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ON SOME PROPERTIES OF QUADRATIC STOCHASTIC PROCESSES

Abstract. In this paper we prove that every measurable quadratic stochastic process $X:\mathbf{R}^N \times \Omega \to \mathbf{R}$ is continuous and has the form

$$X(x,\cdot) = \sum_{i,j=1}^{N} x_i x_j Y_{i,j}(\cdot) \quad \text{(a.e.)},$$

where $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$ and $Y_{i,j}:\Omega \to \mathbf{R}$ are random variables. Moreover, we give a proof of the stability of the quadratic stochastic processes.

The subject of the present paper is to exhibit some properties of quadratic stochastic processes. Theorems 1, 5, 6 and 7 give some conditions for a quadratic process to be continuous. Similar theorems for convex functions were proved, among others, by Bernstein and Doetsch [1], Ostrowski [10] and Sierpiński [11] and for quadratic functionals by Kurepa [5]. In the case of additive stochastic processes such theorems were proved by Nagy [7]. Theorem 8 concerns the stability of quadratic stochastic processes and it yields an analogue of the theorem of Hyers [4] for additive functions.

Let $(\Omega, \mathscr{A}, P)$ be an arbitrary probability space. A function $X:\mathbf{R}^N \times \Omega \to \mathbf{R}$ ($\mathbf{R}$ denotes here the set of all real numbers) is called a *stochastic process* iff for all $x \in \mathbf{R}^N$ the function $X(x,\cdot):\Omega \to \mathbf{R}$ is a *random variable*, i.e. it is an $\mathscr{A}$-measurable function. A stochastic process $X:\mathbf{R}^N \times \Omega \to \mathbf{R}$ is called

— *quadratic* iff for all $x, y \in \mathbf{R}^N$

$$X(x+y,\cdot)+X(x-y,\cdot) = 2X(x,\cdot)+2X(y,\cdot) \quad \text{(a.e.)}; \tag{1}$$

— *P-bounded on a non-empty set* $A \subset \mathbf{R}^N$ iff

$$\lim_{n\to\infty} \sup_{x\in A} \{P(\{\omega \in \Omega: |X(x,\omega)| \geqslant n\})\} = 0;$$

— *continuous at a point* $x_0 \in \mathbf{R}^N$ iff

$$P\text{-}\lim_{x\to x_0} X(x,\cdot) = X(x_0,\cdot),$$

where P-lim denotes the limit in probability.

In a similar way as in the case of quadratic functionals (cf. e.g. [5]) one can prove the following

LEMMA 1. *If a stochastic process* $X:\mathbf{R}^N \times \Omega \to \mathbf{R}$ *is quadratic, then* $X(qx,\cdot) = q^2 X(x,\cdot)$ *(a.e.) for all rational* q *and* $x \in \mathbf{R}^N$.

Received March 15, 1982.

AMS (MOS) Subject classification (1980). Primary 39B70. Secondary 60G07.

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LEMMA 2. *If a stochastic process $X:\mathbf{R}^N\times\Omega\to\mathbf{R}$ is quadratic, then for all $x, y, z\in\mathbf{R}^N$*

$$X(x+y+z,\cdot)+X(x,\cdot)+X(y,\cdot)+X(z,\cdot)=$$
$$=X(x+y,\cdot)+X(y+z,\cdot)+X(z+x,\cdot)\quad (a.e.).$$

Proof. Let $x, y, z\in\mathbf{R}^N$. Using equation (1) three times (for suitable variables) we obtain

$$X(x+y,\cdot)+X(y+z,\cdot)+X(z+x,\cdot)=$$
$$=\frac{1}{2}[X(x+2y+z,\cdot)+X(x-z,\cdot)]+X(z+x,\cdot)=$$
$$=\frac{1}{2}[2X(x+y+z,\cdot)+2X(y,\cdot)-X(x+z,\cdot)+X(x-z,\cdot)]+X(z+x,\cdot)=$$
$$=X(x+y+z,\cdot)+X(y,\cdot)+\frac{1}{2}X(x+z,\cdot)+\frac{1}{2}X(x-z,\cdot)=$$
$$=X(x+y+z,\cdot)+X(y,\cdot)+X(x,\cdot)+X(z,\cdot)\quad \text{(a.e.)},$$

which was to be proved.

LEMMA 3. *If a stochastic process $X:\mathbf{R}^N\times\Omega\to\mathbf{R}$ is quadratic and P-bounded on some set $A\subset\mathbf{R}^N$ with non-empty interior, then it is P-bounded on any bounded subset of $\mathbf{R}^N$.*

Proof. Since $\mathrm{Int}A\neq\emptyset$, there exists a ball $K(x_0, r)$ (with $r>0$) contained in A. First we shall show that the process X is P-bounded on the ball $K(0, r)$. For, let us take a point $y\in K(0, r)$. By equation (1) we have

$$|X(y,\cdot)|\leqslant\frac{1}{2}|X(x_0+y,\cdot)|+\frac{1}{2}|X(x_0-y,\cdot)|+X(x_0,\cdot)|\quad \text{(a.e.)},$$

whence, for every $n\in\mathbf{N}$,

$$P(\{\omega\in\Omega: |X(y,\omega)|\geqslant n\})\leqslant P\left(\left\{\omega\in\Omega: |X(x_0+y,\omega)|\geqslant\frac{n}{3}\right\}\right)+$$
$$+P\left(\left\{\omega\in\Omega: |X(x_0-y,\omega)|\geqslant\frac{n}{3}\right\}\right)+P\left(\left\{\omega\in\Omega: |X(x_0,\omega)|\geqslant\frac{n}{3}\right\}\right)\leqslant$$
$$\leqslant 3\sup\left\{P\left(\left\{\omega\in\Omega: |X(x,\omega)|\geqslant\frac{n}{3}\right\}\right): x\in A\right\}.$$

The above inequality holds for all $y\in K(0, r)$; therefore also

$$\sup\{P(\{\omega\in\Omega: |X(y,\omega)|\geqslant n\}): y\in K(0, r)\}\leqslant$$
$$\leqslant 3\sup\left\{P\left(\left\{\omega\in\Omega: |X(x,\omega)|\geqslant\frac{n}{3}\right\}\right): y\in A\right\},$$

which implies that the process X is P-bounded on the ball $K(0, r)$. Now, assume that the set $B \subset \mathbf{R}^N$ is bounded and take a positive rational number q such that $B \subset K(0, qr)$. Then, for every $x \in B$ and $n \in \mathbf{N}$, we have

$$P(\{\omega \in \Omega : |X(x, \omega)| \geqslant n\}) = P\left(\left\{\omega \in \Omega : q^2 \left|X\left(\frac{x}{q}, \omega\right)\right| \geqslant n\right\}\right) \leqslant$$
$$\leqslant \sup\left\{P\left(\left\{\omega \in \Omega : |X(z, \omega)| \geqslant \frac{n}{q^2}\right\}\right) : \ z \in K(0, r)\right\}.$$

Since the process X is P-bounded on the ball $K(0, r)$, this implies that X is P-bounded on the set B too. This ends our proof.

Now we shall prove a theorem giving a characterization of continuous quadratic processes.

THEOREM 1. *If a stochastic process $X: \mathbf{R}^N \times \Omega \to \mathbf{R}$ is quadratic, then the following conditions are equivalent*:

1) *X is continuous at every point $x \in \mathbf{R}^N$,*
2) *X is continuous at some point $x_0 \in \mathbf{R}^N$,*
3) *X is P-bounded on some set $A \subset \mathbf{R}^N$ with non-empty interior,*
4) *there exist random variables $Y_{i,j}: \Omega \to \mathbf{R}$, $i, j = 1, \ldots, N$, such that*

$$X(x, \cdot) = \sum_{i,j=1}^{N} x_i x_j Y_{i,j}(\cdot) \quad \text{(a.e.) } \textit{for every } x = (x_1, \ldots, x_N) \in \mathbf{R}^N.$$

Proof. Implication 1) $\Rightarrow$ 2) is trivial.

To prove the implication 2) $\Rightarrow$ 3), assume that the process X is continuous at a point $x_0 \in \mathbf{R}^N$. Since for any $x \in \mathbf{R}^N$

$$X(x, \cdot) = \frac{1}{2}[X(x_0 + x, \cdot) + X(x_0 - x, \cdot) - 2X(x_0, \cdot)] \qquad \text{(a.e.)},$$

then the process X is also continuous at the point $0 \in \mathbf{R}^N$. We shall show that X is P-bounded on the ball $K(0, 1)$. Suppose the contrary. Then there exist an $\varepsilon > 0$ and a sequence $(x_n)_{n \in \mathbf{N}}$ such that $x_n \in K(0, 1)$ for $n \in \mathbf{N}$, and $P(\{\omega \in \Omega : |X(x_n, \omega)| \geqslant n\}) > \varepsilon$. Now, for every $n \in \mathbf{N}$, take a rational q_n such that $n \cdot q_n^2 \in (1, 2)$. Then $q_n \to 0$, and so $z_n := q_n x_n \to 0$. On the other hand, we have

$$P(\{\omega \in \Omega : |X(z_n, \omega)| \geqslant 1\}) \geqslant P(\{\omega \in \Omega : |X(q_n x_n, \omega)| \geqslant n q_n^2\}) =$$
$$= P(\{\omega \in \Omega : |X(x_n, \omega)| \geqslant n\}) > \varepsilon,$$

which contradicts the continuity of X at 0.

3) $\Rightarrow$ 4). Assume that the process $X: \mathbf{R}^N \times \Omega \to \mathbf{R}$ is quadratic and P-bounded on a set with non-empty interior and consider the process $B: \mathbf{R}^N \times \mathbf{R}^N \times \Omega \to \mathbf{R}$ defined by

$$B(x, y, \omega) := \frac{1}{2}[X(x + y, \omega) - X(x, \omega) - X(y, \omega)], \ (x, y, \omega) \in \mathbf{R}^N \times \mathbf{R}^N \times \Omega.$$

This process is additive with respect to the first and second variable, that is for every $x, y, z \in \mathbf{R}^N$

$$B(x+y, z, \cdot) = B(x, z, \cdot) + B(y, z, \cdot) \quad \text{(a.e.)}$$

and

$$B(x, y+z, \cdot) = B(x, y, \cdot) + B(x, z, \cdot) \quad \text{(a.e.)}.$$

Indeed, by the definition of the process B and Lemma 2 we have

$$2[B(x+y, z, \cdot) - B(x, z, \cdot) - B(y, z, \cdot)] =$$
$$= X(x+y+z, \cdot) - X(x+y, \cdot) - X(z, \cdot) - X(x+z, \cdot) +$$
$$+ X(x, \cdot) + X(z, \cdot) - X(y+z, \cdot) + X(y, \cdot) + X(z, \cdot) = 0 \quad \text{(a.e.)}.$$

The other of the above two equalities follows from the first one, because the mapping B is symmetric with respect to the first two variables. Now, fix a point $y \in \mathbf{R}^N$ arbitrarily. It follows from the definition of B that

$$|B(x, y, \cdot)| \leqslant \frac{1}{2}|X(x+y, \cdot)| + \frac{1}{2}|X(x, \cdot)| + \frac{1}{2}|X(y, \cdot)|,$$

and hence, for every $x \in K(0, 1)$ we have

$$P(\{\omega \in \Omega : |B(x, y, \omega)| \geqslant n\}) \leqslant P\left(\left\{\omega \in \Omega : |X(x+y, \omega)| \geqslant \frac{n}{3}\right\}\right) +$$
$$+ P\left(\left\{\omega \in \Omega : |X(x, \omega)| \geqslant \frac{n}{3}\right\}\right) + P\left(\left\{\omega \in \Omega : |X(y, \omega)| \geqslant \frac{n}{3}\right\}\right) \leqslant$$
$$\leqslant 3 \sup \left\{P\left(\left\{\omega \in \Omega : |X(z, \omega)| \geqslant \frac{n}{3}\right\}\right) : z \in K(0, \|y\| + 1)\right\}.$$

Since the process X is P-bounded on the ball $K(0, \|y\| + 1)$ (Lemma 3), this implies that the process B, as the function of the first variable, is P-bounded on the ball $K(0, 1)$. Because additive stochastic processes P-bounded on a set with non-empty interior are continuous (see Theorem 4 in [8]), the process B is continuous with respect to the first variable. Now consider the processes $B_i : \mathbf{R} \times \Omega \to \mathbf{R}$, $i = 1, \ldots, N$, defined by $B_i(t, \omega) := B(te_i, y, \omega)$, where $\{e_i, i = 1, \ldots, N\}$ is the ortonormal base of the space $\mathbf{R}^N$ over $\mathbf{R}$. These processes are additive and continuous; therefore, by the theorem of Nagy ([7]), $B_i(t, \cdot) = tB_i(1, \cdot)$ (a.e.) for every $t \in \mathbf{R}$. Now, taking a point $x = x_1 e_1 + \ldots + x_N e_N \in \mathbf{R}^N$, we have

$$B(x, y, \cdot) = \sum_{i=1}^{N} B(x_i e_i, y, \cdot) = \sum_{i=1}^{N} B_i(x_i, \cdot) = \sum_{i=1}^{N} x_i B_i(1, \cdot) =$$
$$= \sum_{i=1}^{N} x_i B(e_i, y, \cdot) \quad \text{(a.e.)}$$

Since the process B is symmetric with respect to the first two variables, we have also

$$B(x, y, \cdot) = \sum_{i=1}^{N} y_i B(x, e_i, \cdot) \quad \text{(a.e.)},$$

where $y = y_1 e_1 + \ldots + y_N e_N$. From the equalities obtained above we get, for every $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$,

$$B(x, y, \cdot) = \sum_{i,j=1}^{N} x_i y_j B(e_i, e_j, \cdot) = \sum_{i,j=1}^{N} x_i y_j Y_{i,j}(\cdot) \quad \text{(a.e.)},$$

where $Y_{i,j} := B(e_i, e_j, \cdot) = \frac{1}{2}[X(e_i + e_j, \cdot) - X(e_i, \cdot) - X(e_j, \cdot)]$, $i, j = 1, \ldots, N$.

Since $B(x, x, \cdot) = X(x, \cdot)$ (a.e.), we obtain

$$X(x, \cdot) = \sum_{i,j=1}^{N} x_i x_j Y_{i,j}(\cdot) \quad \text{(a.e.)},$$

which was to be proved.

Now we shall prove the implication 4) ⇒ 1). Let us fix a point $x_0 \in \mathbf{R}^N$ and take a sequence $(x_n)_{n \in \mathbf{N}}$ converging to x_0. Let $x_0 = (x_{0,1}, \ldots, x_{0,N})$ and $x_n = (x_{n,1}, \ldots, x_{n,N})$, $n \in \mathbf{N}$. Then

$$P\text{-}\lim_{n \to \infty} \sum_{i,j=1}^{N} x_{n,i} x_{n,j} Y_{i,j} = \sum_{i,j=1}^{N} x_{0,i} x_{0,j} Y_{i,j},$$

because the sequence of random variables $\left(\sum_{i,j=1}^{N} x_{n,i} x_{n,j} Y_{i,j}\right)_{n \in \mathbf{N}}$ is convergent on Ω to the random variable $\sum_{i,j=1}^{N} x_{0,i} x_{0,j} Y_{i,j}$ and the measure P is finite. Since

$$X(x_n, \cdot) = \sum_{i,j=1}^{N} x_{n,i} x_{n,j} Y_{i,j} \quad \text{(a.e.)}$$

and

$$X(x_0, \cdot) = \sum_{i,j=1}^{N} x_{0,i} x_{0,j} Y_{i,j} \quad \text{(a.e.)},$$

we have also

$$P\text{-}\lim_{n \to \infty} X(x_n, \cdot) = X(x_0, \cdot).$$

This completes the proof of our theorem.

REMARK 1. An analogous theorem for $N = 1$ we have proved in [9]. However, the methods used in that paper are not applicable in the present situation because the basic Lemma 5 from that paper is not longer true in the case $N \geqslant 2$.

Now, we are going to introduce an operation with the aid of which we shall obtain another sufficient conditions for a quadratic stochastic processes to be continuous.

For a set $A \subset \mathbf{R}^N$ let us define

$$H(A) := \{x \in \mathbf{R}^N : A \cap (A+x) \cap (A-x) \neq \emptyset\}.$$

As an immediate consequence of this definition we obtain the following

THEOREM 2. *For any sets $A, B \subset \mathbf{R}^N$:*

a) *if $A \neq \emptyset$, then $0 \in H(A)$;*

b) *the set $H(A)$ is symmetric with respect to* 0;

c) *if $0 \in A$ and A is symmetric with respect to* 0, *then $A \subset H(A)$;*

d) $H(A) \subset H(H(A))$;

e) *if $A \subset B$, then $H(A) \subset H(B)$;*

f) *$H(A \cap B) \subset H(A) \cap H(B)$ and $H(A \cup B) \supset H(A) \cup H(B)$;*

g) *$H(A+a) = H(A)$ for every $a \in \mathbf{R}^N$;*

h) *$H(tA) = tH(A)$ for every $t \in \mathbf{R}$;*

i) *$H(A) \subset A-A$ and $H(A) \subset \frac{1}{2}(A-A)$.*

THEOREM 3. *If a set $A \subset \mathbf{R}^N$ has positive inner Lebesgue measure, then* $\operatorname{Int} H(A) \neq \emptyset$.

Proof. Let us take a compact set $B \subset A$ with positive Lebesgue measure and denote by χ the characteristic function of B. Consider the function $f: \mathbf{R}^N \to \mathbf{R}$ defined by

$$f(x) := m(B \cap (B-x) \cap (B+x)), \quad x \in \mathbf{R}^N,$$

where m denotes the Lebesgue measure in $\mathbf{R}^N$. On account of elementary properties of the Lebesgue integral we have

$$|f(x)-f(0)| = \Big|\int_{\mathbf{R}^N} \chi(t)\chi(t+x)\chi(t-x)\mathrm{d}t - \int_{\mathbf{R}^N} \chi(t)\mathrm{d}t\Big| \leqslant$$

$$\leqslant \Big|\int_{\mathbf{R}^N} \chi(t)\chi(t+x)\chi(t-x)\mathrm{d}t - \int_{\mathbf{R}^N} \chi(t)\chi(t+x)\mathrm{d}t\Big| +$$

$$+ \Big|\int_{\mathbf{R}^N} \chi(t)\chi(t+x)\mathrm{d}t - \int_{\mathbf{R}^N} \chi(t)\mathrm{d}t\Big| \leqslant$$

$$\leqslant \int_{\mathbf{R}^N} |\chi(t)\chi(t+x)\chi(t-x) - \chi(t)\chi(t+x)|\mathrm{d}t + \int_{\mathbf{R}^N} |\chi(t)\chi(t+x) - \chi(t)|\mathrm{d}t =$$

$$= \int_{\mathbf{R}^N} \chi(t)\chi(t+x)|\chi(t-x) - \chi(t)|\mathrm{d}t + \int_{\mathbf{R}^N} \chi(t)|\chi(t+x) - \chi(t)|\mathrm{d}t \leqslant$$

$$\leqslant \int_{\mathbf{R}^N} |\chi(t-x) - \chi(t)|\mathrm{d}t + \int_{\mathbf{R}^N} |\chi(t+x) - \chi(t)|\mathrm{d}t =$$

$$= m((B+x) \dot{-} B) + m((B-x) \dot{-} B),$$

where $\dot{-}$ denotes the symmetric difference. Fix an $\varepsilon > 0$ and take an open set U such that $B \subset U$ and $m(U \setminus B) < \varepsilon$. Since B is compact, we have $d := \operatorname{dist}(B, U') > 0$. Therefore, for $x \in K(0, d)$, we have $B + x \subset U$ and $B - x \subset U$, whence

$$m\big((B+x) \dot{-} B\big) + m\big((B-x) \dot{-} B\big) \leqslant$$
$$\leqslant m(U \setminus B) + m\big(U \setminus (B+x)\big) + m(U \setminus B) + m\big(U \setminus (B-x)\big) < 4\varepsilon.$$

Thus, for every $x \in K(0, d)$, $|f(x) - f(0)| < 4\varepsilon$, which means that f is continuous at 0. Since $f(0) = m(B) > 0$, there exists a ball $K(0, r)$ such that $f(x) > 0$ for $x \in K(0, r)$. This implies that

$$B \cap (B-x) \cap (B+x) \neq \varnothing \quad \text{for } x \in K(0, r),$$

and so

$$A \cap (A-x) \cap (A+x) \neq \varnothing \quad \text{for } x \in K(0, r),$$

because $B \subset A$. Thus $K(0, r) \subset H(A)$, which was to be proved.

REMARK 2. In case $N = 1$, a similar theorem (but under somewhat stronger assumptions) was proved by Kurepa (see Lemma 1 in [6]).

THEOREM 4. *If a set $A \subset \mathbf{R}^N$ is of the second category with the Baire property, then* $\operatorname{Int} H(A) \neq \varnothing$.

Proof. According to our assumptions, there exists an open, non-empty set U and there exist sets S, T of the first category such that $A = (U \setminus S) \cup T$. Let us take an open ball $K = K(x_0, \varepsilon) \subset U$ and put $K_0 := K - x_0$. Fix arbitrary a point $x \in K_0$ and consider the set

$$V := K_0 \cap (K_0 + x) \cap (K_0 - x).$$

This set is open and non-empty (in particular $0 \in V$); therefore, by a theorem of Baire, it is of the second category. On the other hand the sets

$$V \setminus (A - x_0), \qquad V \setminus (A - x_0 + x), \qquad V \setminus (A - x_0 - x)$$

are of the first category, because the set $K \setminus A$ is of the first category. Since

$$V = \big[V \setminus (A - x_0)\big] \cup \big[V \setminus (A - x_0 + x)\big] \cup \big[V \setminus (A - x_0 - x)\big] \cup$$
$$\cup \big[V \cap (A - x_0) \cap (A - x_0 + x) \cap (A - x_0 - x)\big],$$

we must have

$$(A - x_0) \cap (A - x_0 + x) \cap (A - x_0 - x) \neq \varnothing,$$

and so

$$A \cap (A + x) \cap (A - x) \neq \varnothing.$$

Thus $K_0 \subset H(A)$, which means that $\operatorname{Int} H(A) \neq \varnothing$.

Now, we shall introduce the following definitions:

$$H^1(A) := H(A),$$
$$H^{n+1}(A) := H\big(H^n(A)\big), \; n \in \mathbf{N},$$

where A is a subset of $\mathbf{R}^N$.

We have the following

THEOREM 5. *If a stochastic process* $X:\mathbf{R}^N \times \Omega \to \mathbf{R}$ *is quadratic and P-bounded on a set* $A \subset \mathbf{R}^N$ *such that* $\operatorname{Int} H^n(A) \neq \emptyset$ *for some* $n \in \mathbf{N}$, *then it is continuous.*

Proof. First, we shall prove that the P-boundedness of X on the set A implies its P-boundedness on the set $H(A)$. Let $x \in H(A)$. Then there exists a point $y \in \mathbf{R}^N$ such that $y, y-x, y+x \in A$. Hence, because of the inequality

$$|X(x,\cdot)| \leqslant \frac{1}{2}|X(y+x,\cdot)| + \frac{1}{2}|X(y-x,\cdot)| + |X(y,\cdot)| \qquad \text{(a.e.)},$$

we obtain

$$\begin{aligned} P(\{\omega \in \Omega: |X(x,\omega)| \geqslant n\}) &\leqslant P\left(\left\{\omega \in \Omega: |X(y+x,\omega)| \geqslant \frac{n}{3}\right\}\right) + \\ &+ P\left(\left\{\omega \in \Omega: |X(y-x,\omega)| \geqslant \frac{n}{3}\right\}\right) + \\ &+ P\left(\left\{\omega \in \Omega: |X(y,\omega)| \geqslant \frac{n}{3}\right\}\right) \leqslant \\ &\leqslant 3 \sup\left\{P\left(\left\{\omega \in \Omega: |X(z,\omega)| \geqslant \frac{n}{3}\right\}\right): z \in A\right\}. \end{aligned}$$

The latter inequality holds for every $x \in H(A)$; therefore also

$$\begin{aligned} \sup\{P(\{\omega \in \Omega: |X(x,\omega)| \geqslant n\}): x \in H(A)\} &\leqslant \\ \leqslant 3 \sup\left\{P\left(\left\{\omega \in \Omega: |X(z,\omega)| \geqslant \frac{n}{3}\right\}\right): z \in A\right\}, \end{aligned}$$

which implies that X is P-bounded on the set $H(A)$. Now, using the induction principle, we obtain that the process X is also P-bounded on the set $H^n(A)$. Since $\operatorname{Int} H^n(A) \neq \emptyset$, it follows from the implication 3) $\Rightarrow$ 1) of Theorem 1 that the process X is continuous. This completes the proof.

As an immediate consequence of Theorems 3,4 and 5 we obtain

THEOREM 6. *Let* $A \subset \mathbf{R}^N$ *be a set of positive inner Lebesgue measure or of the second category with the Baire property. If a stochastic process* $X:\mathbf{R}^N \times \Omega \to \mathbf{R}$ *is quadratic and P-bounded on A, then it is continuous.*

REMARK 3. It is worth noting that Theorem 5 is essentialy stronger than Theorem 6. Indeed, there exist sets A of the Lebesgue measure zero and of the first category such that $\operatorname{Int} H(A) \neq \emptyset$. This is, for instance, the case for the set A given in the following

EXAMPLE. Let

$$B := \left\{x \in \mathbf{R}: x = \sum_{i=1}^{\infty} \frac{x_i}{3^i},\ x_i \in \{0,1\},\ i \in \mathbf{N}\right\},$$

$$C := \left\{x \in \mathbf{R}: x = \sum_{i=1}^{\infty} \frac{x_i}{3^i},\ x_i \in \{0,2\},\ i \in \mathbf{N}\right\},$$

$$A := B \cup C \cup (C-1).$$

The sets C and B have Lebesgue measure zero and are nowhere dense $\left(C \text{ is the Cantor set and } B = \frac{1}{2}C\right)$, therefore also A has measure zero and is nowhere dense. We shall show that the interval $(0, 1)$ is contained in $H(A)$. For, let us fix a number $x \in (0, 1)$ and take its 3-adic expansion $x = \Sigma_{i=1}^{\infty} x_i/3^i$, where $x_i \in \{0, 1, 2\}$ for $i \in \mathbf{N}$. Note that then $-x = -1 + \Sigma_{i=1}^{\infty} y_i/3^i$, where $y_i := 2 - x_i, i \in \mathbf{N}$. Now, define the point $a = \Sigma_{i=1}^{\infty} a_i/3^i$ by putting

$$a_i := \begin{cases} 0, & \text{if } x_i = 0 \text{ or } x_i = 2, \\ 1, & \text{if } x_i = 1, \end{cases} \qquad i \in \mathbf{N}.$$

Then $a \in A$ (because $a \in B$), $a + x \in A$ (because $a + x \in C$) and $a - x \in A$ (because $a - x \in C - 1$). Therefore

$$A \cap (A + x) \cap (A - x) \neq \emptyset,$$

which means that $x \in H(A)$.

Now we shall introduce the following notations. Let $\mathscr{L}$ denote the σ-algebra of the Lebesgue measurable subsets of $\mathbf{R}^N$, $\mathscr{L} \times_\sigma \mathscr{A}$ — the product σ-algebra in $\mathbf{R}^N \times \Omega$, $\mu = m \times_\sigma P$ — the product measure on $\mathscr{L} \times_\sigma \mathscr{A}$, $\mathscr{B}$ — the completion of $\mathscr{L} \times_\sigma \mathscr{A}$ with respect to μ, and $\bar{\mu}$ — the completion of μ.

A stochastic process $X: \mathbf{R}^N \times \Omega \to \mathbf{R}$ will be called *measurable* iff it is measurable mapping with respect to the σ-algebra $\mathscr{B}$.

The following theorem is an analogue of the famous theorem of Sierpiński [11] for convex functions.

THEOREM 7. *If a stochastic process $X: \mathbf{R}^N \times \Omega \to \mathbf{R}$ is quadratic and if there exist a measurable process $Y: \mathbf{R}^N \times \Omega \to \mathbf{R}$ and a set $A \subset \mathbf{R}^N$ of positive Lebesgue measure such that for any $x \in A$ $|X(x, \cdot)| \leqslant Y(x, \cdot)$ (a.e.), then X is continuous.*

Proof. Since the σ-algebra $\mathscr{B}$ is completion of the σ-algebra $\mathscr{L} \times_\sigma \mathscr{A}$, there exists an $\mathscr{L} \times_\sigma \mathscr{A}$ – measurable process $Y': \mathbf{R}^N \times \Omega \to \mathbf{R}$ which coincides to the process Y except for a $\bar{\mu}$-nullset $\bar{N}$. Then, by Fubini's theorem, there exists a set $M \subset \mathbf{R}^N$ such that $m(M) = 0$ and for all $x \in \mathbf{R}^N \setminus M$

$$P(\bar{N}_x) = P(\{\omega \in \Omega : (x, \omega) \in \bar{N}\}) = 0.$$

Put $S^n := ((x, \omega) \in \mathbf{R}^N \times \Omega : Y'(x, \omega) \geqslant n\}$ and $S_x^n := \{\omega \in \Omega : Y'(x, \omega) \geqslant n\}$. Then, for every $n \in \mathbf{N}$, $S^n \in \mathscr{L} \times_\sigma \mathscr{A}$ and for all $n \in \mathbf{N}$ and $x \in \mathbf{R}^N$, $S_x^n \in \mathscr{A}$. Let us consider the functions $f_n: \mathbf{R}^N \to [0, 1]$, $n \in \mathbf{N}$, defined by

$$f_n(x) := P(S_x^n), \; x \in \mathbf{R}^N.$$

These functions are measurable and for all $x \in \mathbf{R}^N$

$$\lim_{n \to \infty} f_n(x) = 0.$$

The celebrated theorem of Egoroff guarantees the existence of a set $F \subset A \setminus M$ of positive measure, on which this convergence is uniform. Thus we have

$$\forall_{\varepsilon>0}\exists_{n_0\in\mathbf{N}}\forall_{n>n_0}(\sup\{f_n(x): x\in F\} = \sup\{P(\{\omega\in\Omega: Y'(x,\omega)\geqslant n\}): x\in F\} < \varepsilon),$$

which means that the process Y' is P-upper bounded on F. Since

$$\forall_{x\in F}(Y'(x,\cdot) = Y(x,\cdot) \text{ (a.e.)})$$

and

$$\forall_{x\in A}(|X(x,\cdot)| \leqslant Y(x,\cdot) \text{ (a.e.)}),$$

it follows that the process X is P-bounded on F. Because the measure of the set F is positive, the process X is continuous. This finishes the proof.

As an immediate consequence of this theorem we obtain

COROLLARY 1. *If a stochastic process* $X:\mathbf{R}^N\times\Omega\to\mathbf{R}$ *is quadratic and measurable, then it is continuous.*

Now we shall prove a theorem which concerns the stability (in the sense of Ulam) of quadratic stochastic processes. This theorem is an analogue of the theorem of Hyers [4] for additive functions. In the deterministic case such theorem has been independently proved by Cholewa [2].

THEOREM 8. *If a stochastic process* $X:\mathbf{R}^N\times\Omega\to\mathbf{R}$ *fulfils the condition*

$$\forall_{x,y\in\mathbf{R}^N}(|X(x+y,\cdot)+X(x-y,\cdot)-2X(x,\cdot)-2X(y,\cdot)| \leqslant \varepsilon \text{ (a.e.)}), \tag{2}$$

where ε *is a positive constant, then there exists a quadratic stochastic process* $Y:\mathbf{R}^N\times\Omega\to\mathbf{R}$ *such that*

$$\forall_{x\in\mathbf{R}^N}(|X(x,\cdot)-Y(x,\cdot)| \leqslant \frac{\varepsilon}{2} \text{ (a.e.)}). \tag{3}$$

Moreover, if $Y_1:\mathbf{R}^N\times\Omega\to\mathbf{R}$ *is another quadratic stochastic process satisfying condition* (3), *then for every* $x\in\mathbf{R}^N$ $Y_1(x,\cdot) = Y(x,\cdot)$ (a.e.).

Proof. Using (2) for $x = y = 0$, we have

$$|X(0,\cdot)| \leqslant \frac{\varepsilon}{2} \quad \text{(a.e.)}.$$

From here and from (2) for $x = y$ we obtain

$$|X(2x,\cdot)-4X(x,\cdot)| \leqslant |X(2x,\cdot)+X(0,\cdot)-4X(x,\cdot)|+|X(0,\cdot)| \leqslant \varepsilon+\frac{\varepsilon}{2} \text{ (a.e.)},$$

whence, for every $x\in\mathbf{R}^N$,

$$\left|\frac{1}{4}X(2x,\cdot)-X(x,\cdot)\right| \leqslant \frac{1}{4}\left(\varepsilon+\frac{\varepsilon}{2}\right) \text{ (a.e.)}.$$

Applying the induction principle, we can show easily that for any $n \in \mathbf{N}$ and $x \in \mathbf{R}^N$

$$(4) \qquad \left|\frac{1}{4^n} X(2^n x, \cdot) - X(x, \cdot)\right| \leqslant \left(\frac{1}{4} + \ldots + \frac{1}{4^n}\right)\left(\varepsilon + \frac{\varepsilon}{2}\right) \leqslant \frac{\varepsilon}{2} \text{ (a.e.).}$$

Now, fix a point $x \in \mathbf{R}^N$ and take the sequence of random variables $\left(\frac{1}{4^n} X(2^n x, \cdot)\right)_{n \in \mathbf{N}}$. In view of (4) we have

$$\left|\frac{1}{4^{n+m}} X(2^{n+m} x, \cdot) - \frac{1}{4^n} X(2^n x, \cdot)\right| = \frac{1}{4^n}\left|\frac{1}{4^m} X(2^m 2^n x, \cdot) - X(2^n x, \cdot)\right| \leqslant \frac{\varepsilon}{2 \cdot 4^n} \text{ (a.e.),}$$

which implies that this sequence is a Cauchy sequence with respect to the measure P. Therefore, by the theorem of Riesz (see [3], Theorem E, § 22), it have to be convergent with respect to the measure P. Let us consider the stochastic process $Y: \mathbf{R}^N \times \Omega \to \mathbf{R}$ defined by

$$Y(x, \cdot) := P\text{-}\lim_{n \to \infty} \frac{1}{4^n} X(2^n x, \cdot), \quad x \in \mathbf{R}^N.$$

This process is quadratic because, for every $x, y \in \mathbf{R}^N$, we have

$$|Y(x+y, \cdot) + Y(x-y, \cdot) - 2Y(x, \cdot) - 2Y(y, \cdot)| =$$

$$= \left|P\text{-}\lim_{n \to \infty} \frac{1}{4^n}\left[X(2^n(x+y), \cdot) + X(2^n(x-y), \cdot) - 2X(2^n x, \cdot) - 2X(2^n y, \cdot)\right]\right| =$$

$$= P\text{-}\lim_{n \to \infty} \frac{1}{4^n}\left|X(2^n x + 2^n y, \cdot) + X(2^n x - 2^n y, \cdot) - 2X(2^n x, \cdot) - 2X(2^n y, \cdot)\right| \leqslant$$

$$\leqslant P\text{-}\lim_{n \to \infty} \frac{\varepsilon}{4^n} = 0 \text{ (a.e.).}$$

Moreover, using (4), we get for any $x \in \mathbf{R}^N$

$$|X(x, \cdot) - Y(x, \cdot)| = P\text{-}\lim_{n \to \infty}\left|X(x, \cdot) - \frac{1}{4^n} X(2^n x, \cdot)\right| \leqslant \frac{\varepsilon}{2} \text{ (a.e.).}$$

Now assume that $Y_1: \mathbf{R}^N \times \Omega \to \mathbf{R}$ is another quadratic stochastic process satisfying the condition (3). Then, for any $x \in \mathbf{R}^N$ and $n \in \mathbf{N}$, we obtain

$$|Y(x, \cdot) - Y_1(x, \cdot)| = \frac{1}{n^2}|Y(nx, \cdot) - Y_1(nx, \cdot)| \leqslant$$

$$\leqslant \frac{1}{n^2}\left[|Y(nx, \cdot) - X(nx, \cdot)| + |X(nx, \cdot) - Y_1(nx, \cdot)|\right] \leqslant \frac{\varepsilon}{n^2} \text{ (a.e.).}$$

This implies that $Y_1(x, \cdot) = Y(x, \cdot)$ (a.e.) for any $x \in \mathbf{R}^N$ and the theorem follows.

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