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ON THE EXISTENCE OF OPTIMAL CONTROL FOR GENERAL STOCHASTIC EQUATIONS

Abstract. In this paper we consider the problem of optimal control for general stochastic differential equation of Itô type. We prove the existence of solutions of this equation under weaker assumptions than in [2]. Moreover, we prove the compactness of the space of solutions and the existence of optimal control.

In this paper we consider the problem of optimal control for general stochastic differential equation of Itô type. The equation of this type has been considered in the paper [2] by Fleming and Nisio, where under certain assumptions the existence and uniqueness of solutions and existence of optimal control were proved.

In the present paper using Opial's theorem on the differential inequalities and some ideas of the paper [4] we prove the existence of solutions of this equation under weaker assumptions than in [2]. Similar assumptions are made for stochastic differential equation without delay and control by Błaż in [1]. Moreover, in this paper we prove the compactness of the space of solutions and obtain the theorem on the existence of optimal control similar to the one of [2].

Preliminaries. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Given a stochastic process $X(t)$, $-\infty < t < \infty$, denote by $\mathscr{B}_{u,v}(X)$ the least σ-algebra for which $X(t)$ is measurable for $t \in [u,v]$. The Wiener process is denoted by $B(t)$, $-\infty < t < \infty, B(0) \equiv 0$. $\mathscr{B}_{u,v}(\mathrm{d}B)$ denotes the least σ-algebra generated by $\{B(t)-B(s), u \leqslant s \leqslant t \leqslant v\}$. The least σ-algebra that contains $\mathscr{B}_1, \mathscr{B}_2, \ldots$ is denoted by $\mathscr{B}_1 \vee \mathscr{B}_2 \vee \ldots$.

For fixed s, we define the process $\Pi_s X$ by:

$$(\Pi_s X)(t) = X(s+t), \qquad t \leqslant 0. \tag{1.1}$$

By $\mathscr{C}_-$ we denote the space of all real continuous functions defined on the negative half-line $(-\infty, 0]$ with the metric ϱ_-,
where

$$\varrho_-(f,g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f-g\|_m}{1+\|f-g\|_m} \tag{1.2}$$

with

$$\|h\|_m = \sup_{t \in [-m,0]} |h(t)|.$$

Let $a(t,f)$ and $b(t,f,g)$ be real valued continuous functionals defined on $[0,\infty) \times \mathscr{C}_-$ and $[0,\infty) \times \mathscr{C}_- \times \mathscr{C}_-$, respectively. Let $X_-(t), t \leqslant 0$, be a con-

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tinuous stochastic process. A stochastic process $U(t), t \geqslant 0$, is called an *admissible control*, or to be more precise, the triple (X_-, U, B) is called an *admissible system* if with probability one:

$$(1.3) \qquad |U(t)-U(s)| \leqslant |t-s|, \quad 0 \leqslant t, s < \infty, U(0) = 0,$$

and if

$$(1.4) \qquad \mathscr{B}(X_-) \vee \mathscr{B}_{0,t}(U) \vee \mathscr{B}_{-\infty,t}(B) \text{ is independent of } \mathscr{B}_{t,\infty}(\mathrm{d}B)$$

for every $t \geqslant 0$.

A continuous stochastic process $X(t)$ is called a *solution of a stochastic differential equation* (for an admissible system (X_-, U, B))

$$(1.5) \qquad \mathrm{d}X(t) = a(t, \Pi_t X)\mathrm{d}U(t) + b(t, \Pi_t X, \Pi_t B)\mathrm{d}B(t)$$

with the past condition X_-, if

$$(1.6) \qquad X(t) = X_-(t), \quad t \leqslant 0,$$

$$(1.7) \qquad \mathscr{B}_{-\infty,t}(X) \vee \mathscr{B}_{0,t}(U) \vee \mathscr{B}_{-\infty,t}(B) \text{ is independent of } \mathscr{B}_{t,\infty}(\mathrm{d}B)$$

for every $t \geqslant 0$, and if with probability one

$$(1.8) \qquad X(t) = X(0) + \int_0^t a(\tau, \Pi_\tau X)\mathrm{d}U(\tau) + \int_0^t b(\tau, \Pi_\tau X, \Pi_\tau B)\mathrm{d}B(\tau).$$

Itô's formula [3]. *Let $f:\mathbf{R} \to \mathbf{R}$ be a twice continuously differentiable function and let $\alpha(t)$ be a continuous stochastic process which may be represented as the difference of two increasing processes. Suppose that*

$$(2.1) \qquad \xi(t) = \xi(u) + \int_u^t \mathscr{R}(s)\mathrm{d}\alpha(s) + \int_u^t A(s)\mathrm{d}B(s).$$

Then we have

$$(2.2) \qquad f(\xi(t)) - f(\xi(u)) = \int_u^t f'_x(\xi(s))\mathscr{R}(s)\mathrm{d}\alpha(s) + \\ + \frac{1}{2}\int_u^t f''_{xx}(\xi(s))A^2(s)\mathrm{d}s + \int_u^t f'_x(\xi(s))A(s)\mathrm{d}B(s).$$

In particular, if $f(x) = x^4$ then

$$(2.3) \qquad [\xi(t)]^4 = [\xi(u)]^4 + 4\int_u^t \xi^3(s)\mathscr{R}(s)\mathrm{d}\alpha(s) \\ + 6\int_u^t \xi^2(s)A^2(s)\mathrm{d}s + 4\int_u^t \xi^3(s)A(s)\mathrm{d}B(s).$$

If

$$\int_u^t E[\xi^6(s)A^2(s)]\mathrm{d}s < \infty$$

then

$$(2.4) \qquad E[\xi(t)]^4 = E[\xi(u)]^4 + 4E\int_u^t \xi^3(s)\mathscr{R}(s)\mathrm{d}\alpha(s) + 6\int_u^t E[\xi^2(s)A^2(s)]\mathrm{d}s.$$

Using $4\xi^3\mathscr{R} \leqslant 3\xi^4 + \mathscr{R}^4$ and $2\xi^2A^2 \leqslant \xi^4 + A^4$ we have

$$E[\xi(t)]^4 \leqslant E[\xi(u)]^4 + 3E\int_u^t [\xi^4(s) + \mathscr{R}^4(s)]\mathrm{d}\alpha(s) + 3\int_u^t E[\xi^4(s) + A^4(s)]\mathrm{d}s.$$

If in particular $\alpha(s)$ with probability one satisfies the Lipschitz condition

$$|\alpha(t) - \alpha(s)| \leqslant |t-s|$$

then with probability one $\alpha(t)$ has almost everywhere a derivative $\alpha'(t)$ bounded by one and if a sum $\xi^4(s) + \mathscr{R}^4(s)$ is continuous then

$$\int_u^t [\xi^4(s) + \mathscr{R}^4(s)]\mathrm{d}\alpha(s) \leqslant \int_u^t [\xi^4(s) + \mathscr{R}^4(s)]\mathrm{d}s.$$

By (2.4) we have

$$(2.5) \qquad E[\xi(t)]^4 \leqslant E[\xi(u)]^4 + 6\int_u^t E[\xi^4(s) + \mathscr{R}^4(s) + A^4(s)]\mathrm{d}s.$$

Prohorov Metric. Let Σ be a separable complete metric space with the metric ϱ and $\mathscr{B}_\varrho$ the σ-algebra of Borel sets on Σ. Given two probability measures μ_1, μ_2 on Σ, we define the Prohorov metric $L(\mu_1, \mu_2)$. Let ε_{12} be the infimum of ε such that for every closed subset F of Σ

$$\mu_1(F) \leqslant \mu_2(O_\varepsilon(F)) + \varepsilon$$

where $O_\varepsilon(F)$ is the ε-neighborhood of F. Define ε_{21} by changing μ_1 on μ_2 and μ_2 on μ_1 in the definition of ε_{12}. Set

$$L(\mu_1, \mu_2) = \max(\varepsilon_{12}, \varepsilon_{21}).$$

The set of all probability measures on $(\Sigma, \mathscr{B}_\varrho)$ with metric L is a separable complete metric space.

Let $X(\omega)$ be a Σ-valued random variable defined on a probability space $(\Omega, \mathscr{F}, P)$. The random variable X defines a probability measure μ_X on Σ

$$\mu_X(B) = P(\{\omega\colon\ X(\omega) \in B\}) \text{ for } B \in \mathscr{B}_\varrho.$$

Let $\chi(\Sigma)$ be the system of all Σ-valued random variables (they need not be defined on the same probability space). We define a distance between two random variables $X_1, X_2 \in \chi(\Sigma)$ by:

$$L(X_1, X_2) = L(\mu_{X_1}, \mu_{X_2}).$$

In this way, we can define L-convergence, L-compactness, etc., on $\chi(\Sigma)$. Moreover, we have the following

THEOREM (Skorohod, [4]). *If $X_n, n = 1,2, \dots$ (not necessarily defined on the same probability space) is an L-Cauchy sequence, then there are a probability space $(\Omega, \mathscr{F}, P)$ and a sequence of random variables $Y, Y_n, n = 1,2, \dots$ defined on Ω such that*

$$L(Y_n, X_n) = 0 \tag{3.1}$$

and

$$P(\varrho(Y_n, Y) \to 0) = 1. \tag{3.2}$$

So if in $\chi(\Sigma)$ we identify random variables X, Y which have the same probability law then Skorohod's theorem implies that $(\chi(\Sigma), L)$ is a complete space. The convergence in the sense of the metric L means the weak convergence.

A subsystem $\mathscr{H} = \{X_\alpha : \alpha \in A\}$ of $\chi(\Sigma)$ is *weakly compact* if $\mathscr{H}$ is compact under weak convergence.

We shall use the following

THEOREM (Prohorov, [4]). *In order for $\mathscr{H} = \{X_\alpha : \alpha \in A\}$ to be weakly compact in $\chi(\Sigma)$, it is necessary and sufficient that for every $\varepsilon > 0$, there exists a compact subset K_ε of Σ such that*

$$P(X_\alpha \in K_\varepsilon) > 1 - \varepsilon \text{ for every } \alpha \in A. \tag{3.3}$$

Let (Σ_i, ϱ_i), $i = 1,2, \dots, n$ be separable complete metric spaces. Then the direct product space $\Sigma = \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$ is also a separable complete metric space with metric

$$\varrho(x, y) = \sum_{i=1}^{n} \varrho_i(x_i, y_i), \ x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n).$$

Let $\mathscr{H} = \{X_\alpha = (X_{\alpha,1}, \dots, X_{\alpha,n}) : \alpha \in A\}$ be a subsystem of $\chi(\Sigma)$. Then $\mathscr{H}$ is weakly compact if and only if its component $\mathscr{H}_i = \{X_{\alpha,i} : \alpha \in A\}$ is weakly compact for every $i = 1,2, \dots, n$.

In this paper we consider also $(\mathscr{C}_+, \varrho_+)$ and $(\mathscr{C}, \varrho)$-spaces of all continuous functions on $[0, \infty)$ and $(-\infty, \infty)$, respectively, where

$$\varrho_+(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f-g\|_m}{1+\|f-g\|_m}, \ \|h\|_m = \max_{t \in [0,m]} |h(t)|;$$

$$\varrho(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f-g\|_m}{1+\|f-g\|_m}, \ \|h\|_m = \max_{t \in [-m,m]} |h(t)|.$$

They are separable complete metric spaces.

We have the following useful condition for weakly compactness of $\mathscr{C}_+$.

LEMMA 1 [4, Lemma 3.2]. *$\mathscr{H} \subset \chi(\mathscr{C}_+)$ is weakly compact if there exist $c > 0$ and $c_m > 0, m = 1,2, \dots$ such that, for every $X = (X(t) : t \geqslant 0) \in \mathscr{H}$,*

$$EX^4(0) < c, \tag{3.4}$$

$$E|X(t) - X(s)|^4 \leqslant c_m |t-s|^{3/2}, \ 0 \leqslant t, s \leqslant m. \tag{3.5}$$

This condition holds also for $\mathscr{H} \subset \chi(\mathscr{C}_-)$ and $\mathscr{H} \subset \chi(\mathscr{C})$.

Aproximate sums of a stochastic integral. Let A be a parameter set. For each $\alpha \in A$ we have a continuous stochastic process $X_\alpha = X_\alpha(t)$, $-\infty < t < \infty$, an admissible control $U_\alpha = U_\alpha(t), t \geqslant 0$ and a Wiener process B_α such that $\mathscr{B}_{-\infty,t}(X_\alpha) \vee \mathscr{B}_{0,t}(U_\alpha) \vee \mathscr{B}_{-\infty,t}(B_\alpha)$ is independent of $\mathscr{B}_{t,\infty}(\mathrm{d}B_\alpha)$ for every $t \geqslant 0$.

Let $a(t,f)$ and $b(t,f,g)$ be continuous for $t \in [0,\infty)$ and $f, g \in \mathscr{C}_-$. The following stochastic integral is defined:

$$J_\alpha = \int_0^t a(\tau, \Pi_\tau X_\alpha)\mathrm{d}U_\alpha(\tau) + \int_0^t b(\tau, \Pi_\tau X_\alpha, \Pi_\tau B_\alpha)\mathrm{d}B_\alpha(\tau). \tag{4.1}$$

Let $\Delta = \{0 = s_0 < s_1 < \ldots < s_n = t\}$ and $J_\alpha(\Delta)$ be an approximate sum of J_α for Δ:

$$\begin{aligned} J_\alpha(\Delta) = &\sum_{l=0}^{n-1} a(s_l, \Pi_{s_l} X_\alpha)\left[U_\alpha(s_{l+1}) - U_\alpha(s_l)\right] + \\ &+ \sum_{l=0}^{n-1} b(s_l, \Pi_{s_l} X_\alpha, \Pi_{s_l} B_\alpha)\left[B_\alpha(s_{l+1}) - B_\alpha(s_l)\right]. \end{aligned} \tag{4.2}$$

By the definition of stochastic integral J_α we have that $J_\alpha(\Delta) \to J_\alpha$ in probability for each α as $\|\Delta\| = \max(s_{l+1} - s_l) \to 0$, i.e. there exists $\delta = \delta(\varepsilon, \alpha)$ such that $\|\Delta\| < \delta$ implies

$$P(|J_\alpha(\Delta) - J_\alpha| > \varepsilon) < \varepsilon.$$

LEMMA 2 [2, Lemma 4]. *Let $a(t,f)$ be a continuous functional on $[0,\infty) \times \mathscr{C}_-$. Then $a(t, \Pi_t \varphi)$ is continuous in (t, φ) of $[0,\infty) \times \mathscr{C}$. Similarly $b(t, \Pi_t \varphi, \Pi_t \psi)$ is continuous in (t, φ, ψ)* of $[0,\infty) \times \mathscr{C} \times \mathscr{C}$.

LEMMA 3 [2, Lemma 6]. *If $\{X_\alpha : \alpha \in A\}$ is weakly compact then there is a $\delta = \delta(\varepsilon)$ independent of α such that $\|\Delta\| < \delta$ implies*

$$P(|J_\alpha(\Delta) - J_\alpha| > \varepsilon) < \varepsilon \textit{ for every } \alpha \in A. \tag{4.3}$$

Existence of solution. We consider the stochastic differential equation

$$\mathrm{d}X(t) = a(t, \Pi_t X)\mathrm{d}U(t) + b(t, \Pi_t X, \Pi_t B)\mathrm{d}B(t) \text{ for } t \geqslant 0$$

with past condition

$$X(t) = X_-(t) \text{ for } t \leqslant 0.$$

Let us impose the following assumptions:

(A.1) $a: [0,\infty) \times \mathscr{C}_- \to \mathbf{R}$, $b: [0,\infty) \times \mathscr{C}_- \times \mathscr{C}_- \to \mathbf{R}$ are continuous;

(A.2) there exist a bounded measure $\mathrm{d}K_1$ on $(-\infty, 0]$ and a function $\Phi_1: \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$a^4(t,f) \leqslant \Phi_1\Big(t, \int_{-\infty}^{0} |f(s)|^4 \mathrm{d}K_1(s)\Big);$$

(A.3) there exist a positive integer M, two bounded measures dK_2, dK_3 on $(-\infty, 0]$, an increasing function $G(t)$ and a function $\Phi_2: \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$b^4(t,f,g) \leqslant \Phi_2(t, \int_{-\infty}^{0} |f(s)|^4 dK_2(s) + \int_{-\infty}^{0} g^{2M}(s) dK_3 s))$$

and

$$\sup_{0 \leqslant \tau \leqslant t} \int_{-\infty}^{0} |s+\tau|^M dK_3(s) \leqslant G(t);$$

(A.4) for all $t \in [0, \infty)$ and $y \in \mathbf{R}^+$ functions Φ_1, Φ_2 are increasing and there exist two positive constans V_1, V_2 such that for every random variable $\xi: \Omega \to \mathbf{R}^+, E\xi < \infty$

$$E\Phi_i(t, \xi) \leqslant V_i \Phi_i(t, E\xi), \quad i = 1, 2, \ t \in [0, \infty);$$

(A.5) $EX_-^4(t) \leqslant c < \infty, \ t \leqslant 0;$

(A.6) the right-hand maximum solution $M(t; 0, c)$ of deterministic differential equation

$$y' = 6\Phi(t, y),$$

where

$$\Phi(t, y) = y + V_1 \Phi_1(t, \|K_1\| y) + V_2 \Phi_2(t, \|K_2\| y + G(t)),$$

$$\|K_1\| = \int_{-\infty}^{0} dK_1(s), \quad \|K_2\| = \int_{-\infty}^{0} dK_2(s), \quad A = (2M-1)\cdot(2M-3)\cdot\ldots\cdot 3\cdot 1,$$

with initial condition $(0, c)$ exists in the interval $[0, \infty)$.

THEOREM 1. *Under assumptions* (A.1)—(A.6) *there exists a solution* $X(t)$ *of equation* (1.5), (1.6) *and the inequality*

$$E[X(t)]^4 \leqslant M(t; 0, c), \ t \geqslant 0, \tag{5.1}$$

holds.

Proof. Take $h > 0$ and define an approximate solution $X_h(t)$ by Cauchy's polygonal method:

$$X_h(t) = \begin{cases} X_-(t), & t \leqslant 0, \\ X_h(nh) + a(nh, \Pi_{nh} X_h)(U(t) - U(nh)) & \\ \qquad + b(nh, \Pi_{nh} X_h, \Pi_{nh} B)(B(t) - B(nh)), & nh \leqslant t \leqslant (n+1)h, \\ & n = 0, 1, \ldots \end{cases} \tag{5.2}$$

Let

$$\varphi_h(t) = nh \text{ for } t \in [nh, (n+1)h), \ n = 0, 1, \ldots \tag{5.3}$$

Then $X_h(t)$ satisfies

(5.4)
$$X_h(t) = X_h(0) + \int_0^t a(\varphi_h(s), \Pi_{\varphi_h(s)} X_h) \mathrm{d}U(s)$$
$$+ \int_0^t b(\varphi_h(s), \Pi_{\varphi_h(s)} X_h, \Pi_{\varphi_h(s)} B) \mathrm{d}B(s), \ t \geqslant 0.$$

Let

(5.5)
$$c_h(t) = \sup_{s \leqslant t} E[X_h(s)]^4, \ t \geqslant 0.$$

We shall show that $c_h(t) < \infty$ and $c_h(t) \leqslant M(t; 0, c)$. Since $c_h(t)$ is increasing, to prove that it is finite it is enough to show that $c_h(t) < \infty$ for $t = nh$, by induction. By (A.5) we have

$$c_h(0) = \sup_{s \leqslant 0} E[X_h(s)]^4 = \sup_{s \leqslant 0} E[X_-(s)]^4 \leqslant c < \infty.$$

If $c_h(nh) < \infty$ then $c_h((n+1)h) < \infty$ because we have, for $t \in [nh, (n+1)h]$,

$$E[X_h(t)]^4 \leqslant 27\{E[X_h(nh)]^4 + Ea^4(nh, \Pi_{nh} X_h)h^4 + 3Eb^4(nh, \Pi_{nh} X_h, \Pi_{nh} B)h^2\} \leqslant$$
$$\leqslant 27\{c_h(nh) + E\Phi_1(nh, \int_{-\infty}^0 X_h^4(s+nh) \mathrm{d}K_1(s))h^4 +$$
$$+ 3E\Phi_2(nh, \int_{-\infty}^0 X_h^4(s+nh) \mathrm{d}K_2(s) + \int_{-\infty}^0 B^{2M}(s+nh) \mathrm{d}K_3(s))h^2\} \leqslant$$
$$\leqslant 27\{c_h(nh) + V_1 \Phi_1(nh, c_h(nh)\|K_1\|)h^4 + 3h^2 V_2 \Phi_2(nh, c_h(nh)\|K_2\| +$$
$$+ AG(nh))\} < \infty.$$

Moreover, by (5.4) and (2.5)

$$E[X_h(v)]^4 \leqslant E[X_h(0)]^4 + 6\int_0^v E[X_h^4(s) + a^4(\varphi_h(s), \Pi_{\varphi_h(s)} X_h) +$$
$$+ b^4(\varphi_h(s), \Pi_{\varphi_h(s)} X_h, \Pi_{\varphi_h(s)} B)] \mathrm{d}s \leqslant$$
$$\leqslant c + 6\int_0^v [c_h(s) + V_1 \Phi_1(\varphi_h(s), c_h(s)\|K_1\|) +$$
$$+ V_2 \Phi_2(\varphi_h(s), c_h(s)\|K_2\| + AG(s))] \mathrm{d}s \leqslant$$
$$\leqslant c + 6\int_0^v \Phi(s, c_h(s)) \mathrm{d}s.$$

We have the integral inequality

(5.6)
$$c_h(t) \leqslant c + 6\int_0^t \Phi(s, c_h(s)) \mathrm{d}s.$$

The Opial's theorem [6, Theorem 52.1] implies that

(5.7) $$c_h(t) \leqslant M(t;0,c),\ t \geqslant 0.$$

This estimation does not depend on h.

Next we shall prove that

(5.8) $$E|X_h(t)-X_h(s)|^4 \leqslant c_n|t-s|^{3/2},\ 0 \leqslant s < t \leqslant n,\ n = 1,2,\ldots$$

Indeed,

$$X_h(t)-X_h(s) = \int_s^t a(\varphi_h(\tau), \Pi_{\varphi_h(\tau)}X_h)dU(\tau)+$$
$$+\int_s^t b(\varphi_h(\tau), \Pi_{\varphi_h(\tau)}X_h, \Pi_{\varphi_h(\tau)}B)dB(\tau),$$

$$E(X_h(t)-X_h(s))^4 \leqslant 8(t-s)^3 \int_s^t Ea^4(\varphi_h(\tau), \Pi_{\varphi_h(\tau)}X_h)d\tau +$$
$$+8\cdot 6(t-s)\int_s^t Eb^4(\varphi_h(\tau), \Pi_{\varphi_h(\tau)}X_h, \Pi_{\varphi_h(\tau)}B)d\tau$$

because

$$\int_s^t Eb^4(\varphi_h(\tau), \Pi_{\varphi_h(\tau)}X_h, \Pi_{\varphi_h(\tau)}B)d\tau \leqslant V_2\int_s^t \Phi_2(\varphi_h(\tau), c_h(\tau)\|K_2\| + AG(\tau)d\tau < \infty.$$

Hence

$$E(X_h(t)-X_h(s))^4 \leqslant 8(t-s)^3 V_1 \int_s^t \Phi_1(\varphi_h(\tau), c_h(\tau)\|K_1\|)d\tau +$$
$$+48(t-s)V_2\int_s^t \Phi_2(\varphi_h(\tau), c_h(\tau)\|K_2\| + AG(\tau))d\tau \leqslant$$
$$\leqslant 8(t-s)^3 V_1 \int_s^t \Phi_1(\tau, \|K_1\|\cdot M(\tau;0,c))d\tau +$$
$$+48(t-s)V_2\int_s^t \Phi_2(\tau, \|K_2\|\cdot M(\tau;0,c)+AG(\tau))d\tau \leqslant$$
$$\leqslant 8(t-s)^4 V_1 \max_{\tau\in[s,t]} \Phi_1(\tau, \|K_1\|\cdot M(\tau;0,c))+$$
$$+48(t-s)^2 V_2 \max_{\tau\in[s,t]} \Phi_2(\tau, \|K_2\|\cdot M(\tau;0,c)+AG(\tau)) \leqslant$$
$$\leqslant c_n|t-s|^{3/2}.$$

Applying Lemma 1 to the class of stochastic processes $\{X_{+h} \equiv (X_h(t):t > 0):h > 0\}(\subset \gamma(\mathscr{C}))$ we can see that $\{X_{+h}:h > 0\}$ is weakly compact. It is obvious that $\{B_h \equiv B : h > 0\} (\subset \chi(\mathscr{C}))$ and $\{X_{-h} \equiv X_- : h > 0\} (\subset \chi(\mathscr{C}_-))$ are also weakly compact. Let

$$\mathscr{D}_+ = \{h \in \mathscr{C}_+ : \ |h(t) - h(s)| \leqslant |t-s|, \ t, s \geqslant 0\}. \tag{5.9}$$

It is clear that $\{U_h \equiv U : h > 0\}$ is weakly compact subset of $\mathscr{D}_+$. Hence $\{(X_h, B, U, X_-) : h > 0\}$ is weakly compact subset of $\chi(\mathscr{C} \times \mathscr{C} \times \mathscr{D}_+ \times \mathscr{C}_-)$. So that we can find an L-Cauchy sequence $(X_{h(n)}, B, U, X_-)$ with $h(n) \downarrow 0$. By Skorohod's theorem we can construct (Y_n, B_n, U_n, Y_{-n}), $n = 1, 2, \ldots, \infty$ on a certain probability space such that

$$L((X_{h(n)}, B, U, X_-), (Y_n, B_n, U_n, Y_{-n})) = 0 \tag{5.10}$$

and

$$P((Y_n, B_n, U_n, Y_{-n}) \to (Y_\infty, B_\infty, U_\infty, Y_{-\infty})) = 1, \tag{5.11}$$

where the convergence is to be understood in the sense of the metric in $\mathscr{C} \times \mathscr{C} \times \mathscr{D}_+ \times \mathscr{C}_-$. Since, by (5.10),

$$L((B_n, U_n, Y_{-n}), (B, U, X_-)) = 0$$

and, by (5.11),

$$P((B_n, U_n, Y_{-n}) \to (B_\infty, U_\infty, Y_{-\infty})) = 1,$$

we get

$$L((B_\infty, U_\infty, Y_{-\infty}), (B, U, X_-)) = 0. \tag{5.12}$$

If we can prove that

$$\mathscr{B}_{-\infty, t}(Y_\infty) \vee \mathscr{B}_{0, t}(U_\infty) \vee \mathscr{B}_{-\infty, t}(B_\infty) \text{ is independent of } \mathscr{B}_{t, \infty}(\mathrm{d}B_\infty), \tag{5.13}$$

$$Y_\infty(t) = Y_{-\infty}(t), \ t \leqslant 0, \text{ with probability } 1 \tag{5.14}$$

and

$$\begin{aligned} Y_\infty(t) = Y_\infty(0) + \int_0^t a(\tau, \Pi_\tau Y_\infty) \mathrm{d}U_\infty(\tau) + \\ + \int_0^t b(\tau, \Pi_\tau Y_\infty, \Pi_\tau B_\infty) \mathrm{d}B_\infty(\tau) \text{ with probability } 1 \end{aligned} \tag{5.15}$$

then we can conclude that $X(t) \equiv Y_\infty(t)$ is the solution of (1.5). Using some ideas of the paper [4] we shall prove (5.13), (5.14) and (5.15). By the definition of X_h we have that $\mathscr{B}_{-\infty, t}(X_h) \vee \mathscr{B}_{0, t}(U) \vee \mathscr{B}_{-\infty, t}(B)$ is independent of $\mathscr{B}_{t, \infty}(\mathrm{d}B)$ and, by (5.10) and (5.11), $\mathscr{B}_{-\infty, t}(Y_n) \vee \mathscr{B}_{0, t}(U_n) \vee \mathscr{B}_{-\infty, t}(B_n)$ is independent of $\mathscr{B}_{t, \infty}(\mathrm{d}B_n)$ for every n, also for $n = \infty$. (5.14) holds by definition of X_h and the continuity of $Y_\infty(t)$ and $Y_{-\infty}(t)$. It remains to show (5.15). Set

$$J_n = \int_0^t a(s, \Pi_s Y_n) \mathrm{d}U_n(s) + \int_0^t b(s, \Pi_s Y_n, \Pi_s B_n) \mathrm{d}B_n(s) \tag{5.16}$$

and

$$(5.17)\qquad J_n(h) = \int_0^t a(\varphi_h(s), \Pi_{\varphi_h(s)} Y_n)\mathrm{d}U_n(s) +$$

$$+ \int_0^t b(\varphi_h(s), \Pi_{\varphi_h(s)} Y_n, \Pi_{\varphi_h(s)} B_n)\mathrm{d}B_n(s) =$$

$$= \sum_{k=0}^{m-1} a(kh, \Pi_{kh} Y_n)\left[U_n((k+1)h) - U_n(kh)\right] +$$

$$+ a(mh, \Pi_{mh} Y_n)\left(U_n(t) - U_n(mh)\right) +$$

$$+ \sum_{k=0}^{m-1} b(kh, \Pi_{kh} Y_n, \Pi_{kh} B_n)\left[B_n((k+1)h) - B_n(kh)\right] +$$

$$+ b(mh, \Pi_{mh} Y_n, \Pi_{mh} B_n)\left(B_n(t) - B_n(mh)\right).$$

$J_n(h)$ is an approximate sum of J_n for $\Delta = \{0 < h < 2h < \ldots < mh < t\}$. Since $P(\varrho(Y_n, Y_\infty) \to 0) = 1, \{Y_n : n = 1, 2, \ldots, \infty\}$ is weakly compact and by Lemma 3 for $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $|h| < \delta$ implies

$$(5.18)\qquad P(|J_n(h) - J_n| > \varepsilon) < \varepsilon, \ n = 1, 2, \ldots, \infty.$$

We have, by (5.4), (5.10) and (5.17),

$$Y_n(t) = Y_n(0) + J_n(h(n))$$

and, by (5.11),

$$P(|Y_\infty(t) - Y_\infty(0) - J_\infty| > 6\varepsilon) \leqslant P(|Y_\infty(t) - Y_n(t)| > \varepsilon) +$$

$$+ P(|Y_\infty(0) - Y_n(0)| > \varepsilon) + P(|J_\infty - J_n(h(n))| > 4\varepsilon) <$$

$$< 2\varepsilon + P(|J_\infty - J_n(h(n))| > 4\varepsilon), \ n > N_1.$$

By (5.18) we have

$$P(|J_\infty - J_n(h(n))| > 4\varepsilon) \leqslant P(|J_\infty - J_\infty(h)| > \varepsilon) +$$

$$+ P(|J_\infty(h) - J_n(h)| > \varepsilon) + P(|J_n(h) - J_n| > \varepsilon) +$$

$$+ P(|J_n - J_n(h(n))| > \varepsilon) <$$

$$< 3\varepsilon + P(|J_\infty(h) - J_n(h)| > \varepsilon)$$

for $h < \delta(\varepsilon)$ and $n > N_2$ such that $h(n) < \delta(\varepsilon)$ for $n > N_2$. By (5.17), (5.11) and the continuity of $a(t, \Pi_t \varphi)$ and $b(t, \Pi_t \varphi, \Pi_t \psi)$, $J_n(h) \to J_\infty(h)$ with probability one. Therefore

$$(5.19)\qquad P(|Y_\infty(t) - Y_\infty(0) - J_\infty| > 6\varepsilon) < 6\varepsilon.$$

Since ε is arbitrary, (5.19) implies (5.15). Moreover, because

$$E(X_h(t))^4 \leqslant M(t;0,c) \text{ for } h>0$$

and

$$E(Y_\infty(t))^4 \leqslant \lim_{n\to\infty} E(Y_n(t))^4 = \lim_{n\to\infty} E(X_{h(n)}(t))^4 \leqslant M(t;0,c)$$

we have the estimation (5.1).

Compactness of the solution space. Let $\mathscr{M}$ denotes a set of all admissible systems $S=(X_-,U,B)$. Let $\mathscr{N}=\{X_S: S\in\mathscr{M}\}$ where X_S denotes a solution of equation (1.5) for admissible system S. For $X\in\mathscr{N}$ we have

$$\begin{aligned} E(X(t)-X(s))^4 &\leqslant 8E\Big(\int_s^t a(\tau,\Pi_\tau X)\mathrm{d}U(\tau)\Big)^4+8E\Big(\int_s^t b(\tau,\Pi_\tau X,\Pi_\tau B)\mathrm{d}B(\tau)\Big)^4 \leqslant \\ &\leqslant 8(t-s)^3\int_s^t Ea^4(\tau,\Pi_\tau X)\mathrm{d}\tau+48(t-s)\int_s^t Eb^4(\tau,\Pi_\tau X,\Pi_\tau B)\mathrm{d}\tau \leqslant \\ &\leqslant 8(t-s)^3\int_s^t V_1\Phi_1(\tau,\|K_1\|\cdot M(\tau;0,c))\mathrm{d}\tau+ \\ &\quad +48(t-s)\int_s^t V_2\Phi_2(\tau,\|K_2\|\cdot M(\tau;0,c)+AG(\tau))\mathrm{d}\tau \leqslant \\ &\leqslant c_n|t-s|^{3/2}, \qquad 0\leqslant t,s\leqslant n. \end{aligned}$$

From Lemma 1, recalling (A.5) we conclude that $\mathscr{N}$ is weakly compact subset of $\chi(\mathscr{C})$. $\mathscr{N}\times\mathscr{M}$ is also weakly compact subset of $\chi(\mathscr{C}\times\mathscr{C}_-\times\mathscr{D}_+\times\mathscr{C})$. We can find an L-Cauchy sequence (X_m,S_m). By Skorohod's theorem there exist a certain probability space $(\Omega,\mathscr{F},P)$ and $(Y_m,Y_{-m},\tilde U_m,\tilde B_m)$, $m=0,1,2,\ldots$, such that

$$L((Y_m,Y_{-m},\tilde U_m,\tilde B_m),(X_m,X_{-m},U_m,B_m))=0$$

and

$$P((Y_m,Y_{-m},\tilde U_m,\tilde B_m)\to(Y_0,Y_{-0},\tilde U_0,\tilde B_0))=1$$

where the convergence is to be understood in the sense of the metric in $\mathscr{C}\times\mathscr{C}_-\times\mathscr{D}_+\times\mathscr{C}$. In similar way as in existence theorem we prove that $(Y_0, Y_{-0},\tilde U_0,\tilde B_0)$ is a solution of (1.5). This denotes that $\mathscr{N}\times\mathscr{M}$ is compact.

Existence of optimal control. Let $\psi(f,h)$ be a functional on $\mathscr{C}\times\mathscr{D}_+$, $0\leqslant\psi(f,h)\leqslant+\infty$. We have a theorem analogous to Theorem 3 in the paper [2].

THEOREM 2. *Let $\mathscr{M}_1\subset\mathscr{M}$ be closed in metric L and ψ be lower semi-continuous on $\mathscr{C}\times\mathscr{D}_+$. Then there exists $S_0\in\mathscr{M}_1$ such that*

$$E\psi(X_0,U_0)\leqslant E\psi(X,U),\qquad S\in\mathscr{M}_1,$$

where X_0 and X are the solutions of (1.5) corresponding respectively to S_0 and S.

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