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A NOTE ON THE EXPONENTIAL DISTRIBUTION

Abstract. A characterization of the exponential distribution via a functional equation on a restricted domain is given.

Let $X \geqslant 0$ be a random variable with distribution function

$$F(x) = 1-e^{-bx}, \qquad x \in [0, \infty),$$

where $b > 0$ is a constant. Then F is called the *exponential distribution.*

We say that X or the distribution function F of X, has the lack of memory if F satisfies the functional equation

$$(1) \qquad 1-F(x+y) = \big(1-F(x)\big)\big(1-F(y)\big)$$

for all $x, y \in [0, \infty)$.

The following result gives all solutions of (1) (see for example [1, Theorem 1.3.1.]).

THEOREM 1. *Let F be the distribution function of a random variable $X \geqslant 0$ and let F satisfy equation* (1) *for all $x, y \in [0, \infty)$. Then either F is the exponential distribution or F is degenerate at zero* (*that is $F(0) = 0$ and $F(x) = 1$ if $x > 0$*).

From some applications the following question arises: If the functional equation (1) holds only for all $(x, y) \in [0, \infty)^2 \backslash B$, where B is a Lebesgue null set in $[0, \infty)^2$, what can be said about the distribution function F? Could we expect in this case that there is a distribution function G satisfying (1) for all $x, y \in [0, \infty)$ such that $G = F$ almost everywhere in $[0, \infty)$? A positive answer to this question — in a much more general setting — was given by R. Ger [2] (cf. also [3], p. 490—493; moreover see [3], p. 443 for further problems concerning functional equations on restricted domains).

In this note we consider a special case of the above problem by assuming that the equation (1) is valid for all $x, y \in [0, \infty)\backslash A$ where A is a Lebesgue null set in $[0, \infty)$. We give a brief proof for the known fact that this extended form of the lack of memory property characterizes the exponential distribution among all nondegenerate distributions.

THEOREM 2. *Let $X \geqslant 0$ be a random variable with distribution function F. If F is not degenerate at zero and if there is a Lebesgue null set A in $[0, \infty)$ such that* (1) *is valid for all $x, y \in [0, \infty)\backslash A$ then F is the exponential distribution.*

Proof. Let us introduce the notations $G = 1-F$, $\mathbf{R}_0 = [0, \infty)$ and $B = \mathbf{R}_0 \backslash A$. By hypothesis we have

$$(2) \qquad G(x+y) = G(x)G(y), \quad x, y \in B.$$

We shall show that (2) is even satisfied for all $x, y \in \mathbf{R}_0$. If $x = y = 0$ or if $x = 0$ and $y > 0$ then — because of $F(0) = 0$ — equation (2) is fulfilled.

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Now let $x > 0$, $y > 0$ be arbitrary but fixed elements. Since $C = [A \cup (y-A)] \cap \mathbf{R}_0$ is of measure zero, $[0, y] \setminus C$ is of positive measure, so that there is an element

$$t \in [0, y] \setminus C. \tag{3}$$

Completely analogous $D = [A \cup (x-A) \cup (x+t-A) \cup (A-y+t)] \cap \mathbf{R}_0$ is of measure zero which implies the existence of an element

$$s \in [0, x] \setminus D. \tag{4}$$

Using $t \leqslant y$ (3) implies $t \in \mathbf{R}_0$, $t \notin A$, $t \in y - \mathbf{R}_0$ and $t \notin y - A$ that is

$$t \in B \cap (y-B). \tag{5}$$

In the same manner we conclude from (4) that

$$s \in B \cap (x-B). \tag{6}$$

Because of $t \in \mathbf{R}_0$, $x-s \in \mathbf{R}_0$ and $s \in \mathbf{R}_0$, $y-t \in \mathbf{R}_0$ we get from (4)

$$x-s+t \in \mathbf{R}_0, \; s \notin x+t-A \; \text{ and } \; s+y-t \in \mathbf{R}_0, \; s \notin A-y+t$$

that is

$$x-s+t \in B \; \text{ and } \; s+y-t \in B. \tag{7}$$

But (5) and (6) imply that there are elements $u, v \in B$ with

$$x = u+s \; \text{ and } \; y = v+t \tag{8}$$

so that (7) leads to

$$x-s+t = u+t \in B \; \text{ and } \; s+y-t = s+v \in B. \tag{9}$$

Now using (2), (8), (9) and the fact that $s, t, u, v \in B$ we have

$$G(x+y) = G(u+s+v+t) = G\big((u+t)+(s+v)\big) = G(u+t)G(s+v) =$$
$$= G(u)\,G(s)\,G(v)\,G(t) = G(u+s)\,G(v+t) = G(x)\,G(y)$$

so that indeed (2) is valid for all $x, y \in \mathbf{R}_0$. Thus because of Theorem 1 the proof is finished.

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