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THE DISTRIBUTIONAL SOLUTIONS OF SOME SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Abstract. In this paper we prove a theorem on the existence and uniqueness of the solution of the Cauchy problem for linear differential equations with distributional coefficients.

1. Introduction. One of the possibilities of generalization of the classical solution of a differential equation is the solution in the distributional sense. Distributional solutions of ordinary differential equations have not been studied sufficiently, due to certain difficulties in defining operations on distributions: some operations (e.g. multiplication, substitution, definite integral) cannot be defined for all distributions in a natural way. A particular large numbers of papers have been devoted to linear differential equations (see [6]—[11], [16]—[21], [24], [25], [27]—[31]. Another possibilities are considered in [12]—[15], [23] and [26].

In this note we consider the system of equations

$$y'(t) = A'(t)y(t) + B'(t), \tag{1.0}$$

where all elements $A_{i,j}$ of the matrix A are functions of locally bounded variation on the interval $(a, b) \subseteq \mathbf{R}^1$ $(i, j = 1, \ldots, n)$, the components B_i of the vector B are continuous functions on (a, b), the derivative is understood in the distributional sense and the product of two distributions is understood as generalized operation (see [3], [4]). The vector $y = (y_1, \ldots, y_n)$ is an unknown distribution. We prove a theorem on the existence and uniqueness of the solution of the Cauchy problem for system (1.0) applying the sequential theory of distributions.

In papers [6]—[11], [16]—[21], [27]—[31] it has been examined linear differential equations with distributional coefficients but under another assumptions than in our note.

2. Notation. We shall denote by $\mathscr{V}_{\mathrm{loc}}(a, b)$ the space of real functions which are of bounded variation on every compact interval $[c, d] \subset (a, b)$.

We say that a distribution p is a *measure* on (a, b) if p is the first distributional derivative of a function of the class $\mathscr{V}_{\mathrm{loc}}(a, b)$. The symbol $\mathscr{M}(a, b)$ denotes the set of all measures defined on (a, b).

By $\mathscr{C}(a, b)$ we denote the space of all continuous real functions on (a, b) and by $\mathscr{V}\mathscr{C}(a, b)$ we denote the set of all functions z such that $z = v \cdot c$, where $v \in \mathscr{V}_{\mathrm{loc}}(a, b)$ and $c \in \mathscr{C}(a, b)$.

A matrix $A(t) = (A_{i,j}(t))$ belongs to $\mathscr{V}_{\mathrm{loc}}^{n \times n}(a, b)$ if and only if $A_{i,j} \in \mathscr{V}_{\mathrm{loc}}(a, b)$ for $i, j = 1, \ldots, n$.

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We put

$$\mathscr{V}^n_{\mathrm{loc}}(a,b) = \underbrace{\mathscr{V}_{\mathrm{loc}}(a,b)\times\ldots\times\mathscr{V}_{\mathrm{loc}}(a,b)}_{n\text{ times}},$$

$$\mathscr{C}^n(a,b) = \underbrace{\mathscr{C}(a,b)\times\ldots\times\mathscr{C}(a,b)}_{n\text{ times}},\quad \mathscr{V}\mathscr{C}^n(a,b) = \underbrace{\mathscr{V}\mathscr{C}(a,b)\times\ldots\times\mathscr{V}\mathscr{C}(a,b)}_{n\text{ times}},$$

$$A'(t) = \big(A'_{i,j}(t)\big),\qquad A(t+) = \big(A_{i,j}(t+)\big),\qquad A(t-) = \big(A_{i,j}(t-)\big),$$

$$\Delta A(t) = A(t+)-A(t-),\qquad y_0 = (y_1^0,\ldots,y_n^0),$$

$$A^*(t) = \big(A^*_{i,j}(t)\big),\qquad y^*(t) = \big(y_1^*(t),\ldots,y_n^*(t)\big),$$

where

$A\in\mathscr{V}^{n\times n}_{\mathrm{loc}}(a,b)$, $y_i^0\in\mathbf{R}^1$, $y_i\in\mathscr{V}\mathscr{C}(a,b)$, $y_i^*(t) = \dfrac{y_i(t+)+y_i(t-)}{2}$, $t\in(a,b)$, $y_i(t+)$, $(y_i(t-))$ denotes the right (resp. left) hand side limit of the function y_i at the point t for $i = 1,\ldots,n$.

A sequence of smooth, non–negative functions $\{\delta_k\}$ satisfying:

$$\int_{-\infty}^{\infty}\delta_k(t)\mathrm{d}t = 1, \tag{2.1}$$

$$\delta_k(t) = \delta_k(-t)\ \text{ for all }\ t\in\mathbf{R}^1, \tag{2.2}$$

$$\delta_k(t) = 0\ \text{ for }\ |t|\geqslant\alpha_k, \tag{2.3}$$

where $\{\alpha_k\}$ is a sequence of positive numbers with $\alpha_k\to 0$ as $k\to\infty$, is called a *δ–sequence*.

The product of two distributions, the modulus of a distribution and inequalities between two distributions will be understood in this note as generalized operations (see [1], [2], [4], [5]).

Let $P\in\mathscr{V}\mathscr{C}(a,b)$. Then we define

$$\int_c^d P'(t)\mathrm{d}t = P^*(d)-P^*(c), \tag{2.4}$$

where $c,d\in(a,b)$.

It is easy to observe that if $P\in\mathscr{V}_{\mathrm{loc}}(a,b)$ or $P\in\mathscr{C}(a,b)$, then $\int_c^d P'(t)\mathrm{d}t$ exists for every $c,d\in(a,b)$ (because $P\in\mathscr{V}\mathscr{C}(a,b)$).

In the case when $P,Q\in\mathscr{V}_{\mathrm{loc}}(a,b)$ and $u\in\mathscr{C}(a,b)$, then it has been proved in [1], [3], [4] that

$$P\cdot Q'\in\mathscr{M}(a,b),\qquad u\cdot Q'\in\mathscr{M}(a,b), \tag{2.5}$$

$$P'(Q\cdot u) = (P'\cdot Q)u = (P'\cdot u)Q, \tag{2.6}$$

$$\Big|\int_c^d P(t)Q'(t)\mathrm{d}t\Big|\leqslant\Big|\int_c^d|P|(t)|Q'|(t)\mathrm{d}t\Big|\leqslant\sup_{t\in[c,d]}|P|^*(t)\Big|\int_c^d|Q'|(t)\mathrm{d}t\Big| \tag{2.7}$$

and

$$\lim_{k\to\infty}\int_{-\alpha_k}^{\alpha_k} P(t-s)\delta_k(s)\mathrm{d}s = P^*(t) \tag{2.8}$$

for every $t\in(a,b)$.

It is easy to check that if $P'\geqslant 0$ on (a,b) and $a<c<d<b$, then

$$\int_c^d P'(t)\mathrm{d}t = P^*(d)-P^*(c)\geqslant 0 \tag{2.9}$$

(see [2], Theorem 6). Hence

$$\int_c^d P'(t)\mathrm{d}t \leqslant \int_c^d Q'(t)\mathrm{d}t$$

where $P', Q'\in\mathscr{M}(a,b)$, $P'\leqslant Q'$ and $a\prec c<d<b$.

3. Cauchy problem for system (1.0). Let $A\in\mathscr{V}_{\mathrm{loc}}^{n\times n}(a,b)$, $B\in\mathscr{C}^n(a,b)$ and $y\in\mathscr{V}\mathscr{C}^n(a,b)$. Moreover, let y satisfy system (1.0) in the distributional sense. Then we call y the *solution of equation* (1.0).

THEOREM 3.1. *Assume that*

$$A\in\mathscr{V}_{\mathrm{loc}}^{n\times n}(a,b), \tag{3.1}$$

for every $t\in(a,b)$

$$\det\bigl(2I-\Delta A(t)\bigr)\neq 0 \text{ and } \det\bigl(2I+\Delta A(t)\bigr)\neq 0, \tag{3.2}$$

where I *denotes the identity matrix,*

$$B\in\mathscr{C}^n(a,b). \tag{3.3}$$

Then the problem

$$\begin{cases} y'(t) = A'(t)y(t)+B'(t) \\ y^*(t_0) = y_0 \end{cases} \tag{3.4}$$

has exactly one solution in the class $\mathscr{V}\mathscr{C}^n(a,b)$.

REMARK 3.1. Assumption (3.2) is essential. This can be observed from the following examples:

$$\begin{cases} y'(t) = 2\delta(t)y(t) \\ y^*(-1) = 0 \end{cases} \tag{3.5}$$

and

$$\begin{cases} z'(t) = -2\delta(t)y(t) \\ z^*(1) = 0 \end{cases} \tag{3.6}$$

where δ denotes the Dirac's delta distribution. It is not difficult to show that the distributions

$$y = cH \text{ and } z = c(H-1)$$

are solutions of problem (3.5) and (3.6) respectively (H denotes the Heaviside's function).

We shall prove some lemmas before giving the proof of Theorem 3.1.

LEMMA 3.1. *If $F \in \mathscr{C}(a,b)$ and $g \in \mathscr{V}_{\mathrm{loc}}(a,b)$, then the product $F' \cdot g$ exists.*

Proof of Lemma 3.1. Let $F_k = F*\delta_k$ and $g_k = g*\delta_k$, where $\{\delta_k\}$ is an arbitrary δ–sequence (the asterisk denotes the convolution). Then

$$(F_k g_k)' = F_k' g_k + F_k g_k'. \tag{3.7}$$

Hence, by (2.5), we have

$$F'g = (Fg)' - Fg', \tag{3.8}$$

which completes the proof of Lemma 3.1.

LEMMA 3.2. *If $G' \in \mathscr{M}(a,b)$, $G' \geqslant 0$ $(\text{on } (a,b))$ and $F \in \mathscr{C}(a,b)$, then there exists a number μ_1, such that*

$$\int_c^d F(t)G'(t)\mathrm{d}t = \mu_1 \int_c^d G'(t)\mathrm{d}t = \mu_1\big(G^*(d) - G^*(c)\big), \tag{3.9}$$

where $a < c < d < b$, $\mu_1 \in [m, M]$, $m = \inf\limits_{t\in[c,d]} F(t)$ and $M = \sup\limits_{t\in[c,d]} F(t)$.

Proof of Lemma 3.2. If $G^*(d) - G^*(c) = 0$, then by (2.9), Lemma is obvious. Let $G^*(d) - G^*(c) > 0$ and let $F_k = F*\delta_k$, $G_k' = G'*\delta_k$, where $\{\delta_k\}$ denotes the δ–sequence. For a fixed $\varepsilon > 0$ there exists a number k_0 such that

$$(m-\varepsilon)\int_c^d G_k'(t)\mathrm{d}t \leqslant \int_c^d F_k(t)G_k'(t)\mathrm{d}t \leqslant (M+\varepsilon)\int_c^d G_k'(t)\mathrm{d}t \tag{3.10}$$

if $k \geqslant k_0$. We shall prove that

$$\lim_{k\to\infty} \int_c^d F_k(t)G_k'(t)\mathrm{d}t = \int_c^d F(t)G'(t)\mathrm{d}t.$$

In this purpose we consider the sequence $\{g_k\}$ defined as follows

$$g_k = u_k + v_k, \tag{3.11}$$

where

$$u_k = \int_c^d \big(F_k(t) - F(t)\big)G_k'(t)\mathrm{d}t \text{ and } v_k = \int_c^d F(t)\big(G_k'(t) - G'(t)\big)\mathrm{d}t.$$

Since $|F_k - F| \rightrightarrows 0$ on (a,b) (almost uniformly) and $\{\int_c^d |G_k'(t)|\mathrm{d}t\}$ is a bounded sequence (see [1]), we see by (2.7) that

$$\lim_{k\to\infty} u_k = 0. \tag{3.12}$$

We claim that

$$\lim_{k\to\infty} v_k = 0. \tag{3.13}$$

In fact, for $\bar{\varepsilon} > 0$ there is a positive number γ such that

$$|F(t)-F(\bar{t})| < \text{for } |t-\bar{t}| < \gamma \text{ and } t,\ \bar{t}\in[c,\ d].$$

Now, we divide the interval $[c, d]$ into a finite numbers of intervals $[t_{r-1}, t_r]$ such that $c = t_0 < t_1 \dots < t_p = d$, $|t_{r-1}-t_r| < \gamma$ and $r = 1, \dots, p$. Thus

$$v_k = \sum_{r=1}^{p} \int_{t_{r-1}}^{t_r} \left[\left(F(t)-F(t_r)\right)\left(G_k'(t)-G'(t)\right)+F(t_r)\left(G_k'(t)-G'(t)\right)\right]\mathrm{d}t.$$

This implies (3.13) (by (2.7)—(2.8)). By relations (2.8), (3.10)—(3.13), we obtain

$$m\int_c^d G'(t)\mathrm{d}t \leqslant \int_c^d F(t)G'(t)\mathrm{d}t \leqslant M\int_c^d G'(t)\mathrm{d}t.$$

Putting in the last inequalities

$$\mu_1 = \frac{\int_c^d F(t)G'(t)\mathrm{d}t}{G^*(d)-G^*(c)}$$

we have (3.9) which completes the proof of Lemma 3.2.

LEMMA 3.3. *Let $F\in\mathscr{C}(a, b)$ and let $g\in\mathscr{V}_{\mathrm{loc}}(a, b)$. Then the function T defined as follows*

$$T(t) = \int_c^t F'(s)g(s)\mathrm{d}s = F(t)g^*(t)-F(c)g^*(c)-\int_c^d F(s)g'(s)\mathrm{d}s \tag{3.14}$$

is continuous on the interval (a, b).

Proof of Lemma 3.3. Let $g = g_1-g_2$, where $g_1' \geqslant 0$ anc $g_2' \geqslant 0$ on (a, b). Then, by (3.14), we have

$$T(t)-T(\bar{t}) = \int_{\bar{t}}^{t} F'(s)g(s)\mathrm{d}s = \left(F(t)-F(\bar{t})\right)g^*(t)+F(\bar{t})\left(g^*(t)-g^*(\bar{t})\right)-$$

$$-\int_{\bar{t}}^{t} F(s)g_1'(s)\mathrm{d}s+\int_{\bar{t}}^{t} F(s)g_2'(s)\mathrm{d}s.$$

Hence, by (3.9), we infer that

$$T(t)-T(\bar{t}) = \left(F(t)-F(\bar{t})\right)g^*(t)+F(\bar{t})\left(g_1^*(t)-g_1^*(\bar{t})\right)-\mu_1\left(g_1^*(t)-g_1^*(\bar{t})\right)+$$

$$+F(\bar{t})\left(g_2^*(\bar{t})-g_2^*(t)\right)+\mu_2\left(g_2^*(t)-g_2^*(\bar{t})\right) = \left(F(t)-F(\bar{t})\right)g^*(t)+$$

$$+\left(F(\bar{t})-\mu_1\right)\left(g_1^*(t)-g_1^*(\bar{t})\right)+\left(F(\bar{t})-\mu_2\right)\left(g_2^*(\bar{t})-g_2^*(t)\right),$$

where

$$\mu_1, \mu_2 \in [\inf_{s\in[t,\bar{t}]} F(s), \sup_{s\in[t,\bar{t}]} F(s)].$$

The last equality finishes the proof of Lemma 3.3.

LEMMA 3.4. *If* $y_1, y_2 \in \mathscr{V}_{\mathrm{loc}}(a,b)$ *and* $c \in \mathscr{C}(a,b)$, *then*

$$(y_1 y_2)c' = y_1(y_2 c'). \tag{3.15}$$

Proof of Lemma 3.4. The existence of the products $(y_1 y_2)c'$ and $y_1(y_2 c')$ follows from Lemmas 3.1 and 3.3. We get, by (3.8) and (2.6)

$$(y_1 y_2)c' = (y_1 y_2 c)' - (y_1 y_2)'c = (y_1 y_2 c)' - (y_1' y_2)\, c - y_1(y_2' c) \tag{3.16}$$

and

$$\begin{aligned} y_1(y_2 c') &= y_1[(y_2 c)' - y_2' c] = y_1(y_2 c)' - y_1(y_2' c) = \\ &= (y_1 y_2 c)' - y_1'(y_2 c) - y_1(y_2' c) = (y_1 y_2 c)' - (y_1' y_2)c - y_1(y_2' c), \end{aligned} \tag{3.17}$$

which completes the proof of Lemma 3.4.

LEMMA 3.5. *Let* $A = (A_{i,j})$ *satisfies assumptions* (3.1)—(3.2) *and let* $Z = (z_{i,j})$ *be a matrix such that* $Z \in \mathscr{V}_{\mathrm{loc}}^{n\times n}(a,b)$, $Z^*(t) = (z_{i,j}^*(t))$ *and*

$$\begin{cases} Z'(t) = A'(t)Z(t) \\ Z^*(t_0) = I. \end{cases} \tag{3.18}$$

Moreover, let

$$u(t) = \det(z_{i,j}^*(t)). \tag{3.19}$$

Then

$$u \in \mathscr{V}_{\mathrm{loc}}(a,b), \tag{3.20}$$

for every compact interval $[c,d] \subset (a,b)$

$$\inf_{t\in[c,d]} |u(t)| > 0 \tag{3.21}$$

and

$$\frac{1}{u} \in \mathscr{V}_{\mathrm{loc}}(a,b). \tag{3.22}$$

Proof of Lemma 3.5. The existence and uniqueness of matrix Z follow from [21]. Since $Z \in \mathscr{V}_{\mathrm{loc}}^{n\times n}(a,b)$, we obtain that $u \in \mathscr{V}_{\mathrm{loc}}(a,b)$. It is enough to prove (3.21). Let $[c,d]$ be an arbitrary compact interval such that $[c,d] \subset (a,b)$, $t_0 \in [c,d]$ and $\inf_{t\in[c,d]} |u(t)| = 0$. Then there exists a point $a_1 \in [c,d]$ possessing the following properties:

(i) $u(a_1) = 0$

(ii) there exists a sequence $\{t_k\}$ convergent to a_1 and

$$\lim_{k\to\infty} u(t_k) = u(a_1-) = 0 \quad \text{or} \quad \lim_{k\to\infty} u(t_k) = u(a_1+) = 0.$$

Hence, we have

$$\sum_{j=1}^{n} c_j z_j^*(a_1) = 0 \quad \text{or} \quad \sum_{j=1}^{n} c_j z_j(a_1-) = 0 \quad \text{or} \quad \sum_{j=1}^{n} c_j z_j(a_1+) = 0, \tag{3.23}$$

where $c_1^2+\ldots+c_n^2 > 0$, $z_j = (z_{1,j}, \ldots, z_{n,j})$, $0 = (0, \ldots, 0)$ and $j = 1, \ldots, n$. We put

$$v(t) = \sum_{j=1}^{n} c_j z_j^*(t) \text{ for } t \in (a,b). \tag{3.24}$$

Then v is a solution of the problem

$$\begin{cases} v'(t) = A'(t)v(t) \\ L(y) = 0, \end{cases} \tag{3.25}$$

where $L(v) = v^*(a_1)$ or $L(v) = v(a_1-)$ or $L(v) = v(a_1+)$. Taking into account [21] we infer that

$$\sum_{j=1}^{n} c_j z_j^*(t) = 0 \text{ for } t \in (a,b),$$

which is impossible (because $u(t_0) = 1$). Thus our assertion follows.

LEMMA 3.6. *Let $A = (A_{i,j})$ satisfies assumptions* (3.1)—(3.2). *Then the problem*

$$\begin{cases} y'(t) = A'(t)y(t) \\ y^*(t_0) = 0, \quad t_0 \in (a,b) \end{cases} \tag{3.26}$$

has only the zero solution in the class $\mathscr{V}\mathscr{C}^n(a,b)$.

Proof of Lemma 3.6. If $y \in \mathscr{V}\mathscr{C}^n(a,b)$ and if y satisfies (3.26), then

$$y_i'(t) = \sum_{j=1}^{n} A_{i,j}'(t)\big(v_j(t)c_j(t)\big),$$

where $A_{i,j}' \in \mathscr{M}(a,b)$, $v_j \in \mathscr{V}_{\mathrm{loc}}(a,b)$, $c_j \in \mathscr{C}(a,b)$ and $i,j = 1, \ldots, n$. By (2.5)—(2.6) we obtain that $y_i' \in \mathscr{M}(a,b)$ $(i = 1, \ldots, n)$. Hence $y \in \mathscr{V}_{\mathrm{loc}}^n(a,b)$. An application of [21] (Theorem 2.1) completes the proof of Lemma 3.6.

The matrix $Z = (z_{i,j})$ defined in Lemma 3.5 will be called the *fundamental matrix of solutions.*

Proof of Theorem 3.1. Let $Z(t) = (z_{i,j}^*(t))$ be the fundamental matrix of solutions. Then there exists the inverse matrix $Z^{-1}(t)$ for $t \in (a,b)$ (by Lemma 3.5) and $Z^{-1} \in \mathscr{V}_{\mathrm{loc}}^{n\times n}(a,b)$. We claim that

$$y = Zg, \tag{3.27}$$

where

$$g'(t) = Z^{-1}(t)B'(t), \quad g^*(t_0) = y_0 \tag{3.28}$$

is a solution of problem (3.4). In fact, by the equality

$$(Z^{-1}B') = (Z^{-1}B)' - (Z^{-1})'B,$$

Lemmas 3.1, 3.3, and 3.5 and relation (2.5) we obtain that $g \in \mathscr{C}^n(a, b)$. Using Lemma 3.4 we infer that Zg satisfies (1.0). Moreover, by (3.27)—(3.28), we get $g^*(t_0) = y_0$. Uniqueness assertion follows from Lemma 3.6.

REMARK 3.2. Let all assumptions of Theorem 3.1 be satisfied. Then it is not difficult to show (by Lemmas 3.1, 3.3—3.6) that the system (1.0) with the condition $y(t_0+) = y_0$ or $y(t_0-) = y_0$ has exactly one solution in the class $\mathscr{V}\mathscr{C}^n(a, b)$.

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