advanced argument is not very rich. In particular, the studies are limited to the problem of existence of solutions of the Cauchy or the Nicoletti problem (see for example [2], [3], [4], [6], [7], [8]).

1. Notations and definitions. Let $(B, |\cdot|)$ be a Banach space and **O** be the zero of B. We shall write

— $K(a, r)$ for the open ball centered at a and of radius r,

— $\mathcal{C}(\boldsymbol{R}^+, B)$ for the set of all continuous functions mapping $\boldsymbol{R}^+$ into B,

— $\mathcal{C}^1(\boldsymbol{R}^+, B)$ for the subset of $\mathcal{C}(\boldsymbol{R}^+, B)$ consisting of continuously differentiable functions,

— $X(s) = \{x(s) : x \in X \text{ where } X \subset \mathcal{C}(\boldsymbol{R}^+, B) \text{ and } s \in \boldsymbol{R}^+\}$.

2. Existence theorem.

HYPOTHESIS (H).

(i). The function f maps $\boldsymbol{R}^+ \times B$ into B, is uniformly continuous and satisfies the inequality

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$$|f(t,y)| \leqslant M(t) + N(t) \cdot |y|^{\alpha}, \quad t \in \boldsymbol{R}^{+},$$

where $M, N: \boldsymbol{R}^{+} \to (0, \infty)$ are continuous functions and $0 \leqslant \alpha \leqslant 1$. We put $L = M + N$ and

$$\Lambda(t) = \int_{0}^{t} L(\tau)\, d\tau, \qquad t \in \boldsymbol{R}^{+}.$$

(ii). The function h maps $\boldsymbol{R}^{+}$ continuously into $\boldsymbol{R}^{+}$ and satisfies the inequality

$$\Lambda\bigl(t + h(t)\bigr) \leqslant \alpha^{-1} \bigl(\Lambda(t) + k^{-1} \ln k\bigr), \qquad t \in \boldsymbol{R}^{+}$$

for a $k > 1$.

(iii). The function r maps $\boldsymbol{R}^{+} \times \boldsymbol{R}^{+}$ into $\boldsymbol{R}^{+}$ in such a manner that, for any $t \in \boldsymbol{R}^{+}$, $r(t, 0) = 0$, $r(t, \cdot)$ is non-decreasing, and there exists a constant $0 \leqslant V \leqslant 1$ such that $\mathop{\mathrm{V}_0}\limits_{s=0}^{h(t)} \; r(t,s) \leqslant V,\ t \in \boldsymbol{R}^{+}$, and

$$\lim_{t \to u} \int_{0}^{h(t)} |r(t,s) - r(t,u)|\, ds = 0.$$

(iv). $N: \mathcal{C}(\boldsymbol{R}^{+}, B) \to B$ is a Nicoletti linear bounded operator, with the norm $\|N\|$ and such that for any constant function $x \in \mathcal{C}(\boldsymbol{R}^{+}, B)$ we have $Nx = \xi$, where $x(t) = \xi,\ \ t \in \boldsymbol{R}^{+}$.

(v). There exist positive numbers β and $A \geqslant \max(1, |\eta| + \beta)$ such that the condition

$$|v'(t)| \leqslant k \cdot A \cdot L(t) \cdot \exp\bigl(k \cdot \Lambda(t)\bigr) \Rightarrow |v(t) - Nv| \leqslant \beta \cdot \exp\bigl(k \cdot \Lambda(t)\bigr)$$

holds for every function $v \in \mathcal{C}(\boldsymbol{R}^{+}, B)$ and $t \in \boldsymbol{R}^{+}$.

(vi). Let μ be a sublinear measure of noncompactness and let Φ be the set of all functions $x \in \mathcal{C}(\boldsymbol{R}^{+}, B)$ which satisfy the inequality

$$|x(t)| \leqslant A \cdot \exp\left(k \cdot \int_{0}^{t} L(\tau)\, d\tau\right), \qquad t \in \boldsymbol{R}^{+}.$$

Assume that there exists a locally integrable function $P: \boldsymbol{R}^{+} \to \boldsymbol{R}^{+}$ satisfying the condition

$$\int_{t}^{t+h(t)} P(\tau) \quad d\tau \leqslant c, \qquad t \in \boldsymbol{R}^{+},$$

for a positive number c and such that

$$\mu\bigl(f(t, X)\bigr) \leqslant P(t)\mu(X), \qquad t \in \boldsymbol{R}^{+}, \qquad X \subset \Phi.$$

Fix a positive number λ and denote by E the set of all functions $x \in \mathcal{C}^{1}(\boldsymbol{R}^{+}, B)$ such that

$$\|x\| = \sup\left\{|x(t)| \exp\left(-k \cdot \int_{0}^{t} L(\tau)\, d\tau - \lambda t\right) : t \in \boldsymbol{R}^{+}\right\} < +\infty.$$

Then (see [3]) the set Φ is a non-empty, closed, bounded, and convex subset of E. For all $X \subset \Phi$ put

$$M_\gamma(X) = \sup\left\{\mu(X(t))\cdot\exp\left(-\int_0^t L(\tau)\,d\tau\right) : t\in\mathbf{R}^+\right\},$$

where γ is a positive constant such that $0 \leqslant \exp(c^\gamma)\gamma < 1$. One can easily verify that the function M_γ is a sublinear measure of noncompactness.

REMARK. The hypothesis (H) is sufficient for the continuity of the function $z : \mathbf{R}^+ \to \mathbf{R}$ given by

$$z(t) = \int_0^{h(t)} f(t, x(t+s))\, d_s r(t,s), \quad t\in\mathbf{R}^+.$$

The argument does not differ from that one given by A. Bielecki and M. Maksym [3] in the case of the space $\mathbf{R}^n$.

DEFINITION. Any function $x\in\mathcal{C}(\mathbf{R}^+, B)$ satisfying equalities (1) and (2) is said to be a *solution of problem* (1)—(2).

THEOREM. *If the hypothesis* (H) *holds, then there exists a solution* x *of problem* (1) — (2) *satisfying the inequality*

$$|x(t)| \leqslant A\cdot\exp\left(k\cdot\int_0^t L(\tau)\,d\tau\right).$$

In the proof of Theorem we shall need the following lemma.

LEMMA. *Let* $0 \leqslant a < b < \infty$ *and let* $g : [a,b] \to \mathbf{R}$ *be a non-decreasing function. Then for every sublinear measure of noncompactness* μ *on* B *and for any equicontinuous and bounded set* $X \subset \mathcal{C}(\mathbf{R}^+, B)$ *we have*

$$\mu\left(\int_a^b X(s)\,dg(s)\right) \leqslant \int_a^b \mu(X(s))\,dg(s),$$

where

$$\int_a^b X(s)\,dg(s) = \left\{\int_a^b x(s)\,dg(s) : x\in X\right\}.$$

Proof. By the definition of the Stieltjes integral we have

$$\int_a^b x(s)\,dg(s) = \lim_{\lambda_n\to 0}\sum_{i=0}^{n-1} x(\xi_i)\cdot\left[g(x_{i+1}) - g(x_i)\right], \quad x\in X,$$

where $\lambda_n = \max_{i\in\overline{0,n-1}}(x_{i+1} - x_i)$, $a = x_0 < x_1 < \ldots < x_n = b$, $\xi_i\in\left[x_i, x_{i+1}\right]$, $i\in\overline{0, n-1}$, $n\in\mathbf{N}$. For every $\varepsilon > 0$, using the equicontinuity of X, we can choose a sequence $(t_i : i\in\mathbf{N}_0)$ such that the set $\{t_i : i\in\mathbf{N}_0\}$ is dense in $[a,b]$, for every $n\in\mathbf{N}$, $a = t_0 \leqslant \xi_0 \leqslant t_1 \leqslant \xi_1 \leqslant\ldots\leqslant \xi_{n-1} \leqslant t_n = b$ and

$$\left| \int_a^b x(s)\, dg(s) - \sum_{i=0}^{n-1} x(\xi_i) \cdot \left[g(t_{i+1}) - g(t_i) \right] \right| \leqslant \varepsilon, \quad x \in X.$$

We have also

$$\int_a^b X(s)\, dg(s) \subset \left\{ \int_a^b x(s)\, dg(s) - \sum_{i=0}^{n-1} x(\xi_i) \cdot \left[g(t_{i+1}) - g(t_i) \right] : x \in X \right\}$$

$$+ \left\{ \sum_{i=0}^{n-1} x(\xi_i) \cdot \left[g(t_{i+1}) - g(t_i) \right] : x \in X \right\}, \quad n \in \mathbf{N},$$

whence

$$\mu\left(\int_a^b X(s)\, dg(s) \right) \leqslant \varepsilon \cdot K(0,1) + \sum_{i=0}^{n-1} x(\xi_i) \cdot \left[g(t_{i+1}) - g(t_i) \right], \quad n \in \mathbf{N},$$

and, consequently,

$$\mu\left(\int_a^b X(s)\, dg(s) \right) \leqslant \int_a^b \mu\left(X(s) \right) dg(s).$$

Proof of Theorem. We know that x is a solution of problem (1) — (2) if and only if x is a fixed point of the operator

$$H : \mathcal{C}\, (\mathbf{R}^+, B) \to \mathcal{C}\, (\mathbf{R}^+, B)$$

defined by

$$Hx = \eta - N(Tx) + Tx,$$

where

$$(Tx)\,(t) = \int_0^t \left\{ \int_0^{h(u)} f\left(u, x(u+s)\right) d_s r(u,s) \right\} du.$$

a) We shall show that $Hx \in \Phi$ for every $x \in \Phi$. At first we estimate $|(Tx)'\,(t)|$. We have

$$|(Tx)'\,(t)| \leqslant \sup_{0 \leqslant s \leqslant h(t)} \left| f\left(t, x(t+s)\right) \right| \bigvee_{s=0}^{h(t)} r(t,s).$$

Using (H)(i) and (H)(iii) we obtain

$$|(Tx)'(t)| \leqslant V \cdot \left[M(t) + N(t) \cdot \sup_{0 \leqslant s \leqslant h(t)} |x(t+s)|^{\alpha}\right].$$

Taking into account that $x \in \Phi$ and using the hypothesis (H)(ii) and the inequalities $A \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$ we deduce that

$$\sup_{0 \leqslant s \leqslant h(t)} |x(t+s)| \leqslant k \cdot A \cdot \exp\left(k \cdot \Lambda(t)\right).$$

Finally we obtain the estimation

$$|(Tx)'(t)| \leqslant A \cdot k \cdot L(t) \cdot \exp\left(\Lambda(t)\right).$$

Since $Tx \in \mathcal{C}^1 (\boldsymbol{R}^+, B)$ we have, by (H)(v),

$$|(Tx)(t) - N(Tx)| \leqslant \beta \exp\left(k \cdot \Lambda(t)\right)$$

and the definition of the operator H implies the condition

$$|(Hx)(t)| \leqslant A \cdot \exp\left(k \cdot \Lambda(t)\right).$$

Consequently $H(\Phi) \subset \Phi$ which was to be proved.

b) Now we shall show the continuity of the operator H. Put $l(\varepsilon) = \lambda^{-1}(\ln 4 - \ln \varepsilon)$, $0 < \varepsilon \leqslant 4$, and let $(x_n : n \in \boldsymbol{N})$ be a sequence of functions from Φ which converges to an $x \in \Phi$ in the topology of $\mathcal{C}(\boldsymbol{R}^+, B)$. In the case $t > l(\varepsilon)$ we establish easily that for any $n \in \boldsymbol{N}$

$$|(Hx_n)(t) - (Hx)(t)| \exp\left(-k \cdot \Lambda(t) - \lambda \cdot t\right) \leqslant \varepsilon \cdot A/2,$$

whence

$$\|Hx_n - Hx\| \leqslant \varepsilon \cdot A/2, \quad n \in \boldsymbol{N}.$$

For $t \in \left[0, l(\varepsilon)\right]$ and $n \in \boldsymbol{N}$ we have the estimation

$$|(Hx_n)(t) - (Hx)(t)| \leqslant \|N\| \cdot \|Tx_n - Tx\| + |(Tx_n - Tx)(t)|$$

and deduce that

$$\|Hx_n - Hx\| \leqslant (\|N\| + 1)\cdot \|Tx_n - Tx\|.$$

Observe also that we have

$$|(Tx_n)(t) - (Tx)(t)| \leqslant V\cdot \int_0^{l(\varepsilon)} \sup_{0\leqslant s\leqslant h(u)} \left|f\big(u,x_n(u+s)\big) - f\big(u,x(u+s)\big)\right| du, \ n\in \boldsymbol{N},$$

and, putting $d = \max_{0\leqslant t\leqslant l(\varepsilon)} h(t)$,

$$|x_n(u+s)| \leqslant A\cdot \exp\Big(\int_0^{l(\varepsilon)+d} L(u)\, du\Big), \ \ n\in \boldsymbol{N},$$

whence the condition

$$\lim_{n\to+\infty} |(Tx_n)(t) - (Tx)(t)| = 0$$

easily follows. This means that H is continuous in the topology of $\mathcal{C}(\boldsymbol{R}^+, B)$.

c) Finally we shall show that there exists a $q\in[0,1)$ such that

$$M_\gamma(HX) \leqslant q\cdot M_\gamma(X), \ \ X\subset \Phi.$$

In fact, since $\eta - N(Tx)\in B$ we have (see [1]) $\mu\big(\eta - N(Tx)\big) = 0$, whence

$$\mu\big((HX)(t)\big) \leqslant \mu\Big(\int_0^t \Big\{\int_0^{h(u)} f\big(u,X(u+s)\big)\, d_s r(u,s)\Big\}\, du\Big).$$

Now, using Lemma, the hypothesis (H)(vi) and the definition of the function M_γ, we obtain

$$\mu\big((HX)(t)\big) \leqslant V\cdot M_\gamma(X)\int_0^t\Big(P(u)\exp\Big(\gamma\int_0^{u+h(u)} P(\tau)\, d\tau\Big)\Big)\, du.$$

(H)(vi) applied to the function P gives the inequality

$$\mu\big((HX)(t)\big) \leqslant V\cdot M_\gamma(X)\, e^{c\gamma}/\gamma.$$

Thus, due to the fact that $0 \leqslant e^{c\gamma}/\gamma \leqslant 1$, we obtain the required property.

Making use of the fixed point theorem of Darbo we complete the proof of Theorem.

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