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EXISTENCE AND UNIQUENESS OF CONTINUOUS SOLUTIONS OF NONLINEAR FUNCTIONAL EQUATIONS ARE GENERIC PROPERTIES

Abstract. Fundamental properties of equations of the form (1) are discussed from the Baire category point of view. After showing that they are generic in a suitable function space the density of the set of equations (1) having no solutions is studied. Results of the paper are "product versions" of these proved in [3].

1. Introduction. Here we study some sets of equations of the form

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where φ is an unknown function. We are interested in such fundamental properties of their continuous solutions as existence, uniqueness, continuous dependence and convergence of successive approximations to a solution. Similar problems for equations of various types have been studied by J. Myjak (e.g. [7]) and many other authors. The present paper refers strongly to results and methods presented in [3]. Assuming f being fixed the author proved there that for almost all (in the sense of the Baire category) elements h of a function space equation (1) has all properties mentioned above. Results of the present paper deal with a set of pairs (f, h) and are "product versions" of those given in [3].

In the whole paper we shall assume that (X, ϱ) is a metric space and $(Y, \|\ \|)$ is a finite-dimensional Banach space.

Given topological spaces $\mathscr{X}$ and $\mathscr{Y}$ denote by $\mathscr{C}(\mathscr{X}, \mathscr{Y})$ the space of all functions mapping $\mathscr{X}$ continuously into $\mathscr{Y}$. In the sequel we shall treat it as a topological space endowed with the compact-open topology (cf. [6, §44]).

Let us fix a point $\xi \in X$ and denote by $\mathscr{F}$ the set of all functions $f \in \mathscr{C}(X, X)$ satisfying the inequality

$$\varrho(f(x), \xi) \leqslant \gamma_f(\varrho(x, \xi)), \qquad x \in X,$$

where γ_f is an increasing and right-continuous real function defined on an interval I containing the origin, and $\gamma_f(t) < t$ for every $t \in I \setminus \{0\}$.

In some important cases the definition of $\mathscr{F}$ becomes more clear due to the following characterization given by K. Baron (cf. [1, Theorem 3.3]).

LEMMA 1. *Suppose that*

(2) *the set* $\{x \in X\colon \varrho(x, \xi) \leqslant \varrho(x_0, \xi)\}$ *is compact for every* $x_0 \in X$.

Then $\mathscr{F}$ *is the set of all functions* $f \in \mathscr{C}(X, X)$ *such that* $f(\xi) = \xi$ *and satisfying the inequality*

$$\varrho(f(x), \xi) < \varrho(x, \xi), \qquad x \in X \setminus \{\xi\}.$$

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The next lemma gives basic properties of elements of the space $\mathscr{F}$ (cf. [3, Remark 1]).

LEMMA 2. *If* $f \in \mathscr{F}$ *then the sequence* $(f^k: k \in \mathbf{N})$*) *converges to* ξ *uniformly on every compact subset of* X *and, in particular,* ξ *is the unique fixed point of* f.

Here, as in [3], we confine ourselves to the study of equation (1) assuming that $f \in \mathscr{F}$. Results of [4] show that equations of the form (1) with a function $f \in \mathscr{C}(X, X)$ may have no continuous solutions for almost all functions h. The behaviour of such a function f must be much more complicated than this of elements of the space $\mathscr{F}$ described in Lemma 2.

Fix a point $\eta \in Y$. We shall look for solutions of equation (1) in the class Φ of all mappings $\varphi \in \mathscr{C}(X, Y)$ satisfying the equality $\varphi(\xi) = \eta$. Because of this and the fact that ξ is a fixed point of f it is natural to confine ourselves only to the functions $h \in \mathscr{C}(X \times Y, Y)$ taking the value η at the point (ξ, η). The set of all such functions will be denoted by $\mathscr{H}$.

REMARKS. 1. *If* X *is a separable locally compact space then* $\mathscr{H}$ *is a topologically complete space (metrizable by the metric of the uniform convergence on all compact sets).*

2. *If* X *is a topologically complete space satisfying condition* (2) *then* $\mathscr{F}$, $\mathscr{H}$ *and* $\mathscr{F} \times \mathscr{H}$ *are topologically complete spaces (metrizable by the metric of the uniform convergence on all compact sets).*

To justify the above remarks we shall need the following simple fact.

LEMMA 3. *If* X *satisfies condition* (2) *then it is a separable locally compact space.*

Proof. If there exists an $x_0 \in X$ such that $\varrho(x, \xi) \leqslant \varrho(x_0, \xi)$ for every $x \in X$ then

$$X = \{x \in X: \varrho(x, \xi) \leqslant \varrho(x_0, \xi)\},$$

whence, in view of (2), X is a compact space.

If for any $x \in X$ there exists an $\bar{x} \in X$ such that $\varrho(\bar{x}, \xi) > \varrho(x, \xi)$ then we may choose a sequence $(x_n: n \in \mathbf{N})$ of points of X satisfying the conditions

$$\lim_{n \to \infty} \varrho(x_n, \xi) = \sup\{\varrho(x, \xi): x \in X\}$$

and

$$\varrho(x_n, \xi) < \varrho(x_{n+1}, \xi), \qquad n \in \mathbf{N}.$$

For every $n \in \mathbf{N}$ the set $C_n = \{x \in X: \varrho(x, \xi) \leqslant \varrho(x_n, \xi)\}$ is compact and

$$C_n \subset \{x \in X: \varrho(x, \xi) < \varrho(x_{n+1}, \xi)\} \subset \operatorname{Int} C_{n+1},$$

thus

$$X \subset \bigcup_{n=1}^{\infty} C_n \subset \bigcup_{n=1}^{\infty} \operatorname{Int} C_n \subset X,$$

and the assertion follows.

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*) For every positive integer k, f^k denotes the k-th iterate of f.

Proof of Remarks. By Lemma 3 each of the assumptions implies that the space $X \times Y$ is separable and locally compact. Thus, as follows from [6, §44, VII, Theorems 1 and 3], the space $\mathscr{C}(X \times Y, Y)$ is completely metrizable by the metric of the uniform convergence on all compact sets. Consequently, $\mathscr{H}$ is a topologically complete space as a closed subset of $\mathscr{C}(X \times Y, Y)$.

If X is a topologically complete space satisfying (2) then, in view of Lemma 3, it is separable and locally compact, so we infer that the space $\mathscr{C}(X, X)$ is completely metrizale by the metric of the uniform convergence on all compact sets. Let $(C_n\colon n \in \mathbf{N})$ be a sequence of compact neighbourhoods of ξ such that

$$X = \bigcup_{n=1}^{\infty} C_n \text{ and } C_n \subset \operatorname{Int} C_{n+1}, \qquad n \in \mathbf{N}, \tag{3}$$

(cf. [6, §41, X, Theorem 8]). In virtue of Lemma 1 and 2 we have

$$\mathscr{F} = \bigcap_{n=1}^{\infty} \{f \in \mathscr{C}(X, X)\colon \varrho(f(x), \xi) < \max\{1/n, \varrho(x, \xi)\},\ x \in C_n\},$$

so $\mathscr{F}$ is a G_δ subset of $\mathscr{C}(X, X)$ and, by Alexandrov Theorem (cf. [5, §33, VI]), is topologically complete.

2. Generic properties. The results of this section are "product versions" of results given in [3] (cf. [3, Lemmas 3 and 4, Theorem 1, Lemma 5, Theorem 3, and Corollary]).

Let us denote by $\mathscr{H}_0$ the subset of $\mathscr{H}$ consisting of all functions taking the value η in a neighbourhood of (ξ, η). For any $(f, h) \in \mathscr{F} \times \mathscr{H}$ define the mapping $T(f, h)\colon \Phi \to \Phi$ by

$$T(f, h)(\varphi)(x) = h(x, \varphi[f(x)]), \qquad x \in X.$$

In the sequel, if $C \subset X$ and φ_1, φ_2 map a subset of X containing C into Y then we shall write

$$d_C(\varphi_1, \varphi_2) = \sup\{\|\varphi_1(x) - \varphi_2(x)\|\colon x \in C\}.$$

LEMMA 4. *Let C be a compact neighbourhood of ξ such that $f(C) \subset C$ for any $f \in \mathscr{F}$. Then, for every $(f, h) \in \mathscr{F} \times \mathscr{H}_0$ and for every positive number ε, there exist open neighbourhoods $\mathscr{U}_C(f, h, \varepsilon) \subset \mathscr{F}$ and $\mathscr{V}_C(f, h, \varepsilon) \subset \mathscr{H}$ of f and h, respectively, such that for every $(f', h') \in \mathscr{U}_C(f, h, \varepsilon) \times \mathscr{V}_C(f, h, \varepsilon)$*

$$\bigwedge_{\varphi \in \Phi} \bigvee_{k_0 \in \mathbf{N}} \bigwedge_{k \geq k_0} d_C(T(f', h')^k(\varphi), T(f, h)^k(\varphi)) < \varepsilon.$$

Proof. Fix a pair $(f, h) \in \mathscr{F} \times \mathscr{H}_0$ and choose an open ball $V \subset C$ centered at ξ and a positive number a such that

$$h(x, y) = \eta, \qquad x \in V,\ \|y - \eta\| \leq a. \tag{4}$$

Denote by $\varphi_{f,h}$ the unique solution of equation (1) in the class Φ (cf. [3, Lemma 2]) and fix a number b in such a manner that

$$a + d_C(\varphi_{f,h}, \eta) < b. \tag{5}$$

Fix a number $\varepsilon \in (0, a)$ and choose an integer n such that (cf. Lemma 2)

$$f^n(C) \subset V. \tag{6}$$

Since the restriction of h to the set $C \times \{y \in Y: \|y-\eta\| \leqslant b\}$ is uniformly continuous, we may find numbers $\varepsilon_0, \ldots, \varepsilon_n$ such that

$$0 < \varepsilon_n < \ldots < \varepsilon_0 = \varepsilon \tag{7}$$

and, for every $i \in \{1, \ldots, n\}$,

(8) $(x_1, y_1), (x_2, y_2) \in C \times \{y \in Y: \|y-\eta\| \leqslant b\}, \varrho(x_1, x_2) < \varepsilon_i, \|y_1 - y_2\| < \varepsilon_i$ imply $\|h(x_1, y_1) - h(x_2, y_2)\| < \varepsilon_{i-1} - \varepsilon_i$.

Put

$$\mathscr{U}_C(f, h, \varepsilon) = \{f' \in \mathscr{F}: f'^n(C) \subset V, \varrho(f'^i(x), f^i(x)) < \varepsilon_{i+1}, x \in C, i = 1, \ldots, n-1\},$$

$$\mathscr{V}_C(f, h, \varepsilon) = \{h' \in \mathscr{H}: \|h'(x, y) - h(x, y)\| < \varepsilon_n, x \in C, \|y-\eta\| \leqslant b\}.$$

Clearly $f \in \mathscr{U}_C(f, h, \varepsilon)$ (cf. (6)) and $h \in \mathscr{V}_C(f, h, \varepsilon)$. We shall show that $\mathscr{U}_C(f, h, \varepsilon)$ is an open set in $\mathscr{F}$. The map $F: \mathscr{F} \to \mathscr{C}(X, X)$, given by $F(f) = f'|_C$, is continuous (cf. [6, §44, III, Theorem 1]), Observe also that

$$\mathscr{U}_C(f, h, \varepsilon) =$$
$$= F^{-1}(\{g \in \mathscr{C}(C, C): g^n(C) \subset V, \ \varrho(g^i(x), f^i(x)) < \varepsilon_{i+1}, x \in C, i = 1, \ldots, n-1\}).$$

Thus $\mathscr{U}_C(f, h, \varepsilon)$, as the counterimage of an open set*) by the continuous mapping F, is an open subset of $\mathscr{F}$. The openness of the set $\mathscr{V}_C(f, h, \varepsilon)$ may be verified in a similar way.

Now fix a pair $(f', h') \in \mathscr{U}_C(f, h, \varepsilon) \times \mathscr{V}_C(f, h, \varepsilon)$ and a $\varphi \in \Phi$ and find a positive integer m in such a manner that

$$\|\varphi(x) - \eta\| < \varepsilon_n, \quad x \in f^m(V) \cup f'^m(V). \tag{9}$$

In virtue of [3, Lemma 2], the sequence $(T(f, h)^k(\varphi): k \in \mathbf{N})$ converges to $\varphi_{f,h}$ uniformly on C, so by (5) we can additionally assume that

$$d_C(T(f, h)^k(\varphi), \varphi_{f,h}) < b - a - d_C(\varphi_{f,h}, \eta), \quad k \geqslant m. \tag{10}$$

Fix an $x \in V$ Since $f^{m-1}(x) \in f^{m-1}(V) \subset V$, we obtain, by (9), (7), and (4)

$$T(f, h)(\varphi)[f^{m-1}(x)] = h(f^{m-1}(x), \varphi[f^m(x)]) = \eta.$$

Similarly

$$T(f', h)(\varphi)[f'^{m-1}(x)] = h(f'^{m-1}(x), \varphi[f'^m(x)]) = \eta,$$

so, by inequalities (9), (5) and by the definition of $\mathscr{V}_C(f, h, \varepsilon)$,

$$\|T(f', h')(\varphi)[f'^{m-1}(x)] - \eta\| =$$
$$= \|h'(f'^{m-1}(x), \varphi[f'^m(x)]) - h(f'^{m-1}(x), \varphi[f'^m(x)])\| < \varepsilon_n.$$

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*) To see this use simply the metric of the uniform convergence in $\mathscr{C}(C, C)$ (cf. [6, §44, V, Theorem 2]).

By induction we get

$$T(f,h)^k(\varphi)[f^{m-k}(x)] = \eta,\ T(f',h)^k(\varphi)[f'^{m-k}(x)] = \eta$$

and

$$\|T(f',h')^k(\varphi)[f'^{m-k}(x)]-\eta\| < \varepsilon_n, \quad k\in\{1,\ldots,m\},$$

whence

$$T(f,h)^m(\varphi)(x) = \eta,\ T(f',h)^m(\varphi)(x) = \eta \text{ and } \|T(f',h')^m(\varphi)-\eta\| < \varepsilon_n.$$

Using induction once more we have

(11)
$$T(f,h)^k(\varphi)(x) = \eta,\ T(f',h)^k(\varphi)(x) = \eta$$
$$\text{and } \|T(f',h')^k(\varphi)(x)-\eta\| < \varepsilon_n, \quad x\in V,\ k\geqslant m.$$

Now we shall verify that, for every $i\in\{0,\ldots,n\}$,

(12)
$$\|T(f',h')^k(\varphi)[f'^{n-i}(x)]-T(f,h)^k(\varphi)[f^{n-i}(x)\| < \varepsilon_{n-i}$$
$$\text{and } \|T(f',h')^k(\varphi)[f'^{n-i}(x)]-\eta\| < b,\ \|T(f,h)^k(\varphi)[f^{n-i}(x)]-\eta\| < b,$$
$$x\in C,\ k\geqslant m+i.$$

For $i = 0$, inequalities (12) follow from (11) and (6). Assume (12) for an $i\in\{0,\ldots,n-1\}$. Since $f'^{n-(i+1)}(x)\in f'^{n-(i+1)}(C)\subset C$, we infer, by the definitions of $\mathscr{U}_C(f,h,\varepsilon)$ and $\mathscr{V}_C(f,h,\varepsilon)$ and from (12), (7) and (8), that for $k\geqslant m+(i+1)$ and $x\in C$

$$\|T(f',h')^k(\varphi)[f'^{n-(i+1)}(x)]-T(f,h)^k(\varphi)[f^{n-(i+1)}(x)]\| =$$
$$= \|h'(f'^{n-(i+1)}(x),\ T(f',h')^{k-1}(\varphi)[f'^{n-i}(x)])-$$
$$-h(f^{n-(i+1)}(x),\ T(f,h)^{k-1}(\varphi)[f^{n-i}(x)]\| \leqslant$$
$$\leqslant \|h'(f'^{n-(i+1)}(x),\ T(f',h')^{k-1}(\varphi)[f'^{n-i}(x)]-$$
$$-h(f'^{n-(i+1)}(x),\ T(f',h')^{k-1}(\varphi)\ [f'^{n-i}(x)])\| +$$
$$+\|h(f'^{n-(i+1)}(x),\ T(f',h')^{k-1}(\varphi)[f'^{n-i}(x)])-$$
$$-h(f^{n-(i+1)}(x),\ T(f,h)^{k-1}(\varphi)[f^{n-i}(x)])\| <$$
$$< \varepsilon_n+(\varepsilon_{n-(i+1)}-\varepsilon_{n-i}) < \varepsilon_{n-(i+1)},$$

whence, by (10) and (7), we have

$$\|T(f',h')^k(\varphi)[f'^{n-(i+1)}(x)]-\eta\| \leqslant$$
$$\leqslant \|T(f',h')^k(\varphi)[f'^{n-(i+1)}(x)]-T(f,h)^k(\varphi)[f^{n-(i+1)}(x)]\| +$$
$$+\|T(f,h)^k(\varphi)[f^{n-(i+1)}(x)]-\varphi_{f,h}(x)\|+\|\varphi_{f,h}(x)-\eta\| <$$
$$< \varepsilon_{n-(i+1)}+b-a < b,$$

and

$$\|T(f,h)^k(\varphi)[f^{n-(i+1)}(x)]-\eta\| \leqslant$$
$$\leqslant \|T(f,h)^k(\varphi)[f^{n-(i+1)}(x)]-\varphi_{f,h}(x)\|+\|\varphi_{f,h}(x)-\eta\| <$$
$$< \varepsilon_{n-(i+1)}+b-a < b,$$

i.e. induction yields (12) for every $i \in \{0, \dots, n\}$. Putting $i = n$ in (12) we get

$$\|T(f',h')^k(\varphi)(x)-T(f,h)^k(\varphi)(x)\| < \varepsilon, \qquad x \in C, k \geqslant m+n,$$

which completes the proof.

Repeating the proof of Lemma 4 of the paper [3] we get the following result (the method used in the proof of [3, Lemma 4] follows the pattern given by J. Myjak in [7, Theorem 1.2]).

LEMMA 5. *Let C be a compact neighbourhood of ξ such that $f(C) \subset C$ for any $f \in \mathscr{F}$. Then, for every element (f, h) of the set*

$$(13) \qquad \mathscr{R}(C) = \bigcap_{k=1}^{\infty} \bigcup_{(f',h') \in \mathscr{F} \times \mathscr{H}_0} \mathscr{U}_C(f',h',1/k) \times \mathscr{V}_C(f',h',1/k),$$

equation (1) *has exactly one solution $\varphi \in \Phi$ and for every $\varphi_0 \in \Phi$ the sequence $(T(f,h)^k(\varphi_0): k \in \mathbf{N})$ of successive approximations converges to φ uniformly on every compact subset of X.*

THEOREM 1. *Suppose that the point ξ has a compact neighbourhood in X. Then the set of all pairs $(f, h) \in \mathscr{F} \times \mathscr{H}$ such that equation* (1) *has exactly one solution $\varphi \in \Phi$ and for every $\varphi_0 \in \Phi$ the sequence $(T(f,h)^k(\varphi_0): k \in \mathbf{N})$ of successive approximations converges to φ uniformly on every compact subset of X is residual in $\mathscr{F} \times \mathscr{H}$.*

Proof. Since there exists a compact neighbourhood of ξ, we can find a compact ball C centered at ξ. Observe that $f(C) \subset C$ for any $f \in \mathscr{F}$. The set $\mathscr{R}(C)$ defined by (13) is a G_δ set. Moreover, since $\mathscr{H}_0$ is a dense subset of $\mathscr{H}$ (cf. [2, Theorem 1]) and $\mathscr{F} \times \mathscr{H}_0 \subset \mathscr{R}(C)$, it is also a dense set. Consequently, the set $\mathscr{R}(C)$ is residual and the theorem follows from Lemma 5.

We shall finish this section giving analogs of results of [3] concerning the problem of the continuous dependence of continuous solutions of equation (1). Their proofs will be omitted, because they can be directly reproduced following these of Lemma 5, Theorem 3 and Corollary from [3].

If $(f, h) \in \mathscr{F} \times \mathscr{H}$ and equation (1) has exactly one solution in the class Φ, then we shall denote it by $\varphi_{f,h}$. Existence of $\varphi_{f,h}$ in Lemma 6 follows from [3, Lemma 2].

LEMMA 6. *Let C be a compact neighbourhood of ξ such that $f(C) \subset C$ for any $f \in \mathscr{F}$. If $(f, h) \in \mathscr{F} \times \mathscr{H}_0$ and ε is a positive number, then*

$$(f',h') \in \mathscr{U}_C(f,h,\varepsilon) \times \mathscr{V}_C(f,h,\varepsilon) \text{ imply } d_C(\varphi_{f,h}, \varphi_{f',h'}) \leqslant \varepsilon.$$

Given a subset C of X denote by Φ_C the set of all restrictions of functions from Φ to the set C.

THEOREM 2. *Let C be a compact neighbourhood of ξ such that $f(C) \subset C$ for any $f \in \mathscr{F}$ and let $\mathscr{R}(C)$ be given by* (13). *Then the map $\Lambda_C: \mathscr{R}(C) \to \Phi_C$, given by*

$$\Lambda_C(f, h) = \varphi_{f,h}|_C,$$

is well defined and continuous in $\mathscr{R}(C)$ (which is a residual subset of $\mathscr{F} \times \mathscr{H}$).

COROLLARY. *Let $(C_n\colon n \in \mathbb{N})$ be a sequence of compact neighbourhoods of ξ satisfying* (3) *and such that $f(C_n) \subset C_n$ for every positive integer n and $f \in \mathscr{F}$. Then the map $\Lambda\colon \bigcap_{n=1}^{\infty} \mathscr{R}(C_n) \to \Phi$, given by*

$$\Lambda(f, h) = \varphi_{f,h},$$

is well defined and continuous in $\bigcap_{n=1}^{\infty} \mathscr{R}(C_n)$ (which is a residual subset of $\mathscr{F} \times \mathscr{H}$).

REMARK 3. If X is a compact space or a closed subset of a finite-dimensional Banach space, then it is enough to define C_n as the closed ball centered at ξ and with the radius n for every positive integer n.

3. A density problem. Now it is natural to raise the question: how the set of all pairs $(f, h) \in \mathscr{F} \times \mathscr{H}$, for which equation (1) has no solution in the class Φ, is scattered in the space $\mathscr{F} \times \mathscr{H}$? It turns out that in some interesting cases this set is dense (cf. Theorem 3 below). But, in general, it is not true. For example, if X is a discrete space then the set $\mathscr{F}$ consists of one element only, viz., the constant function taking the value ξ and, consequently, for every $(f, h) \in \mathscr{F} \times \mathscr{H}$ equation (1) has exactly one solution in the class Φ (namely, the function $h(\cdot, \eta)$).

LEMMA 7. *If X is a convex subset of a normed space, then the set of all functions $f \in \mathscr{F}$ such that the set $\{f^k(x_0)\colon k \in \mathbb{N}\}$ is infinite for an $x_0 \in X$ is dense in $\mathscr{F}$.*

Proof. Suppose that X is a subset of a space endowed with a norm $\|\ \|$. Fix a $\vartheta \in (0, 1)$. Because of convexity of X the formula

$$g(x) = \vartheta(x - \xi) + \xi, \qquad x \in X,$$

defines the function $g\colon X \to X$ and, since $\vartheta \in (0, 1)$, $g \in \mathscr{F}$. Define $\mathscr{F}_0$ as the set of all functions from $\mathscr{F}$ which coincide with g on a neighbourhood of ξ. We shall show that $\mathscr{F}_0$ is a dense subset of $\mathscr{F}$.

Fix a nonvoid open subset $\mathscr{U}$ of $\mathscr{F}$ and let $f \in \mathscr{U}$. There exist a positive integer n, compact subsets $C_1, \ldots, C_n$ and open subsets $U_1, \ldots, U_n$ of X such that

$$f \in \bigcap_{k=1}^{n} \{f' \in \mathscr{F}\colon f'(C_k) \subset U_k\} \subset \mathscr{U}.$$

Put

$$\varepsilon_k = \min\{\|u - v\|\colon u \in f(C_k), \quad v \in X \setminus U_k\}, \qquad k \in \{1, \ldots, n\},$$

and

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}.$$

Since $f(C_k)$ is compact, $X\setminus U_k$ is closed and $f(C_k)\cap(X\setminus U_k)=\emptyset$, ε_k is positive for every $k\in\{1,\ldots,n\}$, and so is ε. Put $C=\bigcup_{k=1}^{n}C_k$. Clearly

$$\{f'\in\mathscr{F}\colon \|f'(x)-f(x)\|<\varepsilon,\ x\in C\}\subset\bigcap_{k=1}^{n}\{f'\in\mathscr{F}\colon f'(C_k)\subset U_k\},$$

whence

$$f\in\{f'\in\mathscr{F}\colon \|f'(x)-f(x)\|<\varepsilon,\ x\in C\}\subset\mathscr{U}. \tag{14}$$

Since $f(\xi)=g(\xi)$, there exists an open neighbourhood U of ξ such that

$$\|g(x)-f(x)\|<\varepsilon,\qquad x\in U. \tag{15}$$

Let F be a closed neighbourhood of ξ contained in U. In view of Urysohn Lemma (cf. [5, §14, IV]) there exists a function $p\in\mathscr{C}(X,[0,1])$ such that

$$p(F)\subset\{0\}\ \text{ and }\ p(X\setminus U)\subset\{1\}. \tag{16}$$

We shall verify that the function $f'=pf+(1-p)g$ belongs to $\mathscr{F}_0\cap\mathscr{U}$. Indeed, since X is convex, f maps X into itself. Moreover, for any $x\in X$ we have

$$\begin{aligned}\|f'(x)-\xi\| &= \|p(x)(f(x)-\xi)+(1-p(x))(g(x)-\xi)\|\leqslant\\ &\leqslant p(x)\,\|f(x)-\xi\|+(1-p(x))\,\|g(x)-\xi\|\leqslant\\ &\leqslant p(x)\gamma_f(\|x-\xi\|)+(1-p(x))\gamma_g(\|x-\xi\|)\leqslant\\ &\leqslant \max\{\gamma_f(\|x-\xi\|),\gamma_g(\|x-\xi\|)\},\end{aligned}$$

whence $f'\in\mathscr{F}$. In view of (16), $f'|_F=g|_F$, so $f'\in\mathscr{F}_0$. Moreover, it follows from (15) and (16) that for every $x\in X$

$$\|f'(x)-f(x)\|=\|(1-p(x))(g(x)-f(x))\|=(1-p(x))\|g(x)-f(x)\|<\varepsilon.$$

Consequently, on account of (14), $f'\in\mathscr{U}$, which completes the proof of density of $\mathscr{F}_0$ in $\mathscr{F}$.

Let f be an element of $\mathscr{F}_0$ and choose a neighbourhood U of ξ such that

$$f(x)=\vartheta(x-\xi)+\xi,\qquad x\in U.$$

We can assume that $f(U)\subset U$. If $x_0\in U\setminus\{\xi\}$ then

$$f^k(x_0)=\vartheta^k(x_0-\xi)+\xi,\qquad k\in\mathbf{N},$$

so the set $\{f^k(x_0)\colon k\in\mathbf{N}\}$ is infinite.

Using this lemma we obtain, as an immediate consequence of [3, Theorem 2], the following result.

THEOREM 3.*) *Suppose that X is a convex subset of a normed space. Then the set of all pairs $(f,h)\in\mathscr{F}\times\mathscr{H}$ for which equation* (1) *has no solution in the class Φ is dense in $\mathscr{F}\times\mathscr{H}$.*

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*) Here $(Y,\|\ \|)$ may be an arbitrary nontrivial normed space.

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