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# ON THE STRUCTURE OF TWO-PERSON, FINITE, ZERO-SUM GAMES

**Abstract.** The algebraical and topological structure of the set $\mathcal{S}$ resp. $\mathcal{S}_1$) of all $n \times m$ matrix games with saddle points (resp. with unique saddle point) in the space $\mathbf{R}^{n \times m}$ of all such games is studied. It has been shown that $\mathcal{S}$ is a closed cone with vertex zero and includes the origin. Moreover, it is neither convex nor dense subset of $\mathbf{R}^{n \times m}$. The set $\mathcal{S}_1$ is a non-convex cone which does not include the origin. It is neither closed nor open.

The concept of "reserve of non-saddlexity" has been also introduced.

**1. Introduction.** In his papers [8, pp. 43—44], [9, p. 9] N. N. Vorobyov is regretted that the authors of the papers from game theory consider only some particular games and the facts related to those games but the general classes or spaces of the games and some particular subsets of these spaces are not in consideration. In this note we study some properties of the subsets of the space $\mathcal{X}$ of all matrix games which contains:

a) the games with saddle points (the set $\mathcal{S}$),

b) the games with unique saddle point (the set $\mathcal{S}_1$).

This is proved that $\mathcal{S}$ and $\mathcal{S}_1$ are cones with vertex 0 ($0 \in \mathcal{S}$ while $0 \notin \mathcal{S}_1$) which are not convex sets. $\mathcal{S}$ is a closed subset of $\mathcal{X}$ which is not dense in $\mathcal{X}$ and — in general — is nowhere dense in $\mathcal{X}$. In general, $\mathcal{S}_1$ is not closed nor open subset of $\mathcal{X}$. The concept of "reserve of non-saddlexity" of the game $A \notin \mathcal{S}$ is introduced as the distance between $A$ and the set $\mathcal{S}$.

**2. Algebraical and topological structure of the sets $\mathcal{S}$ and $\mathcal{S}_1$.** Let us consider the space $\mathcal{X}$ of all two-person, finite, zero-sum games $\Gamma$ with the fixed sets of pure strategies: $\{1, \ldots, n\}$ for the first and $\{1, \ldots, m\}$ for the second player, $n, m > 1$. Let $A = [a_{i,j}]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ be the payoff matrix of the game $\Gamma$; then we identify the game $\Gamma$ with the matrix $A$ (we will write $\Gamma$ and $A$ exchangeable), so, the spaces $\mathcal{X}$ and $\mathbf{R}^{n \times m}$ are isomorphic. The space $\mathcal{X}$ with the usual operations and with the norm

$$\|A\| = \max_i \max_j |a_{i,j}| \tag{1}$$

is a Banach space. Let us denote by $\mathcal{S}$ (resp. $\mathcal{S}_1$) the set of all games with a saddle point (resp. with the unique saddle point). More precisely, $A \in \mathcal{S}$ iff $\max_i \min_j a_{i,j} = \min_j \max_i a_{i,j} = a_{i_0, j_0}$ and $A \in \mathcal{S}_1$ iff there is a unique pair $(i_0, j_0)$ with above property. Obviously, $\mathcal{S}_1 \subset \mathcal{S}$.

In this paper the algebraical and topological structure of the sets $\mathcal{S}$ and $\mathcal{S}_1$ will be studied. From the well-known properties of the max min and min max operations it follows the following

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THEOREM 1. *The set $\mathcal{S}$ is a cone with vertex 0 which includes the origin while $\mathcal{S}_1$ is a cone which does not includes the origin.*

Let $A \in \mathcal{S}$ (resp. $A \in \mathcal{S}_1$), $A \neq 0$ be an arbitrary game. Then the line $\mathcal{L}$ such that $A \in \mathcal{L}$ which is parallel to the line $\mathcal{N}(\Gamma = [b_{i,j}] \in \mathcal{N}$ iff $b_{1,1} = b_{1,2} = \ldots = b_{n,m})$ is also included in the set $\mathcal{S}$, $\mathcal{L} \subset \mathcal{S}$ (resp. $\mathcal{L} \subset \mathcal{S}_1$), so, by Theorem 1

$$(2) \qquad Con(\mathcal{L}) \subset \mathcal{S} \quad (\text{resp. } Con(\mathcal{L}) \subset \mathcal{S}_1),$$

where $Con(\mathcal{Y})$ denotes the conical hull of the set $\mathcal{Y}$. In general $Con(\mathcal{L})$ is a proper subset of the two-dimensional subspace (plane) determined by $\mathcal{L}$ and 0. The sets $\mathcal{S}$ and $\mathcal{S}_1$ are set-theoretical sums of such a parts of those planes.

EXAMPLE 1. Let $n = 2$, $m = 3$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathcal{S}$.

Then $-A = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix} \in \mathcal{S}$ also.

By Theorem 1 there exist a one-dimentional subspace of $\mathbf{R}^6$ which is included in $\mathcal{S}$, but by (2) there exists a two-dimensional subspace with this property.

In the case $n = m$ A. I. Sobolev proved [7] that the maximal dimension of the subspace of $\mathcal{X}$ included in $\mathcal{S}$ is equal $(n-1)^2+1$ for $n \geqslant 3$ and is equal 3 for $n = m = 2$.

REMARK 1. *The sets $\mathcal{S}$ and $\mathcal{S}_1$ are not convex.*

To prove this remark let us consider the following

EXAMPLE 2. Let $n = m$, $A_1 = \operatorname{diag}(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)$, $A_2 = \operatorname{diag}(b_1, \ldots, b_{k-1}, 0, b_{k+1}, \ldots, b_n)$ where $i \neq k$, $a_j, b_l > 0$ for $j \neq i, l \neq k$. We have $A_1, A_2 \in \mathcal{S}_1 \subset \mathcal{S}$ but for $\lambda \in (0,1)$, $\lambda A_1 + (1-\lambda)A_2 \notin \mathcal{S}$.

THEOREM 2. *$\mathcal{S}$ is a closed subset of $\mathcal{X}$.*

Proof. It suffices to prove that $\mathcal{X} \backslash \mathcal{S}$ is open in $\mathcal{X}$. Let $\Gamma = [a_{i,j}] \notin \mathcal{S}$. Denote $k = \max_i \min_j a_{i,j}$ and $K = \min_j \max_i a_{i,j}$. Then $K-k > 0$.

Define $\varepsilon = \frac{1}{2}(K-k)$ and let

$$\mathcal{U}(\Gamma, \varepsilon) \overset{\mathrm{df}}{=} \{[a_{i,j}+c_{i,j}];\ |c_{i,j}| < \varepsilon \text{ for } i = 1, \ldots, n,\ j = 1, \ldots, m\}$$

be an open heighbourhood of the game $\Gamma$. Let $\Gamma_1 = [b_{i,j}] \in \mathcal{U}(\Gamma, \varepsilon)$ be an arbitrary game. Then

$$\max_i \min_j b_{i,j} < \max_i \min_j (a_{i,j}+\varepsilon) = k+\varepsilon,$$

$$\min_j \max_i b_{i,j} > \min_j \max_i (a_{i,j}-\varepsilon) = K-\varepsilon.$$

therefore $\Gamma_1 \notin \mathcal{S}$, so, $\mathcal{U}(\Gamma, \varepsilon) \cap \mathcal{S} = \varnothing$.

COROLLARY. *$\mathcal{S}$ is not a dense subset of $\mathcal{X}$.*

In general, $\mathcal{S}$ is not nowhere dense in $\mathcal{X}$. We prove this fact by contradiction. For $\mathcal{S}$ to be a nowhere dense subset of $\mathcal{X}$ it suffices to prove by [5, Ch. XI, §4, Theorem 3] that in an arbitrary ball $\mathcal{U}(\Gamma, \varepsilon)$ there is a ball $\mathcal{U}'(\Gamma', \varepsilon')$, $\mathcal{U}'(\Gamma', \varepsilon') \subset \mathcal{U}(\Gamma, \varepsilon)$ such that $\mathcal{U}'(\Gamma', \varepsilon') \cap \mathcal{S} = 0$. Let us consider the following

EXAMPLE 3. Let $n = m = 2$, $\Gamma = \begin{bmatrix} -1 & 2 \\ 1 & 1{,}9 \end{bmatrix}$, $\varepsilon < 0{,}45$. Then $\Gamma \in \mathcal{S}$ but in the ball $\mathcal{U}(\Gamma, \varepsilon)$ there is no ball $\mathcal{U}'$ with desirable property.

In general the set $\mathcal{S}_1$ is not closed nor open. Let us consider the following

EXAMPLE 4. Let $n = m = 2$, $\Gamma = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \in \mathcal{S}_1$ and let $0 < \bar{\varepsilon} < \varepsilon < \frac{1}{2}$.

Then $\Gamma_1 = \begin{bmatrix} 1 & 1-\bar{\varepsilon} \\ 0 & 2 \end{bmatrix} \in \mathcal{U}(\Gamma, \varepsilon)$ but $\Gamma_1 \notin \mathcal{S}$, so, $\mathcal{S}_1$ is not open set. But the interior of $\mathcal{S}_1$ is non-empty by Example 3.

**3. The concept of "reserve of non-saddlexity".** Now we introduce one new idea which is related to some ideas known from stability theory and controllability theory of linear autonomuous dynamical systems.

The space of all linear homogeneous autonomuous systems

$$\dot{x} = Ax, \qquad x \in \mathbf{R}^n,$$

is isomorphic with the space $\mathbf{R}^{n \times n}$ of all quadratic matrices $A$. In this space the set $\mathcal{W}$ of all stable systems is an open cone with vertex 0 ($0 \notin \mathcal{W}$). For an arbitrary system $A \in \mathcal{W}$ we may define some number $\delta > 0$ (which is called "*reserve of stability*") as the distance between $A$ and the nearest unstable system. Therefore all systems $\{A + C\colon |c_{i,j}| < \delta$ for all $i, j\}$ are stable still.

Similarly, the space $\mathcal{Y}$ of all linear autonomuous control systems

$$\dot{x} = Ax + Bu, \qquad x \in \mathbf{R}^n, \qquad u \in \mathbf{R}^m,$$

may be identified with the space $\mathbf{R}^{n \times (n+m)}$ of all pairs of matrices $(A, B)$ of the dimensions $n \times n$ and $n \times m$ respectively. In this space the set $\mathcal{Z}$ of all controllable (in Kalman sense) systems is a cone with vertex 0 ($0 \notin \mathcal{Z}$) and is an open set. For the system $(A, B)$ it is defined [3] some number $\gamma > 0$ (which is called "*reserve of controllability*") as the distance between the system $(A, B)$ and the nearest uncontrollable system. Then all perturbed systems $\{(A, B) + (C, D)\}$ such that the norm of perturbation is smaller than $\gamma$ are controllable.

By an analogical way we define the idea of "reserve of non-saddlexity". Let $\Gamma \notin \mathcal{S}$ be an arbitrary game.

DEFINITION. *Reserve of non-saddlexity* of the game $\Gamma$ is the distance $\alpha(\Gamma)$ (in the norm (1)) between the game $\Gamma$ and the set $\mathcal{S}$.

If $\Gamma \notin \mathscr{S}$ then $\alpha(\Gamma) > 0$ and for $\bar{\Gamma} \in \mathscr{S}$ we have $\alpha(\bar{\Gamma}) = 0$. If we perturbe the matrix $A$ by some matrix $G$ such that $|g_{i,j}| < \alpha(\Gamma)$ for all $i,j$ then the perturbed game $\Gamma' = A+G \notin \mathscr{S}$.

Those three ideas are very important if we know the elements of the corresponding matrices with some given accuracy (not exactly) or by some experimental data. In practice, calculation of the numbers $\delta, \gamma, \alpha(\Gamma)$ is very difficult.

REMARK 2. The idea "reserve of saddlexity" which may be define in the similar way is not well-defined: there are some games from $\mathscr{S}$ with positive reserve of saddlexity (as in the Example 4) and some other games with null reserve of saddlexity (as in the Example 3).

**4. Few words about the games without saddle points.** Now we give a few words about those games from $\mathscr{X}$ which have a solution in mixed strategies (denote this set by $\bar{\mathscr{S}}$). Let $\bar{\mathscr{S}}_1$ denotes the set of all games from $\mathscr{X}$ with the unique solution in mixed strategies. Obviously, $\mathscr{S} \subset \bar{\mathscr{S}}$, $\mathscr{S}_1 \subset \bar{\mathscr{S}}_1$, $\bar{\mathscr{S}}_1 \subset \bar{\mathscr{S}}$. By the well-known J. von Neumann theorem we have $\bar{\mathscr{S}} = \mathscr{X}$. The set $\bar{\mathscr{S}}_1$ is a cone with vertex 0 ($0 \notin \bar{\mathscr{S}}_1$), and, by the next example, this is not convex set.

EXAMPLE 5. Let $n = 2$, $m = 3$, $A_1 = \begin{bmatrix} 100 & 170 & 80 \\ 120 & 1000 & 280 \end{bmatrix}$,

$A_2 = \begin{bmatrix} 100 & 10 & 20 \\ 90 & 30 & 110 \end{bmatrix}$, $\lambda = 0{,}1$. We have $A_1, A_2 \in \mathscr{S}_1 \subset \bar{\mathscr{S}}_1$ and denote by $A$ the game

$$A = \lambda A_1 + (1-\lambda) A_2 = \begin{bmatrix} 100 & 26 & 26 \\ 93 & 127 & 127 \end{bmatrix}.$$

The optimal mixed strategies are: $\left(\frac{34}{108}, \frac{74}{108}\right)$ for the first player and $\left(\frac{101}{108}, 0, \frac{7}{108}\right)$ or $\left(\frac{101}{108}, \frac{7}{108}, 0\right)$ for the second one. Therefore $A \notin \bar{\mathscr{S}}_1$.

Topological structure of the set $\bar{\mathscr{S}}_1$ was studied in [1]. This is proved that $\bar{\mathscr{S}}_1$ is an open, dense subset of the space $\mathscr{X}$.

**5. Some other classes of games.** Some similar problems for other classes of games were studied in [2], [4], [6].

In [2] this is proved that in the space of all continuous games over unit square the set of all games with the unique solution is a dense set of $G_\delta$–type.

In [4] some maximal subspace of the linear space of all continuous games over unit square is constructed such that every game from this subspace has a saddle point.

By using of some modification of the well-known Lucas' example the author of the paper [6] gives one hypothesis concerning the set of all cooperative n-person games which have a solution is a dense subset of the space of all cooperative *n*-person games.

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