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SOME REMARKS CONCERNING THE ONE-DIMENSIONAL BURGERS EQUATION

Abstract. The behaviour of solutions of the Burgers system (1)—(3) is studied. In earlier papers [4], [5] the problem of the global stability of the constant solution $(U,v) = \left(\frac{P}{\nu}, 0\right)$ when $\frac{P}{\nu} \leqslant \nu$ was solved. The behaviour of those solutions (U,v) which do not converge to the constant solution when t tends to infinity is studied here. In part 3 some of its properties are studied, while in parts 2 and 4 several a priori estimates needed in the proof of existence of solutions are presented.

1. Introduction. In 1939 J.M. Burgers gave the model of the motion of a viscous fluid in a channel. This model has the form:

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_0^\pi v^2(t,x)\,dx, \quad U(0) = U_0, \tag{1}$$

$$v_t(t,x) = U(t)\,v(t,x) + \nu v_{xx}(t,x) - \left(v^2(t,x)\right)_x, \tag{2}$$

$t \geqslant 0$, $x \in (0,\pi)$, where P,ν are positive constants (pressure, viscosity), with the conditions

$$v(0,x) = \varphi(x), \quad v(t,0) = v(t,\pi) = 0. \tag{3}$$

Notation. The following symbols are used:

$$I = (0,\pi), \quad D = [0,T] \times I, \quad z(t) = \|v(t,\cdot)\|^2_{L^2(I)}.$$

For simplicity partial derivatives are denoted by v_t, v_x etc.. The usual notation is used for the L^p and Sobolev spaces H^1_0, H^2, $W^{m,p}$ ([6], [7], [8], [10]). The $C^{\frac{\alpha}{2},\alpha}(\bar{D})$ space of Hölder continuous functions (denoted [6, p. 61] as $H^{\frac{\alpha}{2},\alpha}$) and the space $C^\alpha(\bar{I})$ are also considered. The symbols $L^p(0,T; B)$ (B is a Banach space) are defined in [7].

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The following estimates are used several times:

Cauchy inequality: $xy \leqslant \frac{\varepsilon}{2} x^2 + \frac{1}{2\varepsilon} y^2$, $\varepsilon > 0$ arbitrary,

a version of the Poincaré inequality (Wirtinger inequality [4]):

$$\forall_{f \in H_0^1(I)} \quad \|f\|^2_{L^2(I)} \leqslant \|f_x\|^2_{L^2(I)} =: \|f\|^2_{H_0^1(I)}$$

Sobolev Imbedding Theorem ([10]): *if G is a smooth bounded domain in $\mathbf{R}^n$, then for $0 < \mu = m - \frac{n}{p} - j < 1$ holds, $C^{j+\mu}(\bar{G}) \subset W^{m,p}(\bar{G})$, and*

$$\exists_{c>0} \quad \forall_{f \in C^{j+\mu}} \quad \|f\|_{C^{j+\mu}} \leqslant c\, \|f\|_{W^{m,p}} .$$

DEFINITION 1 ([4]). By a *weak solution of* (1)—(3) ($\varphi \in L^2(I)$) we mean a pair (U,v), such that $U \in C^1([0,T])$ (one side derivatives in $t = 0, T$), $v \in L^2(0,T; H_0^1(I)) \cap C^0(0,T; L^2(I))$, and (U,v) satisfies (1) and the equalities

$$\int_I v' w \,\mathrm{d}x + \nu \int_I v_x w_x \,\mathrm{d}x + 2 \int_I v\, v_x\, w \,\mathrm{d}x - U \int_I v\, w \,\mathrm{d}x = 0$$

for any $w \in H_0^1(I)$ and almost all $t \in [0,T]$ (time derivative v' is understood here as the distributional derivative with values in $L^2(I)$ [4], [7]).

The existence of such solutions for arbitrary $T > 0$ (*global weak solutions*) shown in [4], allows us to study the asymptotic behaviour of U and v when t tends to infinity.

By a $C^{1,2}(\bar{D})$ *solution of* (2) we mean the classical solution having continuous in $\bar{D}$ derivatives v_t, v_x, v_{xx}.

2. Introductory a priori estimates. We start with the following.

LEMMA 1. *Let (U,v) be the weak solution of (1)—(3) and let $\varphi \in C^0(\bar{I})$. If v is also a $C^{1,2}(\bar{D})$ solution, then (U,v) is bounded globally, more precisely*

$$\exists_{c_1, c_2, c_3 > 0} \ \forall_{\substack{t \geqslant 0 \\ x \in I}} \quad |U(t)| \leqslant c_1, \quad \|v(t,\cdot)\|_{L^2(I)} \leqslant c_2, \quad |v(t,x)| \leqslant c_3 \tag{4}$$

with c_1, c_2, c_3 dependent only on P, ν, U_0 and $\|\varphi\|_{C^0}$ and independent on T.

Proof. It is easy to see that the (Liapunov) function

$$L(t) := U^2(t) + \|v(t,\cdot)\|^2_{L^2(I)} \equiv U^2(t) + z(t)$$

remains bounded as long as U and v exist. In fact, when multiplying (1) by U, multiplying (2) in $L^2(I)$ by v and summing the results we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} L(t) = P\, U(t) - \nu U^2(t) - \nu \int_I (v_x)^2 \,\mathrm{d}x,$$

or with the use of the Cauchy ($\varepsilon = \nu$) and Wirtinger inequalities

(5) $$\frac{1}{2}\ \frac{d}{dt} L(t) \leqslant \frac{1}{2\nu} P^2 + (\frac{\nu}{2} - \nu)\ U^2(t) - \nu \int_I v^2\ (t,x)\ dx \leqslant$$
$$\leqslant - \frac{\nu}{2} L(t) + \frac{P^2}{2\nu}\ .$$

Differential inequality (5) ensures the global boundedness of L

$$L(t) \leqslant \max \left\{ L(0),\ \frac{P^2}{\nu^2} \right\},$$

and hence estimates for both $|U|$ and z simultaneously. To close the proof it remains merely to estimate v in the uniform norm. This estimate is based on an interesting method given by N.D. Alikakos in [1, Theorem 3.1]. The existence of a weak solution of (1)—(3) was shown in [4], hence we will now study the properties of the separate problem (2), (3) thinking about U as a given (as a part of the weak solution) "coefficient" of a class C^1. Multiplying (2) by $v^{2^k - 1}$, $k = 1, 2, \ldots$, and integrating over I we verify that

(6) $$2^{-k}\ \frac{d}{dt} \int_I\ v^{2^k}(t,x)\ dx = U(t) \int_I\ v^{2^k}(t,x)\ dx -$$
$$- \nu \int_I v_x(t,x) \left(v^{2^k - 1}(t,x)\right)_x dx - \int_I \left(v^2(t,x)\right)_x v^{2^k - 1}\ (t,x)\ dx =$$
$$= U(t) \int_I v^{2^k}(t,x)\ dx - \nu\ \frac{2^k - 1}{2^{2k-2}} \int_I \left[(v^{2^{k-1}})_x\right]^2 dx,$$

since

$$\int_I \left(v^2(t,x)\right)_x v^{2^k - 1}(t,x)\ dx = \frac{2}{2^k + 1} \int_I \left(v^{2^k + 1}(t,x)\right)_x dx =$$
$$= \frac{2}{2^k + 1}\ v^{2^k + 1}(t,x)\Big|_{x = 0,\pi} = 0.$$

Denoting

$$v^{*} := v^{2k - 1},\ \nu_k := \nu\ \frac{2^k - 1}{2^{k-1}}\ ,\ a_k := c_1\ 2^{k-1},$$

and remembering that $|U(t)| \leqslant c_1$ for $t \geqslant 0$, we arrive at the estimate

(7) $$\frac{d}{dt} \left(\frac{1}{2} \int_I (v^*)^2\ dx\right) \leqslant - \nu_k \int_I \left[(v^*)_x\right]^2 dx + a_k \int_I (v^*)^2\ dx,$$

which is identical with (3.8) in [1] (the non-negativity of v is not essential; see [2]). Since we have shown previously the global boundedness of the $L^2(I)$ norm of $v(t,\cdot)$, remembering that

$$\|v(t,\cdot)\|_{L^1(I)} \leqslant \sqrt{\pi}\|v(t,\cdot)\|_{L^2(I)},$$

the final estimate of [1, Theorem 3.1] gives

$$\|v(t,\cdot)\|_{L^\infty(I)} \leqslant 2^5\sqrt{\pi}\, c_2 K =: c_3,$$

with

$$K = \max\left\{1, \sup_{t\geqslant 0} \int_I |v(t,x)|\, dx,\ \|\varphi\|_{C^0(\bar{I})}\right\}.$$

REMARK 1. The reason why the Alikakos proof was applicable to our nonlinear problem is that the component in (6) corresponding to $(v^2)_x$ vanishes. It is interesting to note that since the function U has an undetermined sign, the result of Lemma 1 is inaccessible with the use of the classical maximum principle type arguments.

3. Some remarks concerning the instability of the constant solution $\left(\frac{P}{\nu},0\right)$ of (1)—(3).

DEFINITION 2. For a non-zero function $f \in H_0^1(I)$ let us define its *complication*

$$K(f) := \frac{\|f\|^2_{H_0^1(I)}}{\|f\|^2_{L^2(I)}}, \quad K(0) := 1. \tag{8}$$

As a consequence of the Wirtinger inequality, $K(f) \geqslant 1$ for all functions $f \in H_0^1(I)$.

DEFINITION 3. We say that a classical solution (U,v) is *trivial* (or simply v is trivial), if

$$\exists_{t_0 \geqslant 0}\ v(t_0,x) = 0 \text{ for } x \in \bar{I}.$$

It was shown in [4] that the weak solution (U,v) of (1)—(3) is uniquely determined, for $t \geqslant \tau$, by its value $U(\tau) \in \boldsymbol{R}$, $v(\tau,\cdot) \in L^2(I)$. This observation is all the more valid for classical solutions. It is thus easy to see that any trivial classical solution has the form

$$v(t,x) = 0,\quad U(t) = U(t_0)\exp\left(-\nu(t-t_0)\right) + \frac{P}{\nu}\left(1-\exp(-\nu(t-t_0)\right)$$

for $t \geqslant t_0$.

The complication $K_t(v)$ of a $C^{1,2}$ solution which is not trivial, is well defined (the denominator is strictly positive). We have:

THEOREM 1. *Let* $\frac{P}{\nu} > \nu$. *Then, for every existing for all* $t \geqslant 0$ $C^{1,2}$ *solution* v *which is not trivial, one of the alternative conditions*

$$\limsup_{t \to +\infty} K_t(v) \geqslant \frac{P}{\nu^2} \quad \text{or} \quad \limsup_{t \to +\infty} \|v(t,\cdot)\|_{L^2(I)} > 0$$

holds.

Proof. It remains to show the implication

$$\left[\limsup_{t \to +\infty} K_t(v) < \frac{P}{\nu^2}\right] \Rightarrow \left[\sim\left(\|v(t,\cdot)\|_{L^2(I)} \to 0, \quad t \to +\infty\right)\right].$$

Multiplying the equation for $W(t) := U(t) - \frac{P}{\nu}$

(9) $$\frac{dW}{dt} = -\nu W - \int_I v^2(t,x)\, dx$$

by W and multiplying (2) in $L^2(I)$ by v, we get $\left(z(t) = \|v(t,\cdot)\|^2_{L^2(I)}\right)$:

(10) $$\frac{1}{2}\frac{dW^2}{dt} = -\nu W^2 - zW,$$

(11) $$\frac{1}{2}\frac{dz}{dt} = \left(W + \frac{P}{\nu}\right)z - \nu\|v(t,\cdot)\|^2_{H^1_0(I)} - 0.$$

As a consequence of our assumption $K_t(v) < \frac{P-\delta}{\nu^2}$ for sufficiently small postive δ and all $t \geqslant T_0(\delta)$. If, on the contrary, we assume that $z(t) \to 0$, $t \to +\infty$, then

$$\exists_{T_1 \geqslant T_0} \quad \forall_{t \geqslant T_1} \quad 0 < z(t) < \delta$$

(the estimate $z(t) > 0$ is valid for all v which are not trivial). Subtracting (10) from (11), for $t \geqslant T_1$ we get

(12) $$\frac{d}{dt}(z - W^2) = \nu W^2 + 2Wz + \left[\frac{P}{\nu} - \nu K_t(v)\right]z,$$

or further $(0 < z < \delta < 1)$

$$\frac{d}{dt}(z - W^2) > 2Wz + \nu W^2 + \frac{\delta}{\nu} z \geqslant \nu W^2 + 2Wz + \frac{1}{\nu} z^2 =$$
$$= \left(\sqrt{\nu}W + \frac{1}{\sqrt{\nu}} z\right)^2.$$

Hence for $t \geqslant T_1$ the function $(z - W^2)$ is weakly increasing and converges to some $\alpha \in \boldsymbol{R}$. But z tends to 0, hence $W^2(t) \to -\alpha$ when t tends to infinity.

If $\alpha = 0$, then for some $T_2 \geqslant T_1$

$$W(t) \geqslant -\frac{\delta}{2\nu}, \text{ for } t \geqslant T_2,$$

or with the use of (11) and the definition of T_0

$$\frac{1}{2}\frac{dz}{dt} = Wz + \left[\frac{P}{\nu} - \nu K_t(v)\right]z > \frac{\delta}{2\nu} z, \quad t \geqslant T_2,$$

which means $\left(z(T_2) > 0\right)$, that z is unbounded and contradicts Lemma 1. If $\alpha \neq 0$, then by (9)

$$\frac{dW}{dt} = -\nu W - z \to -\nu\alpha, \quad t \to +\infty,$$

hence W is unbounded, which again contradicts Lemma 1. The proof is thus finished.

As was observed in Lemma 1, the nonlinear term corresponding to $(v^2)_x$ vanishes in (6). Thus all the estimates of Lemma 1 and Theorem 1 remain unchanged if instead of (2) we take

(13) $$v_t = Uv + \nu v_{xx} + \lambda (v^2)_x$$

with arbitrary $\lambda \in \boldsymbol{R}$ (the last term in (2) is invalid in these estimates!). We want to express the role of this last component by considering the Fourier coefficients of the solution V_λ of (1), (13), (3). We have

LEMMA 2. *For the Fourier coefficients* $v_k(t) = \int_I V_\lambda(t,x) \sin kx \, dx$ *with the numbers* $k > \sqrt{\frac{2c_1}{\nu}}$ *the following estimate holds:*

(14) $$\limsup_{t \to +\infty} |v_k(t)| \leqslant \sqrt{\frac{2}{\nu}} \frac{|\lambda| c_2^2}{\sqrt{\nu k^2 - 2c_1}}.$$

P r o o f. Multiplying (13) in $L^2(I)$ by $\sin kx$, $k = 1,2,\ldots$, and using the identities

$$\int_I (V_\lambda)_{xx} \sin kx \, dx = -k^2 v_k(t),$$

$$\int_I (V_\lambda^2)_x \sin kx \, dx = -k \int_I V_\lambda^2 \cos kx \, dx,$$

we obtain

$$\frac{dv_k(t)}{dt} = U(t) v_k(t) - \nu k^2 v_k(t) + \lambda k \int_I V^2 \cos kx \, dx. \tag{15}$$

Since the last term is estimated by $|\lambda| k\, c_2^2$ (the bound of Lemma 1 remains valid for all V_λ), then multiplying (15) by $v_k(t)$ we have

$$\frac{1}{2} \frac{d}{dt} \left(v_k^2(t)\right) \leqslant \left[U(t) - \nu k^2\right] v_k^2(t) + |\lambda| k\, c_2^2\, |v_k(t)| \leqslant \tag{16}$$

$$\leqslant \left[c_1 - \nu k^2\right] v_k^2(t) + \left[\frac{\nu k^2}{2} v_k^2(t) + \frac{1}{2\nu} \lambda^2 c_2^4\right].$$

Solving this differential inequality for $k > \sqrt{\dfrac{2c_1}{\nu}}$ we obtain

$$v_k^2(t) \leqslant v_k^2(0) \exp\left[2\left(c_1 - \frac{\nu k^2}{2}\right) t\right] + \frac{\lambda^2 c_2^4}{\nu} \; \frac{1 - \exp\left[2\left(c_1 - \dfrac{\nu k^2}{2}\right) t\right]}{\dfrac{\nu k^2}{2} - c_1},$$

hence further

$$|v_k(t)| \leqslant |v_k(0)| \exp\left[\left(c_1 - \frac{\nu k^2}{\nu}\right) t\right] + \sqrt{\frac{2}{\nu}} \; \frac{|\lambda|\, c_2^2}{\sqrt{\nu k^2 - 2c_1}}.$$

Passing with t to infinity in this last inequality, we get (14). We have thus estimated the rate of decay to zero ($k \to +\infty$) of the Fourier coefficients with large numbers k.

4. Existence of smooth solutions of (1)—(3). We give the proof of existence of classical solutions of (1)—(3) having the additional properties

$$U \in C^{2+\frac{1}{4}}([0,T]), \quad v \in C^{1+\frac{1}{4}, 2+\frac{1}{2}}(\bar{D}). \tag{17}$$

THEOREM 2. *For any initial function* $\varphi \in C^{2+\frac{1}{2}}(\bar{I})$ *satisfying the compatibility conditions* $\varphi(0) = \varphi(\pi) = 0$ *and*

$$U_0 \varphi(x) + \nu \varphi_{xx}(x) - \left(\varphi^2(x)\right)_x \Big|_{x = 0, \pi} = 0,$$

there exists a classical solution of (1)—(3) *satisfying* (17).

The proof is divided into three parts. Fundamental here are the a priori estimates of Lemma 1 and Lemma 3 (below). As in Lemma 1 we restrict our considerations to the problem (2), (3) (with U given in $C^1([0,T])$ as a part of the weak solution).

LEMMA 3. *For any* $C^{1,2}(\bar{D})$ *solution* v *of* (2), (3)

$$\|v_t(t,\cdot)\|_{L^4(I)} \leqslant c_4, \quad t\in[0,T], \quad c_4 = c_4(c_1, c_2, c_3, T, \nu) \tag{18}$$

holds.

Proof. The solution considered does not usually have the derivative v_{tt}; therefore instead we must study the difference quotients for v_t. From (2) (for fixed $h > 0$ the difference quotient is well defined for $t \in [0,T\text{-}h]$, hence also the estimates below works for such t) we deduce

$$\begin{aligned} h^{-1}\Big[v_t(t+h,x) - v_t(t,x)\Big] &= U(t+h)\, h^{-1}\Big[v(t+h,x) - v(t,x)\Big] + \\ &+ v(t,x)\, h^{-1}\Big[U(t+h) - U(t)\Big] + \nu h^{-1}\Big[v(t+h,x) - v(t,x)\Big]_{xx} + \\ &+ \Big[v(t+h,x)\, h^{-1}\big(v(t+h,x) - v(t,x)\big) + \\ &+ v(t,x)\, h^{-1}\big(v(t+h,x) - v(t,x)\big)\Big]_x. \end{aligned} \tag{19}$$

Denoting for simplicity $f_h(t) := h^{-1}\big(f(t+h) - f(t)\big)$, and multiplying (19) in $L^2(I)$ by $v_h^3(t,x)$, we obtain

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_I v_h^4(t,x)\,dx &= U(t+h)\int_I v_h^4(t,x)\,dx + \\ &+ U_h(t)\int_I v(t,x)\, v_h^3(t,x)\,dx - \nu\int_I\Big[v_h(t,x)\Big]_x\Big[v_h^3(t,x)\Big]_x\,dx + \\ &+ \int_I\Big[v(t+h,x)\, v_h(t,x) + v(t,x)\, v_h(t,x)\Big]_x\, v_h^3(t,x)\,dx. \end{aligned} \tag{20}$$

Some of the components in (20) are estimated below. First we have

$$\begin{aligned} |U_h(t)\int_I v(t,x)\, v_h^3(t,x)\,dx| &\leqslant c_5\, \|v(t,\cdot)\|_{L^4(I)} \|v_h^3(t,\cdot)\|_{L^{\frac{4}{3}}(I)} \leqslant \\ &\leqslant c_5\Big[\frac{3}{4}\int_I v_h^4(t,x)\,dx + \frac{1}{4}\int_I v^4(t,x)\,dx\Big], \end{aligned}$$

where the Hölder and Young ([7, p. 74]) inequalities are used, and the constant $\frac{c_5}{2} := P + \nu c_1 + c_2$ dominates (in the presence of (4)) the right hand side of (1) (and hence c_5 alone dominates $U_h(t)$ for $t \in [0,T]$ and for all $h \leqslant h_0$, h_0 small). Further

$$\int_I \left[v_h(t,x)\right]_x \left[v_h^3(t,x)\right]_x \, dx = \frac{3}{4} \int_I \left[\left(v_h(t,x)^2\right)_x\right]^2 dx,$$

and the last two components are estimated in the same way (we consider the first one):

$$\int_I \left[v(t+h,x)\, v_h(t,x)\right]_x v_h^3(t,x)\, dx = -\int_I v(t+h,x)\, v_h(t,x) \left(v_h^3(t,x)\right)_x dx =$$
$$= -\frac{3}{2} \int_I v(t+h,x)\, v_h^2(t,x) \left(v_h^2(t,x)\right)_x dx,$$

then using Hölder and Cauchy inequalities we verify that

$$\left| \int_I \left(v(t+h,x)\, v_h(t,x)\right)_x v_h^3(t,x)\, dx \right| \leqslant$$
$$\leqslant \frac{3}{2} c_3 \left(\int_I v_h^4(t,x)\, dx\right)^{\frac{1}{2}} \left(\int_I \left(v_h^2(t,x)_x\right)^2 dx\right)^{\frac{1}{2}} \leqslant$$
$$\leqslant \frac{3c_3}{2\varepsilon} \int_I v_h^4(t,x)\, dx + \frac{3}{2} c_3 \varepsilon \int_I \left[\left(v_h^2(t,x)\right)_x\right]^2 dx,$$

($\varepsilon > 0$ is arbitrary). Collecting all the estimates we have

$$\frac{d}{dt} \int_I v_h^4(t,x)\, dx \leqslant \left[-3\nu + 12\varepsilon c_3\right] \int_I \left[\left(v_h^2(t,x)\right)_x\right]^2 dx +$$
$$+ 4\left[c_1 + \frac{3}{4} c_5 + \frac{3}{\varepsilon} c_3\right] \int_I v_h^4(t,x)\, dx + c_5\, c_3^4 \pi.$$

The last with $\varepsilon = \dfrac{\nu}{4c_3}$ gives an a priori bound

$$\int_I v_h^4(t,x)\, dx \leqslant \int_I v_h^4(0,x)\, dx \exp(\gamma t) + c_5\, c_3^4 \pi \, \frac{\exp(\gamma t) - 1}{\gamma} \tag{21}$$

with $\gamma = 4\left[c_1 + \frac{3}{4} c_5 + \frac{12 c_3^2}{\nu}\right]$. Passing with h to zero in the estimate (21) we finally obtain

$$\int_I v_t^4(t,x)\, dx \leqslant \int_I v_t^4(0,x)\, dx \exp(\gamma T) + \text{const.} =: c_4 .$$

It is noteworthy that if v is the $C^{1,2}(\bar{D})$ solution, then $v_t(0,x)$ can be found (through the continuity) from the equation (2) and its $L^4(I)$ norm will be estimated proportionally to U_0 and the $W^{2,4}(I)$ norm of φ. The proof is thus completed.

LEMMA 4. *For any $C^{1,2}(\bar{D})$ solution v an a priori estimate*

$$\|v\|_{C^{\frac{1}{4},\frac{1}{4}}(\bar{D})} \leqslant c_5, \quad c_5 = c_5(c_2,c_4)$$

holds.

Proof. Fixing an arbitrary $t \in [0,T]$ we may look at (2) as an elliptic problem (t is a parameter)

$$\nu v_{xx}(t,x) - 2\, v(t,x)\, v_x(t,x) + U(t)\, v(t,x) = v_t(t,x)$$

with the "right side" $v_t(t,\cdot)$ bounded in $L^4(I)$ and the "coefficients" $v(t,\cdot)$, $U(t)$ bounded in $L^\infty(I)$. As a simple consequence of Calderon-Zygmunt type estimates ([9, p. 233]) we have

$$\|v(t,\cdot)\|_{W^{2,4}(I)} \leqslant \text{const.}\left(\|v_t(t,\cdot)\|_{L^4(I)} + \|v(t,\cdot)\|_{L^1(I)}\right), \tag{22}$$

with the right side bounded uniformly for $t \in (0,T]$ (Lemmas 1,3). Then it follows from the Sobolev Imbedding Theorem ($n = 1$) that

$$\|v_x(t,\cdot)\|_{C^{\frac{1}{4}}(\bar{I})} \leqslant \text{const.}\ \|v(t,\cdot)\|_{W^{2,4}(I)}, \tag{23}$$

hence v_x is Hölder continuous in x uniformly for $t \in (0,T]$. As a consequence of Lemma 3, $v_t \in L^\infty(0,T;\ L^4(I)) \subset L^4(D)$, also as a consequence of (22) and since v is a $C^{1,2}(\bar{D})$ solution, then $v_x \in L^\infty(0,T;\ L^4(I)) \subset L^4(D)$, and these two conditions together with the Sobolev Imbedding Theorem ($n = 2$) ensure that $v \in C^{\frac{1}{4},\frac{1}{4}}(\bar{D})$ and

$$\|v\|_{C^{\frac{1}{4},\frac{1}{4}}(\bar{D})} \leqslant \text{const.}\left(\|v_t\|_{L^4(D)} + \|v_x\|_{L^4(D)}\right). \tag{24}$$

The proof of Lemma 3 is then completed.

As is well known (c.f. [8, p. 509]) a priori estimate (24) is equivalent (through the Leray-Schauder Principle) to the $C^{1+\frac{1}{4},2+\frac{1}{4}}(\bar{D})$ solvability of (2),(3). We omit the standard proof here.

We have thus shown, under the conditions specified in Theorem 2, that $v \in C^{1+\frac{1}{4},2+\frac{1}{4}}(\bar{D})$. Now returning to the full system (1)—(3), since $\|v(t,\cdot)\|^2_{L^2(I)} \in C^{1+\frac{1}{4}}([0,T])$ we have

$$U \in C^{2+\frac{1}{4}}([0,T]),$$

which completes the proof of Theorem 2.

REFERENCES

[1] N.D. ALIKAKOS, *An application of the invariance principle to reaction-diffusion equations,* J. Differential Equations 33 (1979), 201—225.

[2] N.D. ALIKAKOS, L^p *bounds of solutions of reaction-diffusion equations,* Comm. Partial Differential Equations 4 (1979), 827—868.

[3] J.M. BURGERS, *Mathematical Examples Illustrating Relations Occurring in the Theory of Turbulent Fluid Motion,* Trans. Roy. Neth. Acad. Sci., Amsterdam, 17 (1939), 1—53.

[4] T. DŁOTKO, *On the one-dimensional Burgers' equation; existence, uniqueness and stability,* Prace. Mat. UJ 23 (1982), 157—172.

[5] T. DŁOTKO, *The classical solution of the one-dimensional Burgers' equation,* Prace Mat. UJ 23 (1982), 173—182.

[6] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type,* Prentice Hall, Inc.,Englewood Cliffs, 1967.

[7] J.L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires,* Dunod Gauthier-Villars, Paris, 1969.

[8] O.A. LADYŽENSKAJA, V.A. SOLONNIKOV, N.N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type,* Nauka, Moscow, 1967.

[9] O.A. LADYŽENSKAJA, N.N. URAL'CEVA, *Linear and Quasilinear Equations of Elliptic Type,* Nauka, Moscow. 1973.

[10] L. NIRENBERG, *Topics in Nonlinear Functional Analysis,* Courant Institute, New York, 1974.