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COMPARISON THEOREMS FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

Abstract. The paper contains a generalization of some uniqueness criterions for the Itô's differential equations established by Skorokhod, Yamada and Watanabe. The results were generalized by applying more general non-linear integral inequalities and hence the stochastic versions of uniqueness criterions for non-random differential equations were obtained.

Introduction. In the present paper we shall discuss a problem of the pathwise uniqueness for solutions of stochastic differential equations. A comparison theorem for solutions of the Itô's stochastic differential equation was established by Skorokhod (see [2]), Yamada and Watanabe (see [4]). We can generalize those results by applying some more general non-linear integral inequalities (see [1], [3]) and hence we get stochastic versions of uniqueness criterions for non-random differential equations. We also consider more general class of equations

$$x_t = x_0 + \int_0^t f\,ds + \int_0^t g\,d\mu_s,$$

where $\int_0^t g\,d\mu_s$ is meant as a stochastic integral and μ_t is a local integrable martingale. In this paper for the proofs we use some ideas from [4].

Definitions and notations. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(\mathcal{F}_t, t \geqslant 0)$ be an increasing family of sub-σ-fields of $\mathcal{F}$. We shall assume that $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$ and that the family $(\mathcal{F}_t, t \geqslant 0)$ is continuous from the right. We shall say that function f belongs to $D[0, T]$ iff f is finite, right continuous and has finite left limits for all $t \in [0, T]$.

Process $(x_t, t \geqslant 0)$ is *cadlag*, if, for almost all ω, the function $t \to x_t(\omega)$ is finite, right continuous and has finite left limits for all $t \in \mathbf{R}_+$. Let $\mathcal{M}_2$ be the set of all martingales μ_t with respect to the family $(\mathcal{F}_t, t \geqslant 0)$, such that μ_t is cadlag and

$$\sup_{t \geqslant 0} E\mu_t^2 < \infty$$

holds true. We shall say that process μ_t is *continuous*, if, for almost all ω the function $t \to x_t(\omega)$ is continuous. Let $\mathcal{M}_2^c$ be a subset of $\mathcal{M}_2$, containing all continuous martingales. For each $\mu_t \in \mathcal{M}_2$ μ_t^2 is a submartingale, and from Meyer's theorem there exists only one integrable process $\langle \mu, \mu \rangle_t$ and a martingale v_t such that

$$\mu_t^2 = v_t + \langle \mu, \mu \rangle_t$$

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holds. Let $\mathcal{M}_2^c$ be a class of all martingales μ_t such that $\langle\mu,\mu\rangle_t$ is continuous. Let $f:[0,\infty)\times\Omega\to\mathbf{R}$ be a random function. We assume that

(i) f is $\mathcal{B}\times\mathcal{F}$–measurable and for every $t, f(t,\cdot)$ is $\mathcal{F}_t$–measurable.

(ii) $P\{\omega\in\Omega: \int_0^\infty f^2(t,\omega)\mathrm{d}\langle\mu,\mu\rangle_t < \infty\} = 1.$

Let us denote by L_μ^2 the class of all random functions satisfying (i) and (ii), and by M_μ^2 we cenote the class of all $f\in L_\mu^2$, satisfying the condition

(iii) $E\int_0^\infty f^2\mathrm{d}\langle\mu,\mu\rangle_t < \infty.$

Let $\int_0^\infty f\,\mathrm{d}\mu_t$ denote stochastic integral, where $\mu_t\in\mathcal{M}_2^c$. It is known that stochastic integral exists for all $f\in L_\mu^2$. If $f\in M_\mu^2$, then for each $t\in\mathbf{R}_+$

$$I_t = \int_0^t f\,\mathrm{d}\mu_s$$

is an integrable martingale and $E[\int_0^t f\,\mathrm{d}\mu_s]^2 = E[\int_0^t f^2\mathrm{d}\langle\mu,\mu\rangle_s]$ holds true.

Process $\mu_t, t\geqslant 0$ is a *local integrable martingale*, if there exists an increasing sequence of stopping times (τ_n) such that $\lim_n \tau_n = +\infty$ a.e. and each τ_n reduces the local martingale μ_t. We recall that τ_n *reduces* μ_t, iff $\mu_{t\wedge\tau_n}$ is an uniformly integrable martingale $\mathcal{F}_{t\wedge\tau_n}$–adapted. The class of all local integrable martingales we denote by $l\mathcal{M}_2$. If $f\in L_\mu^2$, $\mu_t\in\mathcal{M}_2^c$, then $I_t = \int_0^t f\,\mathrm{d}\mu_s\in l\mathcal{M}_2^c$ and if $\mu_t\in l\mathcal{M}_2^c$ then $I_t = \int_0^t f\,\mathrm{d}\mu_s := \lim_n \int_0^t f(s\wedge\tau_n)\mathrm{d}\mu_{s\wedge\tau_n}\in l\mathcal{M}_2^c$.

Uniqueness of solutions of stochastic differential equations. Let f and g be two functions mapping $\mathbf{R}_+\times\mathbf{R}\to\mathbf{R}$. We shall assume that f and g are Borel measurable and bounded on every finite interval. Hence, if $x_t\in D$, then the functions

$$\omega\to f(t,x_t), \qquad \omega\to g(t,x_t)$$

are $\mathcal{F}_t$ – measurable. Processes $f(t,x_t)$ and $g(t,x_t)$ are Borel measurable and locally bounded a.e. so the integrals

$$\int_0^t f(s,x_s)\mathrm{d}s \text{ and } \int_0^t g(s,x_s)\mathrm{d}\langle\mu,\mu\rangle_s$$

exist. Let us consider the equation:

(1) $$x_t(\omega) = x_0 + \int_0^t f(s,x_s)\mathrm{d}s + \int_0^t g(s,x_s)\mathrm{d}\mu_s, \qquad t\geqslant 0,\ \mu_t\in l\mathcal{M}_2^c.$$

DEFINITION 1. By a *solution of the equation* (1) we mean a probability space with an increasing family of sub-σ-fields $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$ and a family of stochastic processes (x_t, μ_t) defined on it such that

(i) with probability one x_t and μ_t belong to D and $\mu_0 = 0$,

(ii) they are adapted to $\mathscr{F}_t$ for each t,

(iii) μ_t is an integrable or local integrable martingale,

(iv) (x_t, μ_t) satisfies

$$x_t - x_0 = \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\mu_s \qquad \text{a.e.}$$

DEFINITION 2. We shall say that the *pathwise uniqueness* holds for (1) if, for any two solutions (x_t, μ_t) and (x_t', μ_t') defined on the same probability space $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$, $x_0 = x_0'$ and $\mu_t \equiv \mu_t'$ imply $x_t \equiv x_t'$.

Let $\mu_t \in \mathscr{M}_2^c$ and f, g are Borel measurable and bounded on every finite interval. We can prove the following.

THEOREM 1. *Let*

$$x(t) = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\mu_s, \qquad t \geqslant 0$$

and assume that

1° *there exists a positive increasing function* $r(u)$, $u \in (0, \infty)$ *such that*

$$|g(s, x) - g(s, y)| \leqslant r(|x-y|), \qquad x, y \in \mathbf{R}$$

and

$$\int_{0^+} r^{-2}(u)du = +\infty,$$

2° *there exists a function* $\Phi: \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, *continuous and non-decreasing in* $x \in \mathbf{R}_+$, *such that for every* (t, x), (t, y), $t \geqslant 0$, $x, y \in \mathbf{R}$

$$|f(t, x) - f(t, y)| \leqslant \Phi(t, |x-y|),$$

3° *for every random variable* $\xi: \Omega \to \mathbf{R}_+$ *such that* $E\xi < \infty$ *the inequality*

$$E\Phi(t, \xi) \leqslant V\Phi(t, E\xi)$$

for some constant V *is true,*

4° *the right – hand maximum solution* $M(t; 0,0)$ *of the non-random differential equation*

$$y' = V\Phi(t, y)$$

through (0,0) *exists in every interval* $[0, t]$, $t \geqslant 0$. *Then for every two solutions* x_t, x_t' *of* (1) *we have*

$$E|x_t - x_t'| \leqslant M(t; 0,0), \qquad t \geqslant 0.$$

Proof. Let $a_0 = 1 > a_1 > a_2 > \ldots > a_k \to 0$ be defined by

$$\int_{a_1}^{a_0} r^{-2}(u)\mathrm{d}u = 1, \int_{a_2}^{a_1} r^{-2}(u)\mathrm{d}u = 2, \ldots \int_{a_k}^{a_{k-1}} r^{-2}(u)\mathrm{d}u = k, \ldots$$

Then there exists twice continuously differentiable function $\psi_k(u)$ on $[0, \infty)$ such that $\psi_k(0) = 0$,

$$\psi_k'(u) = \begin{cases} 0, & 0 \leqslant u \leqslant a_k, \\ \text{between } 0 \text{ and } 1, & a_k < u < a_{k-1}, \\ 1, & a_{k-1} \leqslant u, \end{cases}$$

$$\psi_k''(u) = \begin{cases} 0, & 0 \leqslant u \leqslant a_k, \\ \text{between } 0 \text{ and } \frac{2}{k} r^{-2}(u), & a_k < u < a_{k-1}, \\ 0, & a_{k-1} \leqslant u. \end{cases}$$

We extend $\psi_k(u)$ on $(-\infty, \infty)$ such that

$$\psi_k(u) = \psi_k(|u|).$$

$\psi_k(u)$ is a twice continuously differentiable function on $(-\infty, \infty)$ and $\psi_k(u) \uparrow |u|$.

Let (x_t, μ_t) and (x_t', μ_t') be two solutions of (1) on the same probability space satisfying the following

$$x_0 = x_0' \text{ and } \mu_t \equiv \mu_t'.$$

Then

$$x(t) - x'(t) = \int_0^t [f(s, x_s) - f(s, x_s')]\,\mathrm{d}s + \int_0^t [g(s, x_s) - g(s, x_s')]\,\mathrm{d}\mu_s.$$

By Itô's formula we have

$$\begin{aligned} \psi_n(x(t) - x'(t)) = & \int_0^t \psi_n'(x(s) - x'(s))\,[f(s, x_s) - f(s, x_s')]\,\mathrm{d}s + \\ & + \int_0^t \psi_n'(x(s) - x'(s))\,[g(s, x_s) - g(s, x_s')]\,\mathrm{d}\mu_s + \\ & + \frac{1}{2}\int_0^t \psi_n''(x(s) - x'(s))\,[g(s, x_s) - g(s, x_s')]^2 \mathrm{d}\langle \mu, \mu \rangle_s = \\ = & I_1 + I_2 + I_3. \end{aligned}$$

$I_2 \in l\mathcal{M}_2^c$, hence $E(I_2) = 0$ for $t \leqslant \tau_n$, where τ_n is stopping time reducing I_2.

$$E|I_1| \leqslant E\left\{\int_0^t |f(s, x_s) - f(s, x_s')|\,\mathrm{d}s\right\},$$

$$E|I_3| \leqslant \frac{1}{2} E\{\int_0^t \psi_n''(x(s)-x'(s))r^2(|x_s-x_s'|)\mathrm{d}\langle\mu,\mu\rangle_s\} \leqslant$$

$$\leqslant \frac{1}{2} \max_{a_n \leqslant |u| \leqslant a_{n-1}} \{\psi_n''(u)r^2(|u|)\} E(\langle\mu,\mu\rangle_t - \langle\mu,\mu\rangle_0) \leqslant$$

$$\leqslant \frac{1}{2}\cdot C \cdot \frac{2}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

Also we have by assumption

$$\psi_n(x(s)-x'(s)) \uparrow |x(s)-x'(s)|.$$

Hence, and by Fatou's lemma we have

$$E|x(t)-x'(t)| \leqslant E\int_0^t |f(s,x_s)-f(s,x_s')|\mathrm{d}s$$

and

$$E|x(t)-x'(t)| \leqslant E\int_0^t \Phi(s,|x(s)-x'(s)|)\mathrm{d}s,$$

$$E|x(t)-x'(t)| \leqslant V\int_0^t \Phi(s,E|x(s)-x'(s)|)\mathrm{d}s,$$

therefore by Opial's theorem

(2) $$E|x(t)-x'(t)| \leqslant M(t;0,0), \qquad t \leqslant \tau_n.$$

As τ_n was reducing sequence of stopping times such that $\lim_n \tau_n = +\infty$ a.e. we have for sufficiently N $\tau_n \wedge t = t$ a.e. and that completes the proof.

COROLLARY. *If* $(M(t;0,0) \equiv 0)$, *then, under the assumptions of Theorem* 1, *the pathwise uniqueness holds for solutions of* (1).

THEOREM 2 (Stochastic version of Osgood's criterion). *If the assumptions of Theorem* 1 *are satisfied, and* $\Phi(t,x) = a(t)q(x)$, *where* $a(t)$ *is non-negative, continuous function on* $[0,\infty)$ *and* $q(x)$ *is continuous concave, non-decreasing in* $\mathbf{R}$, $q(x) \neq 0$ *and*

$$\int_{0^+} \frac{1}{q(x)}\mathrm{d}x = +\infty$$

then pathwise uniqueness holds for (1).

Proof. In a similar way as in Theorem 1 we get

$$E|x(t)-x'(t)| \leqslant E\int_0^t a(s)q(|x(s)-x'(s)|)\mathrm{d}s \leqslant$$

$$\leqslant \int_0^t a(s)q(E|x(s)-x'(s)|)\mathrm{d}s,$$

and

$$(3)\qquad E|x(t)-x'(t)| \leqslant \varepsilon + \int_0^t a(s)q(E|x(s)-x'(s)|)ds, \qquad \varepsilon > 0.$$

Inequaluty (3) is Bihari's type, hence we have

$$E|x(t)-x'(t)| \leqslant G^{-1}\left[G(\varepsilon)+\int_0^t a(s)ds\right], \qquad t \geqslant 0,$$

where $G(t) = \int_{-\infty}^{u} \frac{1}{q(s)} ds$, and, for ε tends to 0, we have $G^{-1}\left[G(\varepsilon)+\int_0^t a(s)ds\right] \to 0$.

REMARK. Under the assumptions of Theorem 2, if $a(t) \equiv 1$, we have often used inequality:

$$E|x(t)-x'(t)| \leqslant \int_0^t q(E|x(s)-x'(s)|)ds, \qquad t \geqslant 0,$$

hence pathwise uniqueness holds for (1).

Uniqueness of solutions of stochastic differential equations in multi-dimensional case. Let $\sigma(t,x) = [\sigma_{i,j}(t,x)]$, $b(t,x) = [b_i(t,x)]$, $i = 1, \ldots, n$, $j = 1, \ldots, r$, be defined on $[0,\infty)\times \mathbf{R}^n$, Borel measurable and bounded. We consider the equation:

$$(4)\qquad dx_t = \sigma(t,x_t)d\mu_t + b(t,x_t)dt, \qquad t \geqslant 0, \mu_t \in \mathcal{M}_2^c,$$

or, in component wise

$$dx_i(t) = \sum_{j=1}^{r} \sigma_{i,j}(t,x_t)d\mu_j(t) + b_i(t,x_t)dt, \qquad i = 1, \ldots, n.$$

Let $\mu_t = (\mu_1(t), \ldots, \mu_r(t)) \in \mathcal{M}_2^c$ *and* $\langle \mu_i, \mu_k \rangle_t$ *be absolutely continuous with respect to Lebesgue measure for* $k, i = 1, \ldots, r$. *Let the densities* $\varphi_{i,k}(t)$ *be bounded for* $k, i = 1, \ldots, r$ *and the following assumptions be satisfied*:

1° *there exists a positive, increasing function* $\varrho(x)$, $x \in (0,\infty)$, $\varrho(0) = 0$ *such that*

$$\|\sigma(t,x)-\sigma(t,y)\| \leqslant \varrho(|x-y|), \qquad (t,x) \in [0,\infty)\times \mathbf{R}^n,$$

2° *there exists positive, non-decreasing function* $\varphi(x)$, $x \in [0,\infty)$, *such that*

$$|b(t,x)-b(t,y)| \leqslant \varphi(|x-y|), \qquad x, y \in \mathbf{R}^n,$$

3° *the function*

$$\varrho^2(x)x^{-1} + \varphi(x)$$

is concave,

4° $\int_{0^+} [\varrho^2(x)x^{-1}+\varphi(x)]^{-1}dx = +\infty$.

Then the pathwise uniqueness holds for (4).

Proof. In a similar way as in Theorem 1 we define functions $\psi_m(x)$ on $[0, \infty)$ such that

$$\psi'_m(u) = \begin{cases} 0, & 0 \leqslant u \leqslant a_m, \\ \text{between } 0 \text{ and } \varrho^2(u)u, & a_m < u < a_{m-1}, \\ 0, & a_{m-1} \leqslant u, \end{cases}$$

$$f_m(x) := \psi_m(|x|) \text{ for } x \in \mathbf{R}^n.$$

Let $x(t)$ and $x'(t)$ be the solutions of equation (4). By Itô's formula we have

$$f_m(x(t)-x'(t)) = \text{a martingale} + \sum_{i=1}^{n} \int_0^t \frac{\partial f_m}{\partial x_i}(x_s - x'_s)\left[b_i(s, x_s) - b_i(s, x'_s)\right] ds +$$

$$+\frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial f_m}{\partial x_i \partial x_j}(x_s - x'_s)\Big[\sum_{k=1}^{r} (\sigma_{i,k}(s, x_s) - \sigma_{i,k}(s, x'_s)) *$$

$$*(\sigma_{j,k}(s, x_s) - \sigma_{j,k}(s, x'_s))\Big] d\langle \mu_i, \mu_j \rangle_s = I_1 + I_2 + I_3$$

but as ψ'_m is bounded,

$$\left|\frac{\partial f_m}{\partial x_i}\right| = \left|\psi'_m(|x|)\frac{x_i}{|x|}\right| \leqslant K_0, \qquad K_0 = \text{const.},$$

$$\left|\frac{\partial f_m}{\partial x_i \partial x_j}\right| \leqslant K_1 \frac{1}{|x|}\chi_{[x \neq 0]} + \psi''_m(|x|) \cdot K_2, \qquad K_1, K_2 = \text{const.},$$

$$E(I_1) = 0,$$

$$E(I_3) \leqslant \frac{1}{2} E\left[\int_0^t K_1 \frac{1}{|x_s - x'_s|} \chi_{[x_s \neq x'_s]} \Big\{ \sum_{i,j=1}^{n} \sum_{k=1}^{r} (\sigma_{i,k}(s, x_s) - \sigma_{i,k}(s, x'_s)) *\right.$$

$$\left. *(\sigma_{j,k}(s, x_s) - \sigma_{j,k}(s, x'_s)) d\langle \mu_i, \mu_j \rangle_s \Big\}\right],$$

and from the assumption we have

$$E(I_3) \leqslant \frac{1}{2} E\left[V \int_0^t K_1 |x_s - x'_s|^{-1} \chi_{[x_s \neq x'_s]} \varrho^2(|x_s - x'_s|) ds\right] +$$

$$+\frac{1}{2} E\left[V \int_0^t K_2 \psi''_m(|x_s - x'_s|) \varrho^2(|x_s - x'_s|) ds\right.$$

and

$$\frac{1}{2} E\left[V \int_0^t K_2 \psi''_m(|x_s - x'_s|) \varrho^2(|x_s - x'_s|) ds \leqslant\right.$$

$$\leqslant \frac{1}{2} K_2 V \int_0^t E|x_s - x'_s| \chi_{[a_m \leqslant x_s - x'_s \leqslant a_{m-1}]} ds \leqslant C \cdot t \cdot a_{m-1} \to 0.$$

By Fatou's lemma

$$E|x_t - x'_t| \leqslant C \int_0^t E\{\varphi(|x_s - x'_s|) + |x_s - x'_s|^{-1} \varrho^2(|x_s - x'_s|)\} ds,$$

By Jensen's inequality

$$E|x_t - x'_t| = 0, \qquad t \geqslant 0,$$

which completes the proof.

Uniqueness criterion for some integral equations. We consider the equation

$$x_t = x_0 + \int_0^t g(s, x_s) d\mu_s + \int_0^t f(s, x_s) d\alpha_s, \qquad t \geqslant 0; \tag{5}$$

x_0 is $\mathscr{F}_0$ – measurable random variable; f and g are Borel measurable and bounded on every finite interval, $\mu_t \in l\mathscr{M}_2^c$, $\alpha_t \in V^+ - V^+$ where V^+ is the set of all increasing, adapted, cadlag processes A_t such that $A_0 = 0$. Integral $\int_0^t f(s, x_s) d\alpha_s$ is meant in a Stjeltjes-Lebesgue sense. Under the above assumptions both integrals exist and we always can write the right term of equality (5).

Let $\langle \alpha, \alpha \rangle_t$ be absolute continuous with respect to Lebesgue measure and its density φ_t be bounded in **R** *and let the assumptions of Theorem* 1 *be satisfied. Then pathwise uniqueness for* (5) *holds.*

We can prove it in a similar way as in Theorem 1.

REFERENCES

[1] J. BŁAŻ, *On the estimates of solutions of Itô random integral equations*, Prace Nauk. Uniw. Śląsk., Prace Matematyczne 11 (1981), 26—32.

[2] I. I. GIHMAN, A. V. SKOROKHOD, *Theory of stochastic processes*, Moscow, 1975, (in russian).

[3] J. SZARSKI, *Differential inequalities*, PWN, Warszawa, 1967.

[4] T. YAMADA, S. WATANABE, *On the uniqueness of solutions of stochastic differential equations*, J. Math. Kyoto Univ. 11 (1971), 155—167; 11 (1971), 553—563.