

## THE ESTIMATION OF THE OPERATIONAL FUNCTIONS OF THE TYPE $\exp \left[ \left( \sum_{v=1}^n \beta_v s^{p_v} \right) \lambda \right]$

**Abstract.** J. Mikusiński presented in the paper [2] the estimation of the operational function  $e^{-s^\lambda}$ ,  $\lambda \in (0,1)$ , in some unbounded set of  $\mathbf{R}^+ \times \mathbf{R}^+$ . The results obtained by J. Mikusiński were generalized in the paper [3]. The similar estimation of the function

$$\exp \left[ \left( \sum_{v=1}^n \beta_v s^{p_v} \right) \lambda \right]$$

is made in the above paper, where  $\beta_v \in \mathbf{R}$  and  $p_v \in (0,1)$ ,  $v = 1, \dots, n$ .

This paper is devoted to the investigation of the behaviour more general operational functions, when the coefficients  $\beta_v$  are in  $\mathbf{C}$  and fulfilling some relations concerning their real and imaginary parts.

We present our considerations in the form of the following

**THEOREM.** *The operational function  $\exp \left[ \left( \sum_{v=1}^n \beta_v s^{p_v} \right) \lambda \right]$  fulfilling the following conditions*

$$(1) \quad p_1 \in (0,1), \quad p_v \in (-1,1), \quad p_1 > p_v, \quad v = 2, \dots, n,$$

$$(2) \quad \beta_v \in \mathbf{C}, \quad \beta_v = \xi_v + i\eta_v, \quad \xi_v, \eta_v \in \mathbf{R},$$

$$(3) \quad \xi_1 < 0,$$

$$(4) \quad |\eta_1| < |\xi_1| \left( 1 - \frac{1}{2^{p_1}} \right) \operatorname{ctg} \left( p_1 \frac{\pi}{2} \right),$$

is a parametric function for  $\lambda > 0$  ([4]) and according to the denotation used in operational calculus it can be written as follows

$$\exp \left[ \left( \sum_{v=1}^n \beta_v s^{p_v} \right) \lambda \right] = \{ F(\lambda, t) \}.$$

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Moreover, there exist positive constants  $C_1, C_2, G_1,$  and  $G_2$  such that in the set  $D$  determined by inequalities

$$\lambda, t > 0; \quad \frac{\lambda}{t} > C_1 \quad \text{and} \quad \frac{\lambda}{t^{p_1}} > C_2$$

the following estimation

$$(5) \quad |F(\lambda, t)| < G_1 \left( \frac{\lambda}{t} \right)^{\frac{1}{1-p_1}} \exp \left[ -G_2 \left( \frac{\lambda}{t^{p_1}} \right)^{\frac{1}{1-p_1}} \right]$$

holds.

PROOF. If the conditions (1)–(4) are fulfilled then the function  $\exp \left[ \left( \sum_{\nu=1}^n \beta_\nu z^{p_\nu} \right) \lambda \right]$  when  $\text{Re} z > 0$  is the Laplace transform of the function  $\{F(\lambda, t)\}$  for  $\lambda > 0$  and

$$F(\lambda, t) = \frac{1}{2\pi i} \int_L \exp \left[ zt + \left( \sum_{\nu=1}^n \beta_\nu z^{p_\nu} \right) \lambda \right] dz$$

where  $L = \{z: \text{Re} z = x_0 > 0\}$ .

The proof of the inequality (5) requests the consideration of two cases

$$1^\circ -1 \leq \xi_1 < 0,$$

$$2^\circ \xi_1 < -1.$$

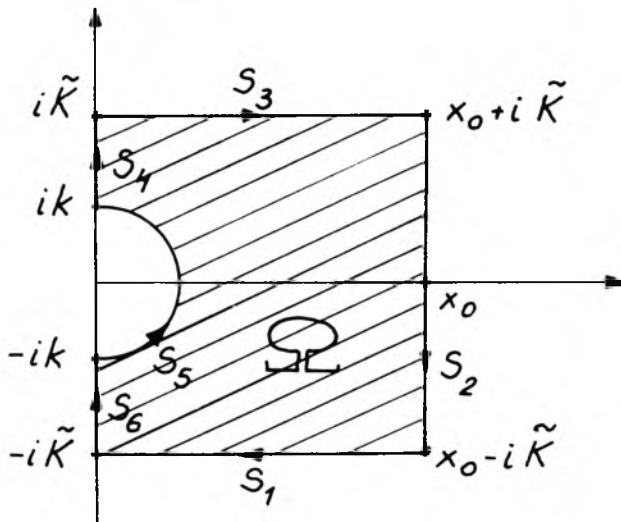
Each of the cases can be proved in the similar way. The difference exists only in the certain arithmetical details. In this paper we shall only present the first case.

Let us put

$$\hat{F}(\lambda, t)(z) = \exp \left[ zt + \left( \sum_{\nu=1}^n \beta_\nu z^{p_\nu} \right) \lambda \right].$$

The symbol  $z^\alpha$  denotes this branch of the power function which is real on the real axis.

Let us consider the contour  $S$



For fixed  $\lambda, t > 0$  the function  $\hat{F}(\lambda, t)(\cdot)$  is holomorphic in the region  $\Omega$  and

$$\int_S \hat{F}(\lambda, t)(z) dz = 0,$$

where  $S$  is the boundary of the region  $\Omega$ . First we shall show that if  $x_0$  is fixed and  $\bar{K} > 0$  then

$$\int_{S_3} \hat{F}(\lambda, t)(z) dz \rightarrow 0 \quad \text{and} \quad \int_{S_1} \hat{F}(\lambda, t)(z) dz \rightarrow 0.$$

The proofs of both cases are similar and we shall give the argumentation for the interval  $S_3$  only.

Let  $z = \varrho + i\bar{K}$ ,  $0 \leq \varrho \leq x_0$ , and let  $\varphi$  denote the main argument of the complex number  $z$ .

$$\begin{aligned} |\hat{F}(\lambda, t)(z)| &= \left| \exp \left[ zt + \left( \sum_{v=1}^n \beta_v z^{p_v} \right) \lambda \right] \right| = \\ &= \exp \left\{ \varrho t + \lambda \left[ \sum_{v=1}^n |z|^{p_v} \left( \xi_v \cos(p_v \varphi) - \eta_v \sin(p_v \varphi) \right) \right] \right\} \leq \\ &\leq \exp(\varrho t) \cdot \exp \left\{ \lambda |z|^{p_1} \cos(p_1 \varphi) \xi_1 \left[ 1 - \frac{\eta_1 \sin(p_1 \varphi)}{\xi_1 \cos(p_1 \varphi)} + \right. \right. \\ &\quad \left. \left. + \sum_{v=2}^n |z|^{p_v - p_1} \frac{\xi_v \cos(p_v \varphi) - \eta_v \sin(p_v \varphi)}{\xi_1 \cos(p_1 \varphi)} \right] \right\}. \end{aligned}$$

From (4) it follows that the following inequalities

$$\sup_{\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \left| \frac{\xi_v \cos(p_v \varphi) - \eta_v \sin(p_v \varphi)}{\xi_1 \cos(p_1 \varphi)} \right| \leq \frac{|\xi_v| + |\eta_v|}{|\xi_1| \cos(p_v \frac{\pi}{2})}$$

for  $v \in \{2, \dots, n\}$  and

$$\sup_{\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \left| \frac{\eta_1 \sin(p_1 \varphi)}{\xi_1 \cos(p_1 \varphi)} \right| \leq \frac{|\eta_1|}{|\xi_1|} \operatorname{tg} \cdot \left( p_1 \frac{\pi}{2} \right) = \gamma < 1$$

hold. According to the assumption (1) there exists a constant  $c > 0$  such that

$$\inf_{\operatorname{Re} z > c} \sum_{v=2}^n |z|^{p_v - p_1} \frac{\xi_v \cos(p_v \varphi) - \eta_v \sin(p_v \varphi)}{\xi_1 \cos(p_1 \varphi)} > -\frac{1}{2} (1 - \gamma).$$

In view of the assumption (3) we get the estimation

$$|\hat{F}(\lambda, t)(z)| \leq \exp(\varrho t) \exp\left[\frac{1}{2} \xi_1 |z|^{p_1} \cos\left(p_1 \frac{\pi}{2}\right) (1 - \gamma)\right].$$

If we choose  $\tilde{K} > 1$  then

$$|\hat{F}(\lambda, t)(z)| \leq \exp(\varrho t) \exp\left[\frac{1}{2} \xi_1 \tilde{K}^{p_1} \cos\left(p_1 \frac{\pi}{2}\right) (1 - \gamma)\right].$$

Hence  $\xi_1 < 0$ , and

$$\frac{1}{2} \tilde{K}^{p_1} \cos\left(p_1 \frac{\pi}{2}\right) (1 - \gamma) > 0.$$

Therefore

$$\lim_{\tilde{K} \rightarrow \infty} \exp\left[\frac{1}{2} \xi_1 \tilde{K}^{p_1} \cos\left(p_1 \frac{\pi}{2}\right) (1 - \gamma)\right] = 0.$$

Hence for each  $\varepsilon > 0$  there exists a  $\delta > 1$  such that for  $\tilde{K} > \delta$

$$|\hat{F}(\lambda, t)(z)| < \varepsilon \cdot \exp(\varrho t), \quad z \in S_3.$$

Hence

$$\left| \int_{S_3} \hat{F}(\lambda, t)(z) dz \right| < \varepsilon \frac{e^{x_0 t} - 1}{t}$$

and then

$$\lim_{\tilde{K} \rightarrow \infty} \int_{S_3} \hat{F}(\lambda, t)(z) dz = 0.$$

Let  $L_1$  and  $L_2$  be the halflines  $L_1 = (iK, \infty) \rightarrow$ ,  $L_2 = (-i\infty, -iK) \rightarrow$  directing to the top of the imaginary axis. Following the fact proved above and the Cauchy theorem we deduce that

$$\int_{L_1 \cup S_5 \cup L_2} \hat{F}(\lambda, t)(z) dz = \int_L \hat{F}(\lambda, t)(z) dz$$

and  $L$  has the direction opposite to that one of  $S_2$ . For fixed  $\lambda$  and  $t$  we shall estimate the modulus of the integral

$$\int_{S_5} \hat{F}(\lambda, t)(z) dz.$$

Let  $z = Ke^{iu}$  and  $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$ . Then the following inequality

$$I_1 = \left| \frac{1}{2\pi i} \int_{S_\delta} \hat{F}(\lambda, t)(z) dz \right| \leq \\ \leq \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left\{ Kt \cos u + \lambda K^{p_1} \xi_1 \cos(p_1 u) \left[ 1 - \frac{\eta_1}{\xi_1} \operatorname{tg}(p_1 u) + \right. \right. \\ \left. \left. + \sum_{v=2}^n K^{p_v - p_1} \frac{\xi_v \cos(p_v u) - \eta_v \sin(p_v u)}{\xi_1 \cos(p_1 u)} \right] du \right\}.$$

holds. By inequality (4) we have

$$\inf_{u \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left[ -\frac{\eta_1}{\xi_1} \operatorname{tg}(p_1 u) \right] \geq -\gamma > \frac{1}{2^{p_1}} - 1$$

and, for  $v \in \{2, \dots, n\}$ ,

$$\sup_{u \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left| \frac{\xi_v \cos(p_v u) - \eta_v \sin(p_v u)}{\xi_1 \cos(p_1 u)} \right| \leq \frac{|\xi_v| + |\eta_v|}{|\xi_1| \cos(p_1 \frac{\pi}{2})}.$$

Let

$$\varepsilon \in \left( 0, 1 - \frac{1}{(1-\gamma)2^{p_1}} \right).$$

There exists a  $K(\varepsilon) > 1$  such that if  $K > K(\varepsilon)$  the following inequality

$$\sup_{u \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left| \sum_{v=2}^n K^{p_v - p_1} \frac{\xi_v \cos(p_v u) - \eta_v \sin(p_v u)}{\xi_1 \cos(p_1 u)} \right| < \varepsilon(1-\gamma)$$

holds. Hence, for each  $K > K(\varepsilon)$ , we get the inequality

$$(6) \quad I_1 < \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left[ Kt \cos u + \lambda \xi_1 K^{p_1} (1-\gamma)(1-\varepsilon) \cos(p_1 u) \right] du.$$

Let us put

$$\alpha = (1 - \gamma)(1 - \varepsilon).$$

It is easy to show that

$$\alpha \in \left( \frac{1}{2^{p_1}}, 1 \right).$$

$\lambda, t$  are fixed. Choosing a suitable  $w$  we may express the radius  $K$  in the form

$$K = \left[ \frac{\lambda}{wt} \right]^{\frac{1}{1-p_1}}$$

Then inequality (6) transforms to

$$I_1 < \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left\{ \left( \frac{\lambda}{wt^{p_1}} \right)^{\frac{1}{1-p_1}} \left[ \cos u - w\alpha|\xi_1| \cos(p_1 u) \right] \right\} du.$$

The function

$$A_w(u) = \cos u - w\alpha|\xi_1| \cos(p_1 u)$$

is defined on  $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$  for

$$w \in \left( 0, \frac{1}{|\xi_1|\alpha p_1 \sin\left(p_1 \frac{\pi}{2}\right)} \right)$$

and takes the greatest value at zero. This value equals

$$A_w(0) = 1 - w|\xi_1|\alpha.$$

Hence we get the following estimation

$$\begin{aligned} I_1 &< \frac{K}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left\{ \left( \frac{\lambda}{wt^{p_1}} \right)^{\frac{1}{1-p_1}} \left[ 1 - w|\xi_1|\alpha \right] \right\} du = \\ &= \frac{K}{2} \exp \left\{ \left( \frac{\lambda}{t^{p_1}} \right)^{\frac{1}{1-p_1}} w^{\frac{1}{1-p_1}} \left[ 1 - w|\xi_1|\alpha \right] \right\} \end{aligned}$$

for

$$w \in \left( 0, \frac{1}{|\xi_1| \alpha p_1 \sin\left(p_1 \frac{\pi}{2}\right)} \right).$$

The inequality (7) holds for each  $w$  belonging to the above interval. We shall choose such a  $w$  for which the estimation of  $I_1$  will be the best. After the standard calculation we infer that the minimum of the function

$$B(w) = w^{-\frac{1}{1-p_1}} \left[ 1 - w |\xi_1| \alpha \right]$$

on the interval

$$\left( 0, \frac{1}{|\xi_1| \alpha p_1 \sin\left(p_1 \frac{\pi}{2}\right)} \right)$$

is taken at the point

$$w_0 = \frac{1}{|\xi_1| \alpha p_1}$$

and then

$$(8) \quad I_1 < \frac{1}{2} \left( \frac{\lambda}{t} \right)^{\frac{1}{1-p_1}} w_0^{-\frac{1}{1-p_1}} \exp \left[ B(w_0) \left( \frac{\lambda}{t^{p_1}} \right)^{\frac{1}{1-p_1}} \right].$$

Let us put

$$K_0 = K_0(\lambda, t) = \left[ \frac{\lambda}{w_0 t} \right]^{\frac{1}{1-p_1}}$$

Of course we assume that  $K_0 > K(\varepsilon)$ . Hence, for  $\lambda, t > 0$ , we have the relation

$$(9) \quad \frac{\lambda}{t} > w_0 [K(\varepsilon)]^{1-p_1}$$

The right side of the above inequality will be denoted by  $C_1$ . Hence, for  $\lambda, t > 0$  and  $\frac{\lambda}{t} > C_1$ , inequality (8) holds.

Now we shall estimate the modulus of the integral

$$\frac{1}{2\pi i} \int_{L_1} \hat{F}(\lambda, t)(z) dz.$$

Let  $z = iu$  and  $u \in [K_0, \infty)$ . Hence

$$\begin{aligned} I_2 &= \left| \frac{1}{2\pi i} \int_{L_1} \hat{F}(\lambda, t)(z) dz \right| \leq \\ &\leq \frac{1}{2\pi} \int_{K_0}^{\infty} \exp \left\{ \lambda \xi_1 u^{p_1} \cos \left( p_1 \frac{\pi}{2} \right) \left[ 1 - \frac{\eta_1}{\xi_1} \operatorname{tg} \left( p_1 \frac{\pi}{2} \right) + \right. \right. \\ &\quad \left. \left. + \sum_{v=2}^n u^{p_v - p_1} \frac{\xi_v \cos \left( p_v \frac{\pi}{2} \right) - \eta_v \sin \left( p_v \frac{\pi}{2} \right)}{\xi_1 \cos \left( p_1 \frac{\pi}{2} \right)} \right] \right\} du. \end{aligned}$$

Thus  $K_0 > K(\varepsilon)$ ,

$$\begin{aligned} -\frac{\eta_1}{\xi_1} \operatorname{tg} \left( p_1 \frac{\pi}{2} \right) + \sum_{v=2}^n u^{p_v - p_1} \frac{\xi_v \cos \left( p_v \frac{\pi}{2} \right) - \eta_v \sin \left( p_v \frac{\pi}{2} \right)}{\xi_1 \cos \left( p_1 \frac{\pi}{2} \right)} &> \\ &> -\gamma - \varepsilon(1 - \gamma) \end{aligned}$$

and

$$I_2 < \frac{1}{2\pi} \int_{K_0}^{\infty} \exp \left[ -\lambda |\xi_1| \alpha u^{p_1} \cos \left( p_1 \frac{\pi}{2} \right) \right] du.$$

Changing the variable in the above integral,  $y = \lambda |\xi_1| \alpha u^{p_1}$ , we get

$$I_2 < \frac{1}{2\pi} \frac{1}{p_1} \left[ \lambda |\xi_1| \alpha \right]^{-\frac{1}{p_1}} \int_{\kappa}^{\infty} y^{\frac{1}{p_1} - 1} \exp \left[ -y \cos \left( p_1 \frac{\pi}{2} \right) \right] dy,$$

where  $\kappa = \lambda \alpha |\xi_1| K_0^{p_1}$ .

Now we shall make the extra assumption (another connection between  $\lambda$  and  $t$ )

$$(10) \quad \frac{\lambda}{t^{p_1}} > \frac{2}{p_1^5 |\xi_1|}$$

and the right side of the inequality (10) we shall denote by  $C_2$ . We have  $-1 \leq \xi_1 < 0$  and  $\frac{1}{2^{p_1}} < \alpha < 1$ , whence



$$\lambda \alpha |\xi_1| K_0^{p_1} > \frac{2}{p_1^4}$$

for  $\lambda$  and  $t$  fulfilling the connections (9) and (10). Taking the advantage of the inequality formulated in the paper [2], i. e.

if  $p \in (0,1)$  and  $q > \frac{2}{p^4}$  then

$$q^{\frac{1}{p}-1} \exp\left[-q \cos\left(p \frac{\pi}{2}\right)\right] < \exp\left[-q(1-p)\right],$$

we get the estimation

$$I_2 < \frac{1}{2\pi} \frac{1}{p_1} \left[\lambda \alpha |\xi_1|\right]^{-\frac{1}{p_1}} \int_x^\infty \exp\left[-y(1-p_1)\right] dy.$$

Let us put

$$\delta_0 = \left[ \frac{2}{p_1^5 |\xi_1|} \right]^{\frac{1}{p_1(1-p_1)}}$$

and

$$\delta_1 = \frac{[\alpha |\xi_1|]^{-\frac{1}{p_1}}}{2\pi p_1 (1-p_1) \delta_0}$$

Then we get the following estimation

$$I_2 < \delta_1 \left(\frac{\lambda}{t}\right)^{\frac{1}{1-p_1}} \exp\left[B(w_0) \left(\frac{\lambda}{t^{p_1}}\right)^{\frac{1}{1-p_1}}\right].$$

The similar estimation may be obtained for the interval

$$\frac{1}{2\pi i} \int_{L_2} \hat{F}(\lambda, t)(z) dz.$$

Putting

$$G_1 = \frac{1}{2} w_0^{\frac{1}{1-p_1}} = 2\delta_1 \quad \text{and} \quad G_2 = -B(w_0)$$

we get, for  $\lambda, t > 0$  fulfilling the inequalities (9) and (10), the last inequality

$$|F(\lambda, t)| < G_1 \left(\frac{\lambda}{t}\right)^{\frac{1}{1-p_1}} \exp\left[-G_2 \left(\frac{\lambda}{t^{p_1}}\right)^{\frac{1}{1-p_1}}\right].$$

In this way we finished the proof in the case  $-1 \leq \xi_1 < 0$ . The proof in the case  $\xi_1 < -1$  is similar.

The inequality proved above may be used in searching properties of solutions of the partial differential equations or operational equations.

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