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REMARKS ON MAPS OF INVERSE LIMITS

Abstract. The aim of this note is to give a short proof of the Ščepin theorem concerning maps of inverse limits. This theorem was generalized by several authors; see e.g. W. Kulpa [5], A. Archangelskii [1] and M.G. Tkačenko [7], [8].

Our method of the proof gives also the most general version of the Ščepin theorem due to Tkačenko. It can be also applied for obtaining in a very general setting the theorem of H.H. Corson and J.R. Isbell [3], [4] concerning maps from products.

LEMMA 1. *Let Z be a Hausdorff space and $f: X \xrightarrow{\text{onto}} Y$, $g: X \to Z$ be continuous maps. If the space Z has a base $\mathcal{B}$ such that for each $U \in \mathcal{B}$ there exists an open set $W \subset Y$ such that*

$$g^{-1}(U) = f^{-1}(W), \tag{1}$$

then there exists a continuous map $h: Y \to Z$ such that $h \circ f = g$.

Proof. Let us verify that for each $y \in Y$ the set $g(f^{-1}(y))$ consists of a single point. It suffices to show that:

(2) if $U \in \mathcal{B}$ and $U \cap g(f^{-1}(y)) \neq \emptyset$ and $g^{-1}(U) = f^{-1}(W)$, where W is open in Y, then $y \in W$.

To show (2), let $z \in U \cap g(f^{-1}(y))$. There exists a point $x \in f^{-1}(y)$ such that $z = g(x)$. Hence $x \in g^{-1}(W)$. Therefore $y = f(x) \in W$. Thus, define $h(y)$ to be the single point in the set $g(f^{-1}(y)$. Clearly, $h \circ f = g$. The map $h: Y \to Z$ is continuous. Indeed, let us fix a point $y \in Y$ and an open set $U \in \mathcal{B}$ such that $h(y) \in U$. By the condition (2), there exists an open subset W of Y such that $y \in W$ and $g^{-1}(U) = f^{-1}(W)$. Hence $h(W) = g(f^{-1}(W)) = g(g^{-1}(U)) \subset U$, i.e. $h(W) \subset U$. The lemma is proved.

An inverse system $\{X_\alpha, p^\beta_\alpha\colon \alpha < \beta < \tau\}$ of topological spaces and continuous maps, where τ is an (uncountable) cardinal, will be called *continuous* if $X_\gamma = \varprojlim \{X_\alpha, p^\beta_\alpha : \alpha < \beta < \gamma\}$ for each limit ordinal $\gamma < \tau$.

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A continuous inverse system $\{X_\alpha, p_\alpha^\beta : \alpha < \beta < \tau\}$ will be called *regular* if τ is a regular cardinal and weight of X_α (denoted by $w(X_\alpha)$) is less than τ.

Let $C(T)$ denote the family of all cozero-sets of the space T.

LEMMA 2. *Let us assume that the inverse system* $\{X_\alpha, p_\alpha^\beta : \alpha < \beta < \tau\}$ *is regular, where all* X_α*'s are compact and let* $X = \varprojlim \{X_\alpha, p_\alpha^\beta : \alpha < \beta < \tau\}$. *If* g *is a continuous map from* X *into a compact space* Z *and there exists a base* $\mathcal{B} \subset C(Z)$ *such that* $|\mathcal{B}| < \tau$, *then there exists an* $\alpha < \tau$ *such that for all* $\beta \geqslant \alpha, \beta < \tau$ *and* $U \in \mathcal{B}$ *there exists a* $W \in C(X_\beta)$ *such that*

$$p_\beta^{-1}(W) = g^{-1}(U).$$

Proof. Let $\mathcal{B} = \{U_\xi : \xi < \gamma\}$, where $\gamma < \tau = \mathrm{cf}(\tau)$. Clearly, $g^{-1}(U_\xi) \in C(X)$ for each ξ. Thus $g^{-1}(U_\xi)$ is an open F_σ-set, i.e. there exists a family $\{F_n : n < \omega\}$ of compact subsets of X such that $g^{-1}(U_\xi) = \bigcup \{F_n : n < \omega\}$. Clearly, sets of the form $p_\alpha^{-1}(W)$, where W is open in X_α for $\alpha < \tau$, form a base in X. Then for each $n < \omega$ there exist an $\alpha_n < \tau$ and an open set $W_n \subset X_{\alpha_n}$ such that $F_n \subset p_{\alpha_n}^{-1}(W_n) \subset g^{-1}(U_\xi)$. Since $\mathrm{cf}(\tau) > \omega$, $\alpha(\xi) = \sup \{\alpha_n : n < \omega\} < \tau$. It is easy to calculate that $g^{-1}(U_\xi) = p_{\alpha(\xi)}^{-1}(W_\xi)$, where $W_\xi = \bigcup \{(p_{\alpha_n}^{\alpha(\xi)})^{-1}(W_n) : n < \omega\}$ is an open subset of $X_{\alpha(\xi)}$.

Let $\beta = \sup \{\alpha(\xi) : \xi < \gamma\}$. Since $\gamma < \mathrm{cf}(\tau)$, we have $\beta < \tau$. Therefore $\{g^{-1}(U_\xi) : \xi < \gamma\} \subset \{p_\beta^{-1}(W) : W \in C(X_\beta)\}$, which means that for some W from $C(X_\beta)$ we have $g^{-1}(U_\xi) = p_\beta^{-1}(W)$. Thus, our lemma is proved.

THEOREM 1 (E.V. Ščepin). *Let us assume that* $X = \varprojlim \{X_\alpha, p_\alpha^\beta : \alpha < \beta < \tau\}$ *and* $Y = \varprojlim \{Y_\alpha, g_\alpha^\beta : \alpha < \beta < \tau\}$, *where all* X_α*'s and* Y_α*'s are compact and the inverse systems are regular. Then for each continuous map (homeomorphism)* $f : X \to Y$ *there exist a closed unbounded set* $S \subset \tau$ *and a family of continuous maps (homeomorphisms)* $f_\alpha : X_\alpha \to Y_\alpha$, $\alpha \in S$, *such that*

$$f_\alpha \circ p_\alpha = q_\alpha \circ f \quad \textit{for } \alpha \in S.$$

Proof. Let $\alpha < \tau$ and $\mathcal{B}_\alpha \subset C(Y_\alpha)$ be a base in Y_α such that $|\mathcal{B}_\alpha| < \tau$.

Let $g = q_\alpha \circ f$. From Lemma 2 it follows that there exists a $\beta(\alpha) < \tau$ such that for each $U \in \mathcal{B}_\alpha$ the exists a $W \in C(X_{\beta(\alpha)})$ such that $g_1^{-}(U) = p_{\beta(\alpha)}^{-1}(W)$. Hence, by Lemma 1, there exists a continuous map $f_\alpha^{\beta(\alpha)} : X_{\beta(\alpha)} \to Y_\alpha$ such that $q_\alpha \circ f = f_\alpha^{\beta(\alpha)} \circ p_{\beta(\alpha)}$. We may assume that $\alpha < \beta(\alpha)$.

We construct by induction a sequence $\{\alpha_n : n < \omega\} \subset \tau$ such that:

a) α_0 is an arbitrary ordinal less than τ

and

b) $\alpha_{n+1} = \beta(\alpha_n)$.

Thus, we get maps $f_n = f_{\alpha n}^{\beta(\alpha n)}$, $n < \omega$, such that

(3) $$q_{\alpha_n} \circ f = f_n \circ p_{\alpha_{n+1}}.$$

Let $\delta = \sup\{\alpha_n : n < \omega\}$. Since $\mathrm{cf}(\tau) > \omega$, then $\delta < \tau$. Since $\alpha_n < \alpha_{n+1}$, then δ is a limit ordinal. Let us set $X_\delta = \varprojlim\{X_{\alpha_n}, p_{\alpha_n}^{\alpha_{n+1}} : n < \omega\}$ and $Y_\delta = \varprojlim\{Y_{\alpha_n}, q_{\alpha_n}^{\alpha_{n+1}} : n < \omega\}$. There exists the limit map $f_\delta = \varprojlim\{f_n : n < \omega\}$. To finish the proof, we shall show that $f_\delta \circ p_\delta = q_\delta \circ f$. Suppose that there exists an $x \in X$ such that $f_\delta(p_\delta(x)) \neq q_\delta(f(x))$. Since these points belong to Y_δ, there exists $n < \omega$ such that $q^\delta_{\alpha_n}(f_\delta(p_\delta(x))) \neq q^\delta_{\alpha_n}(q_\delta(f(x))) = q_{\alpha_n}(f(x))$. Since $f_\delta = \varprojlim\{f_n : n < \omega\}$, $q^\delta_{\alpha_n} \circ f_\delta = f_n \circ p^\delta_{\alpha_{n+1}}$. So $f_n(p^\delta_{\alpha_{n+1}}(p_\delta(x))) = f_n(p_{\alpha_{n+1}}(x)) \neq q_{\alpha_n}(f(x))$, which contradicts the condition (3). The proof is complete.

For every topological space X, let $\mathcal{L}(X)$ be minimal cardinal number τ such that for every open covering $\mathcal{P}$ of X there exists a subcovering $\mathcal{P}' \subset \mathcal{P}$ such that $|\mathcal{P}'| \leqslant \tau$. Note that for every compact space X, $\mathcal{L}(X) \leqslant \aleph_0$.

THEOREM 2 (M.G. Tkačenko). *Let assume that* $X = \varprojlim\{X_\alpha, p^\beta_\alpha : \alpha < \beta < \tau\}$, *where all* X_α*'s are completely regular, and the inverse system is regular and* $\mathcal{L}(X) < \tau$. *Then for each continuous map* $f : X \longrightarrow X$ *there exists a closed unbounded set* S, $S \subset \tau$ *such that for each* $\alpha \in S$ *there exists a continuous map* $f_\alpha : X_\alpha \longrightarrow X_\alpha$ *such that* $f_\alpha \circ p_\alpha = p_\alpha \circ f$ *for each* $\alpha \in S$.

Proof. The complete regularity of X_α and $\mathcal{L}(X) < \tau$ imply the existence of a base $\mathcal{B}_\alpha \subset C(X_\alpha)$ such that $|\mathcal{B}_\alpha| < \tau$. If $U \in \mathcal{B}_\alpha$, then $H = f^{-1}(p_\alpha^{-1}(U))$ is an open F_σ-set. Thus $H = \bigcup_{n<\omega} F_n$, where all F_n's are closed subset of X. The family $\mathcal{P} = \{p_\alpha^{-1}(U) : U \in \mathcal{B}_\alpha \text{ and } \alpha < \tau\}$ is a base in X.

Let $\mathcal{R}_n = \{W \in \mathcal{P} : F_n \cap W \neq \emptyset \text{ and } W \subset H\}$. But $\mathcal{L}(F_n) \leqslant \mathcal{L}(X)$, F_n being closed and $\mathcal{L}(X) < \tau$. Hence there exists $\mathcal{R}'_n \subset \mathcal{R}_n$ such that $|\mathcal{R}'_n| < \tau$ and $F_n \subset \bigcup \mathcal{R}'_n$. Thus there exist $\beta_n < \tau$ and an open set G_n of X_{β_n} such that $\bigcup \mathcal{R}'_n = p_\beta^{-1}(G_n)$.

Let $\beta(U) = \sup\{\beta_n : n < \omega\}$. Then there exists an open set W' of $X_{\beta(U)}$ such that $H = p^{-1}_{\beta(U)}(W')$.

Let $\beta(\alpha) = \sup\{\beta(U) : U \in \mathcal{B}_\alpha\} < \tau$. For each $U \in \mathcal{B}_\alpha$ there exists an open set W of $X_{\beta(\alpha)}$ such that $f^{-1}(p_\alpha^{-1}(U)) = p^{-1}_{\beta(\alpha)}(W)$. Now, it suffices to apply Lemma 1. Thus the theorem is proved.

H.H. Corson [3] and H.H. Corson and J.R. Isbell [4] have proved that every continuous function which maps a product of spaces with countable bases into a metrizable space depends only on countable number of coordinates.

By the use of our methods we are able to prove a more general result.

Note that the relations between the Ščepin theorem and the theorems on maps defined on products were pointed out by Tkačenko [7].

THEOREM 3. *Let $\mathcal{R}$ be a family of topological spaces and $\bar{X} \subset \Pi\{X: X \in \mathcal{R}\}$. If f is a continuous map from $\bar{X}$ into a regular space Z, then there exist a family $\mathcal{R}_1 \subset \mathcal{R}$ and a continuous map $h: \Pi_{\mathcal{R}_1}(X) \longrightarrow Z$ such that $|\mathcal{R}_1| \leqslant w(Z) + \mathcal{L}(\bar{X})$ and $h \circ \Pi_{\mathcal{R}_1} = f$.*

Proof. Let $\mathcal{B}$ be a base in Z such that $|\mathcal{B}| = w(Z)$. Let $\tau = w(Z) + \mathcal{L}(\bar{X})$. Since Z is regular and $w(Z) \leqslant \tau$ for each $U \in \mathcal{B}$, there exists a family $\mathcal{F}_U$ of closed subsets of $\bar{X}$ such that $|\mathcal{F}_U| \leqslant \tau$ and $\bigcup \mathcal{F}_U = f^{-1}(U)$.

Let us fix an element $F \in \mathcal{F}_U$. For each $x \in F$ there exists a basic set $W_x \subset \Pi\{X: X \in \mathcal{R}\}$ such that $x \in W_x \cap \bar{X} \subset f^{-1}(U)$. Then $\{W_x \cap \bar{X}: x \in F\} \cup \{\bar{X} - F\}$ is a covering of $\bar{X}$. Since $\mathcal{L}(\bar{X}) \leqslant \tau$, there exists a family $\mathcal{P}_F$ of basic sets of $\Pi\{X: X \in \mathcal{R}\}$ such that $F \subset \bigcup \mathcal{P}_F \cap \bar{X} \subset f^{-1}(U)$ such that $|\mathcal{P}_F| \leqslant \tau$. Hence there exists a family $\mathcal{P}_U$ of basic subsets of $\Pi\{X: X \in \mathcal{R}\}$ such that $f^{-1}(U) = \bar{X} \cap \bigcup \mathcal{P}_U$ and $|\mathcal{P}_U| \leqslant \tau$, because $|\mathcal{F}_U| \leqslant \tau$. Clearly, each element $H \in \mathcal{P}_U$ is of the form $\Pi^{-1}_{\mathcal{R}'}(G)$, where $\mathcal{R}' \subset \mathcal{R}$ is finite and G is open in $\Pi\{X: X \in \mathcal{R}'\}$. Hence there exists $\mathcal{R}_U \subset \mathcal{R}$ such that $|\mathcal{R}_U| \leqslant \tau$ and $f^{-1}(U) = \Pi^{-1}_{\mathcal{R}_U}(W_U) \cap \bar{X}$, where W_U is an open set in $\Pi\{X: X \in \mathcal{R}_U\}$. Let us set $\mathcal{R}_1 = \bigcup_{U \in \mathcal{B}} \mathcal{R}_U$. Clearly, $|\mathcal{R}_1| \leqslant \tau$ and for each $U \in \mathcal{B}$ there exists an open set $W_U \subset \Pi\{X: X \in \mathcal{R}_1\}$ such that $f^{-1}(U) = \Pi^{-1}_{\mathcal{R}_1}(W_U) \cap \bar{X}$. Now it suffices to apply Lemma 1. Thus the theorem is proved.

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