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THE FUNCTOR K_2 FOR MULTIQUADRATIC NUMBER FIELDS

Abstract. Let F and O_F be a number field and its ring of integers respectively. Let K_2 denote Milnor K-functor. In the paper we describe the structure of the group $K_2 O_F/\mathfrak{R}_2 F$, where $\mathfrak{R}_2 F$ is the Hilbert kernel and F is multiquadratic extension of the rational number field. Moreover, we give some characterization of fields with trivial group $K_2 O_F/\mathfrak{R}_2 F$. At the end we make some remarks on p-rank of $K_2 O_F$ and divisibility of the ideal class group by 2.

1. Introduction. Let F be an algebraic number field, O_F the ring of integers in F and K_2 the Milnor K-functor. In this paper we investigate the group $K_2 O_F/\mathfrak{R}_2 F$, where $\mathfrak{R}_2 F$ is the Hilbert kernel, for multiquadratic extension F of the rational field $\mathbf{Q}$. In Section 2, we describe completely the structure of the group $K_2 O_F/\mathfrak{R}_2 F$ for any multiquadratic extension F, thus extending the result of J. Browkin [1]. In section 3 we characterize the number fields F with $K_2 O_F/\mathfrak{R}_2 F = 0$. The concluding remarks are concerned with the p-rank of $K_2 O_F$ in the case when $K_2 O_F/\mathfrak{R}_2 F$ is trivial. We also estimate the 2-rank of the group $K_2 O_F/\mathfrak{R}_2 F$ is some special cases of multiquadratic number fields. This allows us to produce a series of examples of multiquadratic number fields with even order of the ideal class group.

We use the following notation, terminology and auxiliary facts. F_v denotes the completion of F with respect to the valuation v and μ_v is the group of roots of unity in F_v, m_v being the order of μ_v. Similarly μ and m are the group of roots of unity in F and the order of the group. As proved by D. Quillen in [4], $K_2 O_F$ is the kernel of the homomorphism $\tau\colon K_2 F \to \sqcup \bar{F}_v$, v running through discrete valuations of F, where τ satisfies $\tau(\{\alpha, \beta\}) = ((\alpha, \beta)_v)_v$. Here $(\, , \,)_v$ is the tame symbol, as defined in Milnor [3]. The Hilbert kernel $\mathfrak{R}_2 F$ is defined to be the kernel of the homomorphism $\eta\colon K_2 F \to \sqcup \mu_v$ (v runs through non real valuations of F), satisfying $\eta(\{\alpha, \beta\}) = ([\alpha, \beta]_v)_v$. Here $[\, , \,]_v$ is the Hilbert symbol according to [3]. The Hilbert symbol and the tame symbol are bound to satisfy $[a, b]_v^{m_v/(Nv-1)} = (a, b)_v$ for any discrete valuation v. It follows that $\mathfrak{R}_2 F \subset K_2 O_F$.

J. Browkin [1] has given the following presentation for $K_2 O_F/\mathfrak{R}_2 F$ which will be also the basis for all the computations in this paper.

THEOREM. *The group $K_2 O_F/\mathfrak{R}_2 F$ is isomorphic to the Abelian group with generators g_v, where v runs through all the real valuations of F and these discrete valuations for which $\mathfrak{z}_p \in F_v$ for $v|p$, $\mathfrak{z}_p$ being the primitive p-th root of unity. The generators are subject to the following relations*:

(1) $g_v^2 = 1$ *for real valuations* v,

(2) $g_v^{m_v/(Nv-1)} = 1$ *for discrete* v (*here* $Nv = |\bar{F}_v|$),

(3) $$\prod_{v-real} g_v \cdot \prod_{\substack{v-discr. \\ \mathfrak{z}_p \in F \text{ for } v/p}} g_v^{m_v/m} = 1.$$

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2. The group $K_2O_F/\mathfrak{K}_2F$ for multiquadratic extensions of the rationals. In this section we assume that $F = \mathbf{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})$, where $a_1, \ldots, a_n$ are square-free integers and $[F:Q] = 2^n$.

PROPOSITION 2.1. *Suppose p is a prime, $p \neq 2$ and v is a valuation of F with $v|p$. Then*

(i) *F_v is one of the following extensions of the p-adic field:* $\mathbf{Q}_p, \mathbf{Q}_p(\sqrt{p}), \mathbf{Q}_p(\sqrt{\mathfrak{z}p})$, $\mathbf{Q}_p(\sqrt{\mathfrak{z}}), \mathbf{Q}_p(\sqrt{p}, \sqrt{\mathfrak{z}})$, *where $\mathfrak{z}$ is a primitive root of unity of degree $p-1$.*

(ii) $e(F_v/\mathbf{Q}_p) \leqslant 2$.

Proof. (i) First observe that $F_v = \mathbf{Q}_p(\sqrt{a_1}, \ldots, \sqrt{a_n})$ so that F_v is a multiquadratic extension of $\mathbf{Q}_p$. There are only three quadratic extensions of $\mathbf{Q}_p$ and these are $\mathbf{Q}_p(\sqrt{p}), \mathbf{Q}_p(\sqrt{\mathfrak{z}p}), \mathbf{Q}_p(\sqrt{\mathfrak{z}})$. Adjoining to anyone of these a square root of any element of $\mathbf{Q}_p$ we get either one of the three fields or the unique biquadratic extension of $\mathbf{Q}_p$ equal to $\mathbf{Q}_p(\sqrt{p}, \sqrt{\mathfrak{z}})$. Since $\mathbf{Q}_p(\sqrt{p}, \sqrt{\mathfrak{z}})$ cannot be further extended by adjoining a square root of an element of $\mathbf{Q}_p$, we get (i).

(ii) Each of the fields in (i) has ramification index $\leqslant 2$.

LEMMA 2.2. *If $v|p$ and $\mathfrak{z}_p \in F_v$, then $p = 2$ or $p = 3$.*

Proof. Corollary 2 in [1] implies that $p-1 \mid e(F_v/\mathbf{Q}_p)$ and $e \leqslant 2$ by 2.1. (ii).

The above lemma shows that the generators for $K_2O_F/\mathfrak{K}_2F$ can come only from real valuations and those discrete ones that divide 2 or 3.

LEMMA 2.3. *If $v|3$ and $\mathfrak{z}_3 \in F_v$, then $F_v = \mathbf{Q}_3(\sqrt{-3})$ or $F_v = \mathbf{Q}_3(\sqrt{-3}, \sqrt{3})$.*

Proof. The result follows by inspecting which ones of the fields in 2.1. (i) contain the third root of unity.

PROPOSITION 2.4. *If $v|2$, then*

(i) *F_v is one of the following fields:* $\mathbf{Q}_2, \mathbf{Q}_2(\sqrt{3}), \mathbf{Q}_2(\sqrt{-3}), \mathbf{Q}_2(\sqrt{-1})$, $\mathbf{Q}_2(\sqrt{2}), \mathbf{Q}_2(\sqrt{-2}), \mathbf{Q}_2(\sqrt{6}), \mathbf{Q}_2(\sqrt{-6}), \mathbf{Q}_2(\sqrt{-1}, \sqrt{3}), \mathbf{Q}_2(\sqrt{-1}, \sqrt{2})$, $\mathbf{Q}_2(\sqrt{-1}, \sqrt{6}), \mathbf{Q}_2(\sqrt{2}, \sqrt{3}), \mathbf{Q}_2(\sqrt{-2}, \sqrt{3}), \mathbf{Q}_2(\sqrt{2}, \sqrt{-3}), \mathbf{Q}_2(\sqrt{-2}, \sqrt{-3})$, $\mathbf{Q}_2(\sqrt{-1}, \sqrt{2}, \sqrt{3})$.

(ii) $e(F_v/\mathbf{Q}_2) \leqslant 4, f(F_v/\mathbf{Q}_2) \leqslant 2$.

Proof. We have $F_v = \mathbf{Q}_2(\sqrt{a_1}, \ldots, \sqrt{a_n})$. It is easy to see that the list of fields contains all quadratic and biquadratic extension of $\mathbf{Q}_2$. And $\mathbf{Q}_2(\sqrt{-1}, \sqrt{2}, \sqrt{3})$ is the unique multiquadratic extension of $\mathbf{Q}_2$ of degree 8. It contains all the square roots of elements of $\mathbf{Q}_2$ so that it cannot be extended properly by adjoining square roots of elements of $\mathbf{Q}_2$. This proves (i).

(ii) follows inmediately from (i) by direct computing the degree and ramification index of all the fields listed in (i).

COROLLARY 2.5. *For any valuation v such that $v \mid 2$ we have*

(i) *if $\mathfrak{z}_n \in F_v$ and $2 \nmid n$, then $n = 3$,*

(ii) *if $\mathfrak{z}_{2^k} \in F_v$, then $k \leqslant 3$,*

(iii) $m_v \mid 24$.

Proof. (i) We have g.c.d. $(n, Nv) = 1$, hence $n \mid Nv-1$. Since $Nv = 2^f$ and $f \leqslant 2$ we must have $n = 3$.

(ii) Consider the tower $\mathbf{Q}_2 \subseteq \mathbf{Q}_2(\mathfrak{z}_{2^k}) \subseteq F_v$. It follows that $e(\mathbf{Q}_2(\mathfrak{z}_{2^k})/\mathbf{Q}_2) \mid e(F_v/\mathbf{Q}_2)$. Since $e(\mathbf{Q}_2(\mathfrak{z}_{2^k})/\mathbf{Q}_2) = 2^{k-1}$ and $e(F_v/\mathbf{Q}_2) \leqslant 4$ we get $k \leqslant 3$.

(iii) This follows directly from (i) and (ii).

The number of generators of the group $K_2 O_F/\mathfrak{K}_2 F$ equals the total number of pairwise inequivalent real valuations, the valuations dividing 2 and those valuations v dividing 3 for which $\mathfrak{z}_3 \in F_v$. The relations depend on the values of Nv, m_v and m for these valuations. The theorems below show how all these numbers depend on $a_1, \ldots, a_n$. In order to state the results in a concise form we introduce the following notation:

r — the number of real valuations of the field F,

$g(2)$ — the number of pairwise inequivalent valuations dividing 2,

$g(3)$ — the number of pairwise inequivalent valuations dividing 3 and satisfying $\mathfrak{z}_3 \in F_v$.

THEOREM 2.6.

2.6.1. *If* $a_i < 0$ *for a certain* i, $1 \leqslant i \leqslant n$, *then* $r = 0$.

2.6.2. *If* $a_i > 0$ *for every* i, $1 \leqslant i \leqslant n$, *then* $r = 2^n$.

Proof. This is obvious.

THEOREM 2.7.

2.7.1. *If* $a_i \equiv 1 \pmod 8$ *for every* i, $1 \leqslant i \leqslant n$, *then* $g(2) = 2^n$ *and for every valuation* v *dividing* 2 *we have* $m_v = 2$ *and* $Nv = 2$.

2.7.2. *Suppose* $a_k \not\equiv 1 \pmod 8$ *for a certain* k, $1 \leqslant k \leqslant n$ *and either* $a_i \equiv 1 \pmod 8$ *for all* $i \neq k$ *or* $a_i \equiv a_k \pmod 8$ *for all* $i \neq k$ *and* $2 \nmid a_k$ *or* $a_i \equiv a_k \pmod{16}$ *for all* $i \neq k$ *and* $2 \mid a_k$. *Then* $g(2) = 2^{n-1}$ *and for every valuation* v *dividing* 2 *we have*:

a) $m_v = 2$, $Nv = 2$, *if* $a_k \equiv 3 \pmod 8$,

b) $m_v = 6$, $Nv = 4$, *if* $a_k \equiv 5 \pmod 8$,

c) $m_v = 4$, $Nv = 2$, *if* $a_k \equiv 7 \pmod 8$,

d) $m_v = 2$, $Nv = 2$, *if* $a_k \equiv 2 \pmod 4$.

2.7.3. *Suppose there are* k, l, $1 \leqslant k,l \leqslant n$ *with* $a_k \not\equiv 1 \not\equiv a_l \pmod 8$, $a_k \not\equiv a_l \pmod 8$. *Let* v *be any valuation dividing* 2.

a) *If* $a_i \not\equiv 1, 3, 5, 7 \pmod 8$ *for every* i, $1 \leqslant i \leqslant n$, *then* $m_v = 12$, $Nv = 4$.

b) *If* $a_i = 1, 7 \pmod 8$ *or* $a_i \equiv 2, 14 \pmod{16}$ *for every* i, $1 \leqslant i \leqslant n$, *then* $m_v = 8$, $Nv = 2$.

c) *If* $a_i \equiv 1, 7 \pmod 8$ *or* $a_i \equiv 6, 10 \pmod{16}$ *for every* i, $1 \leqslant i \leqslant n$, *then* $m_v = 4$, $Nv = 2$.

d) *If* $a_i \equiv 1, 3 \pmod 8$ *or* $a_i \equiv 2, 6 \pmod{16}$ *for every* i, $1 \leqslant i \leqslant n$, *then* $m_v = 2$, $Nv = 2$.

e) *If* $a_i \equiv 1, 3 \pmod 8$ *or* $a_i \equiv 10, 14 \pmod{16}$ *for every* i, $1 \leqslant i \leqslant n$, *then* $m_v = 2$, $Nv = 2$.

f) *If* $a_i \equiv 1, 5 \pmod 8$ *or* $a_i \equiv 2, 10 \pmod{16}$ *for every* i, $1 \leqslant i \leqslant n$, *then* $m_v = 6$, $Nv = 4$.

g) *If* $a_i \equiv 1, 5 \pmod 8$ *or* $a_i \equiv 6, 14 \pmod{16}$ *for every* i, $1 \leqslant i \leqslant n$, *then* $m_v = 6$, $Nv = 4$.

In all the cases a) *through* g) *one has* $g(2) = 2^{n-2}$.

2.7.4. *For all the remaining possible values of a_i's* mod 8 *we have* $g(2) = 2^{n-3}$ *and for any valuation* $v \mid 2$, $m_v = 24$ *and* $Nv = 4$.

Proof. 2.7.1. If $a_i \equiv 1 \pmod 8$ for every i, then $F_v = \mathbf{Q}_2(\sqrt{a_1}, \ldots, \sqrt{a_n}) = \mathbf{Q}_2$. Hence $e(F_v/\mathbf{Q}_2) = f(F_v/\mathbf{Q}_2) = 1$. Thus $g(2) = 2^n$.

2.7.2. If $a_i \equiv a_k \pmod 8$ and $2 \nmid a_k$ or $a_i \equiv a_k \pmod{16}$ and $2 \mid a_k$, then $a_i a_k$ is a square in $\mathbf{Q}_2$. It follows that under the assumptions of 2.7.2 F_v is a quadratic extension of $\mathbf{Q}_2$ and in the cases a), b), c) we find out that F_v is $\mathbf{Q}_2(\sqrt{3})$, $\mathbf{Q}_2(\sqrt{-3})$, $\mathbf{Q}_2(\sqrt{-1})$, respectively and in the case d) it is one of the fields $\mathbf{Q}_2(\sqrt{-2})$, $\mathbf{Q}_2(\sqrt{-6})$, $\mathbf{Q}_2(\sqrt{6})$, $\mathbf{Q}_2(\sqrt{2})$. Since $[F_v:\mathbf{Q}_2] = 2$, $g(2) = 2^{n-1}$. It is routine to determine the values of m_v and Nv for the given fields.

2.7.3. As in the case of 2.7.2, we conclude that now F_v contains a biquadratic extension of $\mathbf{Q}_2$. Consider the case a). Since $a_i \equiv 1, 3, 5, 7 \pmod 8$ we have $a_i \in \mathbf{Q}_2^{\cdot 2}$, $3a_i \in \mathbf{Q}_2^{\cdot 2}$, $-3a_i \in \mathbf{Q}_2^{\cdot 2}$, $-a_i \in \mathbf{Q}_2^{\cdot 2}$, respectively. Hence for every i, a_i is a square in the field $\mathbf{Q}_2(\sqrt{-1}, \sqrt{3})$. It follows that $F_v = \mathbf{Q}_2(\sqrt{-1}, \sqrt{3})$. Thus $[F_v:\mathbf{Q}_2] = 4$ and $g(2) = 2^{n-2}$. Moreover, we observe that $\mathfrak{z}_3 \in F_v$, $\mathfrak{z}_4 \in F_v$ and $\mathfrak{z}_8 \notin F_v$ (otherwise $\sqrt{2} \in \mathbf{Q}_2(\sqrt{-1}, \sqrt{3})$ which is impossible). It follows that $m_v = 12$. Since both the residue degree and ramification index are equal 2 we conclude $Nv = 4$. This finishes the proof of a). In the remaining cases we prove analogously that F_v is $\mathbf{Q}_2(\sqrt{-1}, \sqrt{2})$, $\mathbf{Q}_2(\sqrt{-1}, \sqrt{6})$, $\mathbf{Q}_2(\sqrt{2}, \sqrt{3})$, $\mathbf{Q}_2(\sqrt{-2}, \sqrt{3})$, $\mathbf{Q}_2(\sqrt{2}, \sqrt{-3})$, $\mathbf{Q}_2(\sqrt{-2}, \sqrt{-3})$ respectively. In all the cases $[F_v:\mathbf{Q}_2] = 4$, hence $g(2) = 2^{n-2}$. As in a) we determine m_v and Nv.

2.7.4. All the cases when F_v is a quadratic extension of $\mathbf{Q}_2$ have been discussed in 2.7.2 and when F_v is biquadratic extension of $\mathbf{Q}_2$ — in 2.7.3. Now F_v is a unique multiquadratic extension of $\mathbf{Q}_2$ of degree 8, that is $F_v = \mathbf{Q}_2(\sqrt{-1}, \sqrt{2}, \sqrt{3})$. Hence $g(2) = 2^{n-3}$. Since $\mathfrak{z}_3 \in F_v$ and $\mathfrak{z}_8 \in F_v$, we get $m_v = 24$. Further $e(F_v/\mathbf{Q}_2) = 4$, $f(F_v/\mathbf{Q}_2) = 2$ so that $Nv = 4$. This finishes the proof of the Theorem 2.7.

THEOREM 2.8.

2.8.1. *If there is a* k, $1 \leqslant k \leqslant n$, *with* $a_k \equiv 6 \pmod 9$ *and for every* $i \neq k$ *either* $a_i \equiv 1 \pmod 3$ *or* $a_i \equiv 6 \pmod 9$, *then* $g(3) = 2^{n-1}$ *and for any valuation* v *dividing* 3 *we have* $m_v = 6$ *and* $Nv = 3$.

2.8.2. *If there are* k, l, $1 \leqslant k, l \leqslant n$, *with* $a_k \equiv 2 \pmod 3$ *and* $a_l \equiv 3, 6 \pmod 9$ *or* $a_k \equiv 3 \pmod 9$ *and* $a_l \equiv 6 \pmod 9$, *then* $g(3) = 2^{n-2}$ *and* $m_v = 24$, $Nv = 9$ *for any valuation* $v \mid 3$.

2.8.3. *For all the remaining possible values of* a_i*'s* mod 3 *we have* $g(3) = 0$.

Proof. As in the proof of Theorem 2.7 it easy to establish that $F_v = \mathbf{Q}_3(\sqrt{-3})$ in the case 2.8.1 and $F_v = \mathbf{Q}_3(\sqrt{-1}, \sqrt{3})$ in the case 2.8.2 Lemma 2.3 implies that these are unique possible fields with $\mathfrak{z}_3 \in F_v$. Thus in all other cases $g(3) = 0$. Now if $F_v = \mathbf{Q}_3(\sqrt{-3})$, then $[F_v:\mathbf{Q}_3] = 2$ and so $g(3) = 2^{n-1}$. Moreover, $\mathfrak{z}_9 \notin F_v$, since $e(\mathbf{Q}_3(\mathfrak{z}_9)/\mathbf{Q}_3) = 6$ and $e(F_v/\mathbf{Q}_3) = 2$. Also $\mathfrak{z}_4 \notin F_v$.

Hence $m_v = 6$. If $F_v = \mathbf{Q}_3(\sqrt{-1}, \sqrt{3})$, then $e(F_v/\mathbf{Q}_3) = f(F_v/\mathbf{Q}_3) = 2$, hence $Nv = 9$. Further $[F_v:\mathbf{Q}_3] = 4$, hence $g(3) = 2^{n-2}$ and as above we prove $\mathfrak{z}_9 \notin F_v$. Hence $m_v = 24$.

In order to determine the relations between generators we need to know the order of the group of roots of unity in F denoted by m. Since $m \mid m_v$ for any valuation v of F, we note that $m \mid 24$. Moreover,

$\mathfrak{z}_3 \in F$ if and only if F contains $\mathbf{Q}(\sqrt{-3})$,

$\mathfrak{z}_4 \in F$ if and only if F contains $\mathbf{Q}(\sqrt{-1})$,

$\mathfrak{z}_8 \in F$ if and only if F contains $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{2})$,

$\mathfrak{z}_{12} \in F$ if and only if F contains $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{3})$,

$\mathfrak{z}_{24} \in F$ if and only if F contains $\mathbf{Q}(\sqrt{-1})$, $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$.

It follows that m can take only the following values: 2, 4, 6 (when $n \geqslant 1$), 8, 12 (when $n \geqslant 2$), and 24 (when $n \geqslant 3$).

The theorem below gives necessary and sufficient conditions for various roots of unity to belong to F.

THEOREM 2.9.

2.9.1. *$\mathfrak{z}_3 \in F$ if and only if there are distinct indices $i_1, \ldots, i_k$ such that $a_{i_1} \cdot \ldots \cdot a_{i_k} = -3c^2$ for an integer c.*

2.9.2. *$\mathfrak{z}_4 \in F$ if and only if there are distinct indices $i_1, \ldots, i_k$ such that $a_{i_1} \cdot \ldots \cdot a_{i_k} = -c^2$ for an integer c.*

2.9.3. *$\mathfrak{z}_8 \in F$ if and only if there are two sets of distinct indices $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$ such that $a_{i_1} \cdot \ldots \cdot a_{i_k} = -c_1^2$ and $a_{j_1} \cdot \ldots \cdot a_{j_l} = 2c_2^2$ for some integers c_1 and c_2.*

2.9.4. *$\mathfrak{z}_{12} \in F$ if and only if there are two sets of distinct indices $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$ such that $a_{i_1} \cdot \ldots \cdot a_{i_k} = -c_1^2$ and $a_{j_1} \cdot \ldots \cdot a_{j_l} = 3c_2^2$ for some integers c_1 and c_2.*

2.9.5. *$\mathfrak{z}_{24} \in F$ if and only if there are three sets of distinct indices $i_1, \ldots, i_k$, $j_1, \ldots, j_l$ and $k_1, \ldots, k_q$ such that $a_{i_1} \cdot \ldots \cdot a_{i_k} = -c_1^2$, $a_{j_1} \cdot \ldots \cdot a_{j_l} = 2c_2^2$, $a_{k_1} \cdot \ldots \cdot a_{k_q} = 3c_3^2$ for some integers c_1, c_2 and c_3.*

Proof. Everything follows from the remark preceding the statement of the theorem and from the lemma below.

LEMMA 2.10. *Suppose K is a subfield of $F = \mathbf{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})$, $[F:\mathbf{Q}] = 2^n$ and $[K:\mathbf{Q}] = 2$. Then there are distinct indices $i_1, \ldots, i_k$ such that $K = \mathbf{Q}(\sqrt{a_{i_1} \cdot \ldots \cdot a_{i_k}})$.*

Proof. We use elementary Galois theory. For distinct sets of indices $i_1, \ldots, i_k$, the quadratic fields $\mathbf{Q}(\sqrt{a_{i_1} \cdot \ldots \cdot a_{i_k}})$ are distinct and their total number is $2^n - 1$. On the other hand if $[K:\mathbf{Q}] = 2$ and $K \subset F$, then K is the fixed field of a subgroup of index two in $\mathrm{Gal}(F/\mathbf{Q}) = (\mathbf{Z}/2\mathbf{Z})^n$. But subgroups of index 2 in $(\mathbf{Z}/2\mathbf{Z})^n$ can be viewed as hyperplanes in n-dimensional vector space $(\mathbf{Z}/2\mathbf{Z})^n$ over the field $\mathbf{Z}/2\mathbf{Z}$. These hyperplanes are in one-to-one corespondence with homogenous linear equations in n indeterminats over $\mathbf{Z}/2\mathbf{Z}$. Since there are $2^n - 1$ such equations, this is the number of hyperplanes and the number of quadratic extensions of $\mathbf{Q}$ contained in F. But we have specified $2^n - 1$ distinct quadratic extensions contained in F at the beginning of the proof. It follows these are all quadratic extensions in F and the lemma is proved.

We proceed to determine the group $K_2 O_F / \mathfrak{R}_2 F$. Recall that $F = \mathbf{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})$, where $a_1, \ldots, a_n$ are square-free integers and $[F:\mathbf{Q}] = 2^n$. The r generators of the group coming real valuations of F will be denoted $z_1, \ldots, z_r$, the $g(2)$ generators coming from discrete valuation dividing 2 will be written $g_1, \ldots, g_{g(2)}$ and the $g(3)$ generators coming from discrete valuations dividing 3 will be written $h_1, \ldots, h_{g(3)}$.

The classification of possible groups $K_2 O_F / \mathfrak{R}_2 F$ given below is divided into 6 parts depending on the value of m, the order of the group of roots of unity in F. Each part is seperated into several cases depending on the residues of a_i's considered in Theorem 2.6 through 2.9. We use the notation of cases introduced in the theorems.

Part I. $m = 2$. The table below gives the number of generators, the relations and the structure of the group $K_2 O_F / \mathfrak{R}_2 F$ in the case when $g(3) = 0$, that is, in the case 2.8.3. The other two possibilities 2.8.1 and 2.8.2 are discussed below the table.

TABLE 1

Case	$g(2)$	Relations	Structure of $K_2 O_F/\mathfrak{R}_2 F$
		2.6.1 ($r = 0$)	
2.7.1	2^n	$g_i^2 = 1,\ \prod g_i = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^n - 1}$
2.7.2a, 2.7.2d	2^{n-1}	$g_i^2 = 1,\ \prod g_i = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-1} - 1}$
2.7.2b	2^{n-1}	$g_i^2 = 1,\ \prod g_i^3 = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-1} - 1}$
2.7.2c	2^{n-1}	$g_i^4 = 1,\ \prod g_i^2 = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-1} - 1} \oplus (\mathbf{Z}/2\mathbf{Z})$
2.7.3a	2^{n-2}	$g_i^4 = 1,\ \prod g_i^6 = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2} - 1} \oplus (\mathbf{Z}/2\mathbf{Z})$
2.7.3b	2^{n-2}	$g_i^8 = 1,\ \prod g_i^4 = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-2} - 1} \oplus (\mathbf{Z}/4\mathbf{Z})$
2.7.3c	2^{n-2}	$g_i^4 = 1,\ \prod g_i^2 = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2} - 1} \oplus (\mathbf{Z}/2\mathbf{Z})$
2.7.3d, 2.7.3e	2^{n-2}	$g_i^2 = 1,\ \prod g_i = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-2} - 1}$
2.7.3f, 2.7.3g	2^{n-2}	$g_i^2 = 1,\ \prod g_i^3 = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-2} - 1}$
2.7.4	2^{n-3}	$g_i^8 = 1,\ \prod g_i^{12} = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-3} - 1} \oplus (\mathbf{Z}/4\mathbf{Z})$
		2.6.2 ($r = 2^n$)	
2.7.1	2^n	$g_i^2 = z_j^2 = 1,\ \prod g_i \prod z_j = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n+1} - 1}$
2.7.2a, 2.7.2d	2^{n-1}	$g_i^2 = z_j^2 = 1,\ \prod g_i \prod z_j = 1$	$(\mathbf{Z}/2\mathbf{Z})^{3 \cdot 2^{n-1} - 1}$
2.7.2b	2^{n-1}	$g_i^2 = z_j^2 = 1,\ \prod g_i^3 \prod z_j = 1$	$(\mathbf{Z}/2\mathbf{Z})^{3 \cdot 2^{n-1} - 1}$
2.7.2c	2^{n-1}	$g_i^4 = z_j^2 = 1,\ \prod g_i^2 \prod z_j = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-1}} \oplus (\mathbf{Z}/2\mathbf{Z})^{2^n - 1}$
2.7.3a	2^{n-2}	$g_i^4 = z_j^2 = 1,\ \prod g_i^6 \prod z_j = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2}} \oplus (\mathbf{Z}/2\mathbf{Z})^{2^n - 1}$
2.7.3b	2^{n-2}	$g_i^8 = z_j^2 = 1,\ \prod g_i^4 \prod z_j = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-2}} \oplus (\mathbf{Z}/2\mathbf{Z})^{2^n - 1}$
2.7.3c	2^{n-2}	$g_i^4 = z_j^2 = 1,\ \prod g_i^2 \prod z_j = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2}} \oplus (\mathbf{Z}/2\mathbf{Z})^{2^n - 1}$
2.7.3d, 2.7.3e	2^{n-2}	$g_i^2 = z_j^2 = 1,\ \prod g_i \prod z_j = 1$	$(\mathbf{Z}/2\mathbf{Z})^{5 \cdot 2^{n-2} - 1}$
2.7.3f, 2.7.3g	2^{n-2}	$g_i^2 = z_j^2 = 1,\ \prod g_i^3 \prod z_j = 1$	$(\mathbf{Z}/2\mathbf{Z})^{5 \cdot 2^{n-2} - 1}$
2.7.4	2^{n-3}	$g_i^8 = z_j^2 = 1,\ \prod g_i^{12} \prod z_j = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-3}} \oplus (\mathbf{Z}/2\mathbf{Z})^{2^n - 1}$

When $m = 2$ and 2.8.1 holds, in every case considered above we have $g(3) = 2^{n-1}$ and the additional generators satisfy $h_i^3 = 1$. These generators do not appear in relation (3) stated in Introduction. Hence the group $K_2O_F/\mathfrak{R}_2F$ acquires additional direct summand $(\mathbf{Z}/3\mathbf{Z})^{2^{n-1}}$ in every case. When $m = 2$ and 2.8.2 holds, in every case considered in the table $g(3) = 2^{n-2}$ and the group in the table should be enlarged by a direct summand $(\mathbf{Z}/3\mathbf{Z})^{2^{n-2}}$.

Part II. $m = 4$. Of the two cases of Theorem 2.6 only 2.6.1 can hold. Moreover, since $m \mid m_v$, we conclude that only 2.7.2c, 2.7.3b, 2.7.3a, 2.7.3c, 2.7.4 and 2.8.2, or 2.8.3 can happen. The table below describes the situation under 2.8.3. The other case 2.8.2 is discussed below the table 1.

TABLE 2

Case	r	$g(2)$	$g(3)$	Relations	Structure of $K_2O_F/\mathfrak{R}_2F$
2.7.2c	0	2^{n-1}	0	$g_i^4 = 1,\ \prod g_i = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-1}-1}$
2.7.3a	0	2^{n-2}	0	$g_i^4 = 1,\ \prod g_i^3 = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2}-1}$
2.7.3b	0	2^{n-2}	0	$g_i^8 = 1,\ \prod g_i^2 = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-2}-1} \oplus (\mathbf{Z}/2\mathbf{Z})$
2.7.3c	0	2^{n-2}	0	$g_i^4 = 1,\ \prod g_i = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2}-1}$
2.7.4	0	2^{n-3}	0	$g_i^8 = 1,\ \prod g_i^{12} = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-3}-1} \oplus (\mathbf{Z}/4\mathbf{Z})$

When $m = 4$ and 2.8.2 holds, in every case considered in the table $g(3) = 2^{n-2}$ and $h_i^3 = 1$. Since $\mathfrak{z}_3 \notin F$, the additional generators do not appear in the relation (3) stated in the Introduction. In every case the group in the table should be enlarged by adding the direct summand $(\mathbf{Z}/3\mathbf{Z})^{2^{n-2}}$.

Part III. $m = 6$. Now 2.6.1 holds and only 2.7.2a, 2.7.3a, 2.7.3f, 2.7.3g, 2.7.4, and 2.8.1 or 2.8.2 can happen. Since 2.6.1 holds, in every case considered in the table 3 below we have $r = 0$.

TABLE 3

Case	$g(2)$	$g(3)$	Relations	Structure of $K_2O_F/\mathfrak{R}_2F$
		2.8.1		
2.7.2a	2^{n-1}	2^{n-1}	$g_i^2 = h_j^3 = 1,\ \prod g_i \prod h_j = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-1}-1} \oplus (\mathbf{Z}/3\mathbf{Z})^{2^{n-1}-1}$
2.7.3a	2^{n-2}	2^{n-1}	$g_i^4 = h_j^3 = 1,\ \prod g_i^2 \prod h_j = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2}-1} \oplus (\mathbf{Z}/2\mathbf{Z}) \oplus$ $\oplus (\mathbf{Z}/3\mathbf{Z})^{2^{n-1}-1}$
2.7.3f, 2.7.3g	2^{n-2}	2^{n-1}	$g_i^2 = h_j^3 = 1,\ \prod g_i \prod h_j = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-2}-1} \oplus (\mathbf{Z}/3\mathbf{Z})^{2^{n-1}-1}$
2.7.4	2^{n-3}	2^{n-1}	$g_i^8 = h_j^3 = 1,\ \prod g_i^4 \prod h_j = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-3}-1} \oplus (\mathbf{Z}/4\mathbf{Z}) \oplus$ $\oplus (\mathbf{Z}3\mathbf{Z})^{2^{n-1}-1}$
		2.8.2		
2.7.2a	2^{n-1}	2^{n-2}	$g_i^2 = h_j^3 = 1,\ \prod g_i \prod h_j^4 = 11$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-1}-1} \oplus (\mathbf{Z}/3\mathbf{Z})^{2^{n-2}-1}$
2.7.3a	2^{n-2}	2^{n-2}	$g_i^4 = h_j^3 = 1,\ \prod g_i^2 \prod h_j^4 = 1$	$(\mathbf{Z}/4\mathbf{Z})^{2^{n-2}-1} \oplus (\mathbf{Z}/2\mathbf{Z}) \oplus$ $\oplus (\mathbf{Z}/3\mathbf{Z})^{2^{n-2}-1}$
2.7.3f, 2.7.3g	2^{n-2}	2^{n-2}	$g_i^2 = h_j^3 = 1,\ \prod g_i \prod h_j^4 = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-2}-1} \oplus (\mathbf{Z}/3\mathbf{Z})^{2^{n-2}-1}$
2.7.4	2^{n-3}	2^{n-1}	$g_i^8 = h_j^3 = 1,\ \prod g_i^4 \prod h_j^4 = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-3}-1} \oplus (\mathbf{Z}/4\mathbf{Z}) \oplus$ $\oplus (\mathbf{Z}/3\mathbf{Z})^{2^{n-2}-1}$

Part IV. $m = 8$. Now 2.6.1 holds and only 2.7.3b, 2.7.4 and 2.8.2 or 2.8.3 can happen. The table shows the situation under 2.8.3.

TABLE 4

Case	r	$g(2)$	$g(3)$	Relations	Structure of $K_2O_F/\mathfrak{R}_2F$
2.7.3b	0	2^{n-2}	0	$g_i^8 = 1,\ \prod g_i = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-2}-1}$
2.7.4	0	2^{n-3}	0	$g_i^8 = 1,\ \prod g_i^3 = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-3}-1}$

When $m = 8$ and 2.8.2 holds, $g(3) = 2^{n-2}$ with $h_i^3 = 1$ and (3) in the Introduction becomes $\prod h_i^3 \cdot \prod g_j^{m_v/m} = 1$. In every case the group in the table should get the additional direct summand $(\mathbf{Z}/3\mathbf{Z})^{2^{n-2}}$.

Part V. $m = 12$. Here 2.6.1 and 2.8.2 hold and only 2.7.3a or 2.7.4 can happen (table 5).

TABLE 5

Case	r	$g(2)$	$g(3)$	Relations	Structure of $K_2O_F/\mathfrak{R}_2F$
2.7.3a	0	2^{n-2}	2^{n-2}	$g_i^4 = h_j^3 = 1,\ \prod g_i \prod h_j^2 = 1$	$(\mathbf{Z}/2\mathbf{Z})^{2^{n-2}-1}\oplus(\mathbf{Z}/3\mathbf{Z})^{2^{n-2}-1}$
2.7.4	0	2^{n-3}	2^{n-2}	$g_i^8 = h_j^3 = 1,\ \prod g_i^2 \prod h_j^2 = 1$	$(\mathbf{Z}/8\mathbf{Z})^{2^{n-3}-1}\oplus(\mathbf{Z}/2\mathbf{Z})\oplus$ $\oplus(\mathbf{Z}/3\mathbf{Z})^{2^{n-2}-1}$

Part VI. $m = 24$. Here 2.6.1, 2.7.4 and 2.8.2 hold. Thus $g(2) = 2^{n-3}$, $g_i^8 = 1$, $g(3) = 2^{n-2}$, $h_j^3 = 1$ and $\prod g_i \cdot \prod h_j = 1$. Hence $K_2O_F/\mathfrak{R}_2F = (\mathbf{Z}/8\mathbf{Z})^{2^{n-3}-1}\oplus(\mathbf{Z}/3\mathbf{Z})^{2^{n-2}-1}$.

3. Number fields with trivial group $K_2O_F/\mathfrak{R}_2F$. In this section we prove the following characterization of fields with trivial group $K_2O_F/\mathfrak{R}_2F$.

THEOREM 3.1. *Let F be normal extension of* $\mathbf{Q}$ *of degree n. The group $K_2O_F/\mathfrak{R}_2F$ is trivial if and only if the following three conditions hold.*

(i) *F is pure imaginary field* (*i.e.* $r = 0$).

(ii) *For every prime number p, if $\mathfrak{z}_p \in F$, then there is exactly one prime ideal in F dividing p.*

(iii) *For every prime number p and every positive integer k such that $\phi(p^k)|n$, if $\mathfrak{z}_{p^k} \notin F$, then $\mathfrak{z}_{p^k} \notin F_v$, for all valuations v dividing p.*

Proof. The sufficiency of (i), (ii) and (iii) follows from the following more general result (put $r = 0$ to obtain $K_2O_F/\mathfrak{R}_2F = 0$).

LEMMA 3.2. *Let F be an extension of* $\mathbf{Q}$ *with r real conjugates. If F satisfies* (ii) *and* (iii), *then* $K_2O_F/\mathfrak{R}_2F = (\mathbf{Z}/2\mathbf{Z})^r$.

Proof of 3.2. Let $p_1 = 2, p_2, \ldots, p_l$ be all prime numbers with $\mathfrak{z}_{p_i^{k_i}} \in F$ for a certain $k_i \in \mathbf{N}$, $i = 1, \ldots, l$. We assume k_i is the largest possible value of the exponent. From (ii) and (iii) we conclude that each prime p_i supplies exactly one generator g_i, $i = 1, \ldots, l$. Further, (iii) implies that $p_i^{k_i}|m$ whenever $p_i^{k_i}|m_{v_i}$ for a valuation v_i dividing p_i. The group $K_2O_F/\mathfrak{R}_2F$ is generated by $g_1, \ldots, g_l$ and by $z_1, \ldots, z_r$ coming from real valuations of F with relations

(3.2.1) $$z_i^2 = 1, \quad i = 1, \ldots, r,$$

(3.2.2) $$g^{p_i^{k_i}} = 1, \quad i = 1, \dots, l,$$

(3.2.3) $$z_1 \cdot \ldots \cdot z_r \cdot g_1^{q_1} \cdot \ldots \cdot g_l^{q_l} = 1,$$

where $q_i = p_i^{k_i}(p_i^{f_i}-1)/m$. We take now any $p_i > 2$ and consider $n_i = m/p_i^{k_i}$. This is an even integer and for $j \neq i$ we have $p_j^{k_j} | n_i$, since $p_j^{k_j} | m$. Thus raising the relation (3.2.3) to the power n_i gives

$$\prod_j g_j^{n_i q_j} = 1.$$

Here $p_j^{k_j} | n_i$ for $j \neq i$, so in view of (3.2.2), this reduces to

$$g_i^{n_i q_i} = 1.$$

But g.c.d. $(n_i q_i, p_i^{k_i})$ = g.c.d. $(p_i^{f_i}-1, p_i^{k_i}) = 1$ and it follows from (3.2.2) that $g_i = 1$. Thus (3.2.3) becomes

$$z_1 \cdot \ldots \cdot z_r \cdot g_1^{q_1} = 1.$$

Now $n_1 = m/2^{k_1}$ is an odd integer and raising the last relation to the power n_1 we get

(3.2.4) $$z_1 \cdot \ldots \cdot z_r = g_1^{n_1 q_1}.$$

As above g.c.d. $(n_1 q_1, 2^{k_1})$ = g.c.d. $(2^{f_1}-1, 2^{k_1}) = 1$, so that there are two integers x and y such that $x(2^{f_1}-1) = 1 + y \cdot 2^{k_1}$. From (3.2.4) we obtain

$$z_1 \cdot \ldots \cdot z_r = g_1^{x(2^{f_1}-1)} = g_1^{1+y \cdot 2^{k_1}} = g_1.$$

Thus (3.2.3) reduces to $z_1 \cdot \ldots \cdot z_r = g_1$ and it follows that $K_2 O_F/\mathfrak{K}_2 F = (\mathbf{Z}/2\mathbf{Z})^r$. This proves the lemma.

Now we prove the necessity of (i), (ii) and (iii). So assume $K_2 O_F/\mathfrak{K}_2 F = 0$. To prove (iii) we assume there is a prime power p^k such that $\phi(p^k) | n$, $\mathfrak{z}_{p^k} \notin F$ and $\mathfrak{z}_{p^k} \in F_v$ for some valuation v dividing p. Then $p | (m_v/m)$. Let g be a generator coming from the prime p. We consider the group H generated by an element h and satisfying $h^{m_v/m} = 1$ and $h^{m_v \to (Nv-1)} = 1$. Since $p |$ g.c.d. $\left(\frac{m_v}{m}, \frac{m_v}{Nv-1}\right)$, the order of h must be divisible by p and so $H \neq 0$. Consider a mapping $f: K_2 O_F/\mathfrak{K}_2 F \to H$ such that $f(g) = h$ and $f(g_i) = 1$ for all the remaining generators g_i. Since the mapping preserves relations it is a group homomorphism and its image H is non-trivial. This contradicts the triviality of the pre-image. Thus we have proved (iii).

To prove (ii) let us assume that $\mathfrak{z}_p \in F$ for a prime p and $v_1 | p$ and $v_2 | p$, where v_1 and v_2 are distinct valuations. The normality of the extension $F/\mathbf{Q}$ implies that $m_{v_1} = m_{v_2} =: m_v$ and $Nv_1 = Nv_2 = Nv$. Let g_1 and g_2 be the generators corresponding to v_1 and v_2, respectively. We consider a group H generated by two elements h_1 and h_2 with relations $h_i^{m_v(Nv-1)} = 1, i = 1, 2$ and $(h_1 h_2)^{m_v/m} = 1$. It follows from (iii) that g.c.d. $\left(\frac{m_v}{Nv-1}, \frac{m_v}{m}\right) = 1$ which implies $h_1 h_2 = 1$. Hence H is a cyclic group of order $\frac{m_v}{(Nv-1)} \neq 1$ (since $\mathfrak{z}_p \in F_{v_i}$). As above we define a group homomorphism $f\colon K_2 O_F/\mathfrak{K}_2 F \to H$ satisfying $f(g_1) = h_1$, $f(g_2) = h_2$ and $f(g_i) = 1$ for the other generators contradicting the triviality of $K_2 O_F/\mathfrak{K}_2 F$. This proves (ii).

Finally, let us assume $r \geqslant 1$. Let g_0 be a generator coming from one of the real valuations and let g_1 be a generator coming from $v|2$. We consider a group H generated by two elements h_0 and h_1, with relations $h_0^2 = 1$, $h_1^{2k} = 1$, $h_0 h_1^{(2^f-1)/m'} = 1$, where f is the residue degree of the valuation v, k is the largest integer such that $\mathfrak{z}_{2^k} \in F$ and $m' = m/2^k$. Using the same arguments as in the final part of the proof of Lemma 3.2 we get $h_0 = h_1$. Thus H is cyclic of order 2. Now sending $g_0 \mapsto h_0$, $g_1 \mapsto h_1$, and $g_i \mapsto 1$ for the other generators we obtain a group homomorphism $K_2 O_F/\mathfrak{K}_2 F \to H$, a contradiction. This proves (i) and finishes the proof of Theorem 3.1.

COROLLARY 3.3. *Let* $F = \mathbf{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})$ *be a multiquadratic extension of* $\mathbf{Q}$ *of degree* 2^n. *Then for* $n > 3$ *the group* $K_2 O_F/\mathfrak{K}_2 F$ *is nontrivial.*

Proof. As observed in Proposition 2.2, for any valuation $v|2$ we have $[F_v:\mathbf{Q}_2] \leqslant 8$. The number of distinct prime ideals dividing 2 equals $2^n/[F_v:\mathbf{Q}_2]$ and so for $n > 3$ this quotient is greater than 1. Theorem 3.1 implies then the assertion.

COROLLARY 3.4. *The group* $K_2 O_F/\mathfrak{K}_2 F$ *is trivial for* $F = \mathbf{Q}(\mathfrak{z}_{2^k})$, $k \geqslant 2$.

Proof. Here (i) and (ii) of Theorem 3.1 are obviously satisfied. We prove (iii).

Suppose for a prime $p \neq 2$ and valuation v dividing p we have $\mathfrak{z}_p \in F_v$. Clearly $F_v = \mathbf{Q}_p(\mathfrak{z}_{2^k})$ and $e(F_v/\mathbf{Q}_p) = 1$. From the tower $\mathbf{Q}_p \subset \mathbf{Q}_p(\mathfrak{z}_p) \subset \mathbf{Q}_p(\mathfrak{z}_{2^k})$ we infer $e(\mathbf{Q}_p(\mathfrak{z}_p)/\mathbf{Q}_p) | e(F_v/\mathbf{Q}_p)$ and this contradictions $e(\mathbf{Q}_p(\mathfrak{z}_p)/\mathbf{Q}_p) = p-1$. For any valuation $v|2$ we have $F_v = \mathbf{Q}_2(\mathfrak{z}_{2^k})$. It is also easy to notice that if $\mathfrak{z}_{2^t} \in F_v$ for an integer t then $t \leqslant k$, and so $\mathfrak{z}_{2^t} \in F$. Thus Theorem 3.1 implies the triviality of $K_2 O_F/\mathfrak{K}_2 F$.

4. Concluding remarks. For a finite Abelian group A let $r_p(A)$ be the p-rank of the group, that is, the number of primary components in a decomposition of the group into direct sum of cyclic groups. J. Browkin [1] proves that for an arbitrary number field F

$$r_2(K_2 O_F) = r - 1 + g(2) + j(2),$$

where r is the number of real valuations, $g(p)$ is the number of distinct discrete valuations dividing p and $j(p) = r_p(\mathrm{Cl}F/\mathrm{Cl}_p F)$, $\mathrm{Cl}F$ being the ideal class group of F and $\mathrm{Cl}_p F$ its subgroup generated by the ideals dividing p. S. Chaładus [2] generalizes this result for any prime number p assuming F contains $\mathfrak{z}_p$. He proves

$$r_p(K_2 O_F) = r - 1 + g(p) + j(p).$$

This result and Theorem 3.1 in the case when the group $K_2 O_F/\mathfrak{K}_2 F$ is trivial show that if $\mathfrak{z}_p \in F$ and $F \supset \mathbf{Q}$ is Galois extension, then $r_p(K_2 O_F) = j(p)$. In particular $r_2(K_2 O_F) = j(2)$.

Let us also remark that the following inequality can be read of from the tables given in Section 2:

$$r_2(K_2 O_F/\mathfrak{K}_2 F) > r-1+g(2)$$

in the following cases: 2.7.2c, 2.7.3a, b, c, 2.7.4 for nonreal and $m = 2$, 2.7.3b, 2.7.4 for $m = 4$, 2.7.3a, 2.7.4 for $m = 6$, 2.7.4 for $m = 12$. Since $r_2(K_2 O_F) \geqslant r_2(K_2 O_F/\mathfrak{K}_2 F)$, we conclude $r_2(K_2 O_F) > r-1+g(2)$. Thus in all the cases considered $j(2) \geqslant 1$, and in particular, the class number of F is even. A specific example is $F = \mathbf{Q}(\sqrt{a_1}, \sqrt{a_2})$, where $a_1 < 0$ or $a_2 < 0$ and $a_1 \equiv 3 \pmod 8$, $a_2 \equiv 5 \pmod 8$, $a_2 \neq -3$. There are many other examples of this kind multiquadratic extensions with even class number.

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