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NOTE ON MULTIPLICATIVE FUNCTIONS

Abstract. The author shows how to fill a gap occurring in [3; § 13.1].

1. In [3] one finds a lemma (Lemma 13.1.4) dealing with the functional equation

$$f(xy) = f(x)f(y). \tag{1}$$

We quote here this lemma as Lemma 1 ($D_0 = D\setminus\{0\}$, $\mathbf{R}$ denotes the set of all real numbers).

LEMMA 1. *If D is one of the sets*

$$(0,1),\ [0,1),\ (-1,1),\ (-1,0)\cup(0,1),\ (1,\infty), \tag{2}$$

and if $f_0: D \to \mathbf{R}$, $f_0 \neq 0$, is a solution of (1), *then the function $f:[D \cup D_0^{-1} \cup \{1\}] \to \mathbf{R}$ given by*

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in D, \\ 1 & \text{if } x = 1, \\ [f_0(x^{-1})]^{-1} & \text{if } x \in D_0^{-1}, \end{cases} \tag{3}$$

satisfies equation (1) *in $D \cup D_0^{-1} \cup \{1\}$ and $f_0 = f \mid D$.*

Unfortunately this lemma is not quite correct, and at any case it does not serve its purpose. Lemma 1 was intended to yield the following result (Corollary 13.1.1 in [3]) which would allow one to replace considering equation (1) on a set D of a form listed in (2) by considering it on one of the sets

$$(0,\infty),\ [0,\infty),\ (-\infty,0)\cup(0,\infty),\ \mathbf{R}. \tag{4}$$

LEMMA 2. *If D is one of sets* (2), *and $f_0: D \to \mathbf{R}$ satisfies equation* (1), *then there exist a set G of form* (4) *and a function $f: G \to \mathbf{R}$ satisfying* (1) *such that $D \subset G$ and $f_0 = f \mid D$.*

The problem lies in the fact that if D is one of

$$(-1,1),\ (-1,0)\cup(0,1), \tag{5}$$

then the set $D \cup D_0^{-1} \cup \{1\}$ is not one of sets (4): it does not contain -1. It is not even closed under multiplication.

This situation can be easily mended. It is enough to define f at the point $x = -1$.

So let D be one of sets (5). For arbitrary $x, y \in D_0$ we have $-x, -y \in D_0$, and by (1)

$$f_0(-x)f_0(y) = f_0(-xy) = f_0(x)f_0(-y).$$

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It was shown in [3] that f_0 is never zero in D_0 (cf. also Lemma 4 below). Hence, for arbitrary $x, y \in D_0$,

$$\frac{f_0(-x)}{f_0(x)} = \frac{f_0(-y)}{f_0(y)}. \tag{6}$$

Relation (6) means that

$$\varepsilon = \frac{f_0(-x)}{f_0(x)}, \qquad x \in D_0, \tag{7}$$

is constant. Setting in (6) $y = -x$ we obtain

$$\varepsilon^2 = 1. \tag{8}$$

Relation (7) may be written as

$$f_0(-x) = \varepsilon f_0(x) \tag{9}$$

for $x \in D_0$. As was shown in [3] (Lemma 13.1.3; cf. also Lemma 3 below) $f_0(0)$ is either 0 or 1; in the latter case $f_0(x) = 1$ for all $x \in D$. Thus (9) is fulfilled also for $x = 0$. Consequently (9) holds in the whole of D.

Now we put

$$f(-1) = \varepsilon, \tag{10}$$

where ε is defined by (7). It remains to verify that f defined by (3) and (10) satisfies equation (1) in $D \cup D_0^{-1} \cup \{1\} \cup \{-1\}$. Beside the cases dealt with in [3] we need consider also the cases where one of x, y, xy is -1.

If $x = -1$, $y \in D$, then (1) results from (3), (9) and (10). If $x = -1$, $y = 1$, then we obtain (1) from (3) and (10). If $x = y = -1$, then (3), (10) and (8) imply (1). If $x = -1$, $y \in D_0^{-1}$, then also $-y \in D_0^{-1}$ and by (3), (9), (8) and (10)

$$f(xy) = f(-y) = [f_0(-y^{-1})]^{-1} = [\varepsilon f_0(y^{-1})]^{-1} = \\ = \varepsilon[f_0(y^{-1})]^{-1} = f(x)f(y).$$

If $x \in D$, $y \in D_0^{-1}$, $xy = -1$, then $y^{-1} = -x$ and

$$f(x)f(y) = f_0(x)[f_0(y^{-1})]^{-1} = f_0(x)[f_0(-x)]^{-1} = \\ = f_0(x)[\varepsilon f_0(x)]^{-1} = \varepsilon = f(xy)$$

in virtue of (3), (9), (8) and (10).

The cases where $y = -1$, or $x \in D_0^{-1}$, $y \in D$, $xy = -1$, follow from those already considered in view of the commutativity of the multiplication of real numbers.

This takes care of the inaccuracies in [3; § 13.1].

2. The consideration of the preceding section may be put into a more general setting.

In the sequel D denotes a multiplicative semigroup of real numbers, i.e. a set of real numbers such that $xy \in D$ whenever $x \in D$ and $y \in D$. We put $D_0 = D\setminus\{0\}$. G_0

denotes the group generated[1] by D_0 and we write $G = G_0 \cup D$. Thus $G = G_0 \cup \cup \{0\}$ if $0 \in D$, and $G = G_0$ if $0 \notin D$.

Sometimes D will be subjected to further conditions.

(i) For every $x \in D \cap (0, \infty)$, $x \neq 1$, there exists an $\alpha_x > 0$ such that if $x < 1$, then $yx^{-1} \in D$ for every $y \in D \cap (0, \alpha_x)$, and if $x > 1$, then $yx^{-1} \in D$ for every $y \in D \cap (\alpha_x, \infty)$.

(ii) If $1 \in D$ and $D_0 \backslash \{-1, 1\} \neq \emptyset$, then there exists an $x \in D_0 \backslash \{-1, 1\}$ such that $x^{-1} \in D$.

REMARK 1. Evidently, if D_0 is a multiplicative *group*, then D fulfils conditions (i) and (ii).

Let F be a group with zero. This means that F is a set endowed with an inner binary operation $\cdot$, and there exists an element $\Theta \in F$ such that $F_0 = F \backslash \{\Theta\}$ with the operation $\cdot$ is a group (not necessarily commutative) and $\Theta a = a\Theta = \Theta$ for every $a \in F$. The neutral element of the group F_0 will be denoted by e: $ea = ae = a$ for every $a \in F$.

We start with two lemmas. Such results were proved in [3] under less general conditions, and although the proofs in the present case do not differ essentially from those in the special case, we give them here for the sake of completeness.

LEMMA 3. *Let D be a multiplicative semigroup of real numbers such that $0 \in D$, and let F be a group with zero. If a function $f: D \to F$ satisfies equation* (1) *in D, then either $f(0) = \Theta$ or $f(0) = e$. In the latter case $f(x) = e$* for all $x \in D$.

Proof. Setting in (1) $x = y = 0$ we obtain $f(0) = f(0)^2$, whence either $f(0) = \Theta$, or $f(0) = e$. In the latter case we have by (1) for an arbitrary $x \in D$

$$f(x) = f(x)e = f(x)f(0) = f(0) = e.$$

COROLLARY. *Let $D \neq \{0\}$ be a multiplicative semigroup of real numbers and let F be a group with zero. If a function $f_0: D \to F$ satisfies equation* (1) *in D, then the conditions*

$$f_0(x) = \Theta \text{ for } x \in D_0 \tag{11}$$

and

$$f_0(x) = \Theta \text{ for } x \in D \tag{12}$$

are equivalent.

LEMMA 4. *Let D be a multiplicative semigroup of real numbers fulfilling conditions* (i) *and* (ii), *and let F be a group with zero. If a function $f: D \to F$ satisfies equation* (1) *in D and $f(u) = \Theta$ for a $u \in D_0$, then $f(x) = \Theta$ for all $x \in D$.*

Proof. By (1) we have $f(u^2) = f(u)^2 = \Theta$. Therefore we may assume that $u > 0$. We will distinguish three cases.

I. $u < 1$. Take an arbitrary $x \in D \cap (0, \alpha_u)$ (cf. (i)). Then

$$f(x) = f(xu^{-1}u) = f(xu^{-1})f(u) = f(xu^{-1})\Theta = \Theta.$$

[1] If $D_0 = \emptyset$ we assume that also $G_0 = \emptyset$.

Thus

(13) $$f(y) = \Theta \quad \text{for } y \in D \cap (0, \alpha_u).$$

For an arbitrary $x \in D \cap (0, 1)$ we may find a positive integer n such that $x^n \in D \cap (0, \alpha_u)$. We get from (1) $f(x^n) = f(x)^n$, whence by (13) $f(x)^n = \Theta$, which implies $f(x) = \Theta$. Consequently

(14) $$f(y) = \Theta \quad \text{for } y \in D \cap (0, 1).$$

Now suppose that $1 \in D$. Observe that $u \in D_0 \setminus \{-1, 1\}$ so that $D_0 \setminus \{-1, 1\} \neq \neq \emptyset$. By (ii) there exists a $t \in D_0 \setminus \{-1, 1\}$ such that $t^{-1} \in D$, and again we may assume that $t > 0$ (replacing, if necessary, t by t^2). Of the two numbers t and t^{-1} one belongs to $D \cap (0, 1)$. By (1) and (14)

(15) $$f(1) = f(tt^{-1}) = f(t)f(t^{-1}) = \Theta.$$

If $z \in D \cap (1, \infty)$, then there exists a positive integer m such that $v = = uz^m \in D \cap (1, \infty)$, whence $f(v) = f(u)f(z^m) = \Theta f(z^m) = \Theta$. Hence for $x \in D \cap \cap (\alpha_v, \infty)$

$$f(x) = f(xv^{-1}v) = f(xv^{-1})f(v) = f(xv^{-1})\Theta = \Theta.$$

Thus

(16) $$f(y) = \Theta \quad \text{for } y \in D \cap (\alpha_v, \infty).$$

For an arbitrary $x \in D \cap (1, \infty)$ we have $x^n \in D \cap (\alpha_v, \infty)$ for a sufficiently large positive integer n, whence by (1) and (16) $f(x)^n = f(x^n) = \Theta$. Consequently $f(x) = \Theta$ and

(17) $$f(y) = \Theta \quad \text{for } y \in D \cap (1, \infty).$$

By (14), (15) and (17)

(18) $$f(y) = \Theta \quad \text{for } y \in D \cap (0, \infty).$$

For an arbitrary $x \in D_0$ we have $x^2 \in D \cap (0, \infty)$ and by (1) and (18) $f(x)^2 = = f(x^2) = \Theta$, whence $f(x) = \Theta$ and consequently

$$f(y) = \Theta \quad \text{for } y \in D_0.$$

In virtue of the Corollary to Lemma 3 we infer hence that $f(x) = \Theta$ for all $x \in D$.

II. $u = 1$. Then we have for every $x \in D$

$$f(x) = f(ux) = f(u)f(x) = \Theta f(x) = \Theta.$$

III. $u > 1$. The proof is similar to that in case I.

THEOREM 1. *Let D be a multiplicative semigroup of real numbers fulfilling conditions* (i) *and* (ii), *and let F be a group with zero. If a function $f_0: D \to F, f_0 \not\equiv 0$, satisfies equation* (1) *in D, then there exists a unique function $f: G \to F$ satisfying equation* (1) *in G and such that $f|D = f_0$.*

Proof. Since $f_0 \not\equiv 0$, we have by Lemma 4

$$f_0(x) \neq \Theta \quad \text{for } x \in D_0. \tag{19}$$

As has been proved in [1] (cf. also [3; Corollary 18.2.1]) there exists a unique extension $f: G_0 \to F_0$ of $f_0 | D_0$ onto G_0 satisfying equation (1) in G_0. It remains to define f at 0 in the case where $0 \in D$.

So suppose that $0 \in D$. We put

$$f(0) = f_0(0). \tag{20}$$

Clearly this is the only possible choice if f is to be an extension of f_0, which (together with the uniqueness of $f | G_0$) proves the uniqueness of the extension. We must check that f satisfies equation (1) in G.

Take arbitrary $x, y \in G$. If $x, y \in G_0$, then (1) results from the properties of $f | G_0$. If one of x, y is zero and $f(0) = \Theta$, then

$$f(x) f(y) = \Theta = f(0) = f(xy).$$

If one of x, y is zero and $f(0) \neq \Theta$, then by (20) and Lemma 3 $f(x) = f_0(x) = e$ for $x \in D$, and by the uniqueness of the extension $f(x) = e$ for $x \in G$. Thus (1) is fulfilled in this case, too.

REMARK 2. In the proof of Theorem 1 conditions (i) and (ii) were used only to derive relation (19). If we assume that f_0 fulfils (19), then conditions (i) and (ii) may be removed from the hypotheses of Theorem 1.

THEOREM 2. *Let D be an arbitrary multiplicative semigroup of real numbers, let F be a group with zero, and let the function $f_0: D \to F$ satisfy equation* (1) *in D. The function f_0 may be extended onto G to a function $f: G \to F$ satisfying equation* (1) *in G if and only if it fulfils either* (11) *or* (19). *When it exists, the extension is unique.*

Proof. If f_0 fulfils (19), then the existence and the uniqueness of the extension f results from Theorem 1 and Remark 2.

If f_0 fulfils (11), then according to the Corollary[2] to Lemma 3 it fulfils also (12). The function

$$f(x) = \Theta \quad \text{for } x \in G \tag{21}$$

is an extension of f_0 satisfying equation (1) in G. Suppose that $\tilde{f}$ is another such extension. In view of Remark 1 we may use Lemma 4 with D replaced by G and f by $\tilde{f}$. By (11) we have for any $u \in D_0 \subset G_0$, since $\tilde{f} | D = f_0$, $\tilde{f}(u) = \Theta$. By Lemma 4

$$\tilde{f}(x) = \Theta \quad \text{for } x \in G. \tag{22}$$

Consequently $\tilde{f} = f$ which means that function (21) is the unique extension of f_0 satisfying equation (1) in G.

Conversely, suppose that f_0 admits an extension onto G to a solution $f: G \to F$ of equation (1). If $f(x) \not\equiv 0$ in G_0, then $f_0 = f | D$ fulfils (19). And if there exists

[2] If $D \neq \{0\}$. If $D = \{0\}$ (i.e. $D_0 = \varnothing$), then the Theorem is trivial and there is nothing to prove.

a $u \in G_0$ such that $f(u) = \Theta$, then by Lemma 4 (compare Remark 1) $f(x) = \Theta$ in G_0 and so f_0 fulfils (11).

3. Let us observe that Lemma 2 follows easily from Theorem 2 and Lemma 4, whereas Theorem 1 replaces Lemma 1. On the other hand the results of §**2** may be applied to many other multiplicative semigroups of real numbers beside (2). (E.g. D may be $(0, s)$ or $(0, s]$, $s < 1$, or (s, ∞) or $[s, \infty)$, $s > 1$, or any of these sets supplemented by 0 and/or its symmetric reflection on the negative axis, and still many other semigroups.)

However, D cannot be quite arbitrary. We are going to show by suitable examples that conditions (i) and (ii) in Lemma 4 and Theorem 1 are essential. In both examples below we put $F = \mathbf{R}$.

EXAMPLE 1. Let $D = \mathbf{N}$ be the set of positive integers. D does not fulfil condition (i). (It does not fulfil condition (ii) either, but this can be avoided if we replace $\mathbf{N}$ by $\mathbf{N} \setminus \{1\}$). In this case G is the group of all positive rationals. The function

$$f_0(x) = \begin{cases} 0 \text{ for even } x, \\ 1 \text{ for odd } x, \end{cases}$$

satisfies equation (1) in D (cf. also [2; p. 195]). Obviously Lemma 4 is not valid in this case. Also it follows from Theorem 2 that f_0 cannot be extended onto G to a function $f: G \to \mathbf{R}$ satisfying equation (1) in G.

EXAMPLE 2. Let $D = (0, 1]$. In this case D fulfils condition (i) ($\alpha_x = x$), but does not fulfil (ii). Now $G = (0, \infty)$. The function

$$f_0(x) = \begin{cases} 0 \text{ for } x \in (0, 1), \\ 1 \text{ for } x = 1, \end{cases}$$

satisfies equation (1) in D. Obviously also in this case Lemma 4 is not valid and f_0 cannot be extended onto G to a function $f: G \to \mathbf{R}$ satisfying equation (1) in G.

Of course, what has been said does not mean that (i) and (ii) cannot be replaced by other conditions. One such possibility is mentioned in Remark 2. But if we postulate (19), then Theorem 1 becomes an almost trivial consequence of the result in [1].

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