JOSEPH L. YUCAS*

A CLASSIFICATION THEOREM FOR QUADRATIC FORMS OVER SEMI-LOCAL RINGS

Abstract. Let R be a semi-local ring with $2 \in U(R)$ and such that all residue class fields of R contain more than 3 elements. It is proved here that bilinear spaces over R are classified by dimension, determinant, Hasse invariant and total signature if and only if the third power of the fundamental ideal of Witt ring $W(R)$ is torsion free. This is a generalization of the same result when R is a field due to Elman and Lam.

1. Introduction. In notations and terminology we primarily follow [10]. Unless otherwise stated we will assume R is a connected semi-local ring with $2 \in U(R)$ and such that all residue class fields of R contain more than 3 elements. As a consequence all bilinear spaces over R are free and can be diagonalized (see [9]). The Witt ring of R will be denoted by $W(R)$ and the ideal of $W(R)$ generated by the bilinear spaces of even dimension will be denoted by $I(R)$.

The purpose of this paper is to prove that bilinear spaces (forms) over R are classified by dimension, determinant, Hasse invariant and total signature if and only if $I^3(R)$ is torsion free. This is a generalization of the same result when R is a field due to Elman and Lam [7, Theorem 3]. Finally we show that this result remains valid if we remove the condition that R is connected.

The author has learned that R. Baeza has independently given another proof of this result (see [2]). It would be interesting to know if such a result could be generalized to an arbitrary abstract Witt ring. The main difficulty in generalizing either proof is that they both depend on the usage of quadratic extentions.

We start by recording some preliminary results due to Elman and Lam whose proofs generalize either verbatim of with a slight modification using [3, Satz 2.7].

For a form φ over R we will denote the set of units represented by φ by $D_R(\varphi) = D(\varphi)$.

PROPOSITION 1.1 ([5, Corollary 2.3]). *Suppose φ is a $2n$-dimensional form over R such that $2\varphi = 0$ in $W(R)$. Then $\varphi \simeq \sum_{i=1}^{n} \langle a_i \rangle \langle\langle -w_i \rangle\rangle$ for suitable $a_i \in U(R)$ and $w_i \in D(\langle 1, 1 \rangle)$.*

Received October 27, 1981.

AMS (MOS) Subject classification (1980). Primary 10 C 03 Secondary 15 A 63.

* Department of Mathematics, Southern Illinois University at Carbondale, Carbondale Illinois 62901, USA.

COROLLARY 1.2. *Let* $\varphi = \langle\langle b_1, \ldots, b_n \rangle\rangle$ *be an n-fold Pfister form over R.* $2\varphi = 0$ *in* $W(R)$ *if and only if* $\varphi \simeq \langle\langle -w, \ldots \rangle\rangle$ *where* $w \in D(\langle 1, 1\rangle)$.

PROPOSITION 1.3 ([7, Lemma 1]). *Let* $n \geqslant 1$. *Suppose there is no anisotropic n-fold Pfister form* φ *such that* $2\varphi = 0$ *in* $W(R)$. *Then there are no anisotropic m-fold torsion Pfister forms for any* $m \geqslant n$.

PROPOSITION 1.4 ([4, Theorem 2.8]). *Let* $\varphi \simeq \sum_{i=1}^{r} \langle a_i \rangle \langle\langle -w_i \rangle\rangle$ *with* $a_i \in U(R)$ *and* $w_i \in D(\infty)$. *If* $\varphi \in I^2(R)$ *then* $\varphi = \sum \langle\langle b_j, -c_j \rangle\rangle$ *in* $W(R)$ *where* $b_j \in U(R)$ *and* $c_j \in D(\infty)$.

PROPOSITION 1.5 ([6, Lemma 2.5]). *Suppose* φ *is a 2n-dimensional form in* $I^2(R)$. *There exist 2-fold Pfister forms* $\varphi_1, \ldots, \varphi_{n-1}$ *and* $a_1, \ldots, a_{n-1} \in U(R)$ *such that* $\varphi = \sum_{i=1}^{n-1} \langle a_i \rangle \varphi_i$ *in* $W(R)$.

2. The classification theorem. Let $d \in U(R)$ and suppose $S = R(\sqrt{d})$. The residue fields of S are field extensions of residue fields of R, thus S is a connetced semi-local ring with $2 \in U(R)$ and each residue field of S contains more than 3 elements.

LEMMA 2.1. *Let* $d \in U(R) - [U(R)]^2$, $S = R(\sqrt{d})$ *and suppose* φ *is a 2-fold Pfister form in* $W(S)$. *There exist* $a_1, a_2 \in U(S)$ *and* $r \in U(R)$ *such that* $\varphi \simeq \langle\langle a_1, a_2 \rangle\rangle$ *and* $\langle\langle a_1 r^{-1}, a_2 \rangle\rangle \simeq \langle\langle s_1, s_2 \rangle\rangle$ *for some* $s_1 \in U(R)$ *and* $s_2 \in U(S)$.

Proof. Write $\varphi = \langle\langle b_1, b_2 \rangle\rangle$ for some $b_1, b_2 \in U(R)$. By [11, Lemma 2.2] $\langle b_1, b_2 \rangle \simeq \langle a_1, a_2 \rangle$ where $a_1 = u_1 + v_1\sqrt{d}$, $a_2 = u_2 + v_2\sqrt{d}$ with v_1, v_2 and $u_1 v_2 - u_2 v_1 \in U(R)$. Let $r = -v_1 v_2^{-1}$. Then $a_1 r^{-1} + a_2 = u_2 - u_1 v_2 v_1^{-1} \in U(R)$. Take $s_1 = a_1 r^{-1} + a_2$, then there exists $s_2 \in U(S)$ such that $\langle a_1 r^{-1}, a_2 \rangle \simeq \langle s_1, s_2 \rangle$. Consequently, $\varphi \simeq \langle\langle a_1, a_2 \rangle\rangle$ and $\langle\langle a_1 r^{-1}, a_2 \rangle\rangle \simeq \langle\langle s_1, s_2 \rangle\rangle$.

Let d be a unit of R which is not a square and suppose $s: R(\sqrt{d}) \to R$ is the R-linear map defined by $s(1) = 0$ and $s(\sqrt{d}) = 1$. s is non-degenerate so that the transfer map $s_*: W(R\sqrt{d}) \to W(R)$ is $W(R)$-linear [8, section 4]. We will denote the inclusion map $R \to R(\sqrt{d})$ by i and the induced map $W(R) \to W(R(\sqrt{d}))$ by i*.

PROPOSITION 2.2. *Let* $d \in U(R)$, $S = R(\sqrt{d})$ *and suppose* $\varphi = \langle\langle x, y, -z \rangle\rangle$ *with* $z \in D_S(\langle 1, 1 \rangle)$ *and* $x, y \in U(S)$. *If there are no torsion 3-fold Pfister forms over R then* $s_*(\varphi) = 0$ *in* $W(R)$.

Proof. By Lemma 2.1 there exist $a_1, a_2 \in U(S)$ and $r \in U(R)$ such that $\langle\langle x, y \rangle\rangle \simeq \langle\langle a_1, a_2 \rangle\rangle$ and $\langle\langle a_1 r^{-1}, a_2 \rangle\rangle \simeq \langle\langle s_1, s_2 \rangle\rangle$ with $s_1 \in U(R)$ and $s_2 \in U(S)$. In $W(R)$,

$$s_*(\langle\langle x, y, -z \rangle\rangle) = s_*(\langle\langle a_1, a_2, -z \rangle\rangle) = s_*(\langle\langle -r, a_2, -z \rangle\rangle) + \langle r \rangle s_*(\langle\langle a_1 r^{-1}, a_2, -z \rangle\rangle)$$

since s_* is $W(R)$-linear. Now, $\langle\langle a_1 r^{-1}, a_2, -z \rangle\rangle \simeq \langle\langle s_1, s_2, -z \rangle\rangle$, thus $s_*(\varphi) = s_*(\langle\langle -r_1, a_2, -z \rangle\rangle) + \langle r \rangle s_*(\langle\langle s_1, s_2, -z \rangle\rangle)$ in $W(R)$. Since r and $s_1 \in U(R)$, to show $s_*(\varphi) = 0$ in $W(R)$ we may assume $x \in U(R)$. Again by Lemma 2.1 there exist $a_1', a_2' \in U(S)$ and $r_0 \in U(R)$ such that $\langle\langle y, -z \rangle\rangle \simeq \langle\langle a_1', a_2' \rangle\rangle$ and $\langle\langle a_1 r_0^{-1}, a_2' \rangle\rangle \simeq \langle\langle s_1', s_2' \rangle\rangle$ for some $s_1' \in U(R)$ and $s_2' \in U(S)$. Since

$$\langle\langle a_1', a_2' \rangle\rangle = \langle\langle -r_0 \rangle\rangle \langle\langle a_2' \rangle\rangle + \langle r_0 \rangle \langle\langle a_1 r_0^{-1} \rangle\rangle \langle\langle a_2' \rangle\rangle$$

in $W(S)$ we have in $W(R)$,

$$s_*(\langle\langle y, -z\rangle\rangle) = s_*(\langle\langle a_1', a_2'\rangle\rangle) = \langle\langle -r_0\rangle\rangle s_*(\langle\langle a_2'\rangle\rangle) + \langle r_0\rangle s_*(\langle\langle a_1 r_0^{-1}\rangle\rangle \langle\langle a_2'\rangle\rangle)$$
$$= \langle\langle -r_0\rangle\rangle s_*(\langle\langle a_2'\rangle\rangle) + \langle r_0\rangle \langle\langle s_1'\rangle\rangle s_*(\langle\langle s_2'\rangle\rangle).$$

Consequently $s_*(\langle\langle y, -z\rangle\rangle) \in I^2(R)$. Now $s_*(\varphi) = \langle\langle x\rangle\rangle s_*(\langle\langle y, -z\rangle\rangle)$ and $2s_*(\langle\langle y, -z\rangle\rangle) = s_*(\langle\langle 1, y, -z\rangle\rangle) = 0$ in $W(R)$ since $z \in D_s(\langle 1, 1\rangle)$. By Proposition 1.1 and Proposition 1.4 we can write $s_*(\langle\langle y, -z\rangle\rangle) = \Sigma \langle\langle b_j, -c_j\rangle\rangle$ where $b_j \in U(R)$ and $c_j \in D_R(\infty)$. It follows that $s_*(\varphi) = \Sigma \langle\langle x, b_j, -c_j\rangle\rangle$ in $W(R)$. By the hypothesis we see that $s_*(\varphi) = 0$ in $W(R)$.

For $a, b \in U(R)$ let $\mu(a, b) = \left(\frac{a,\ b}{R}\right)$ and define $\gamma: I^2(R) \to \mathrm{Br}(R)$ by

$$\gamma(\varphi) = \mathrm{Hasse}\,(\varphi)\,(\mu(-1, -1))^{n(n-2)/8}$$

where $n = \dim\varphi$. In [11, p. 464] Mandleberg observed that γ is a well defined group homomorphism and that $\gamma(I^3(R)) = 1$.

LEMMA 2.3. *If φ is a 2-fold Pfister form in $I^3(R)$ then $\varphi = 0$ in $W(R)$.*

Proof. Write $\varphi = \langle\langle -a, -b\rangle\rangle$. Since $\varphi \in I^3(R)$, $\gamma(\varphi) = 1$ in $\mathrm{Br}(R)$ i.e. $\mu(-1, -1)\,\mu(a, b)\,\mu(-1, -1)^{4(2)/8} = 1$. Consequently $\mu(a, b) = 1$ and $\varphi = 0$ in $W(R)$.

PROPOSITION 2.4. *Let $w \in D_R(\infty)$ and $S = R(\sqrt{w})$. If there are no 3-fold torsion Pfister forms over R then $i^*: I^3(R) \to I^3(S)$ is injective.*

Proof. Let φ be a form in $I^3(R)$ with $i^*(\varphi) = 0$ in $W(S)$. By [1, Korollar 2.9], $\varphi \simeq \langle\langle -w\rangle\rangle\psi$ for some form ψ over R. If $\dim\psi$ is odd then $\det\varphi = -w$ contradicting $\varphi \in I^2(R)$. Consequently $\dim\psi = 2m$ for some $m \in \mathbf{Z}^+$. Write $\psi = \langle\langle (-1)^m d\rangle\rangle + \psi'$ where $\psi' \in I^2(R)$ and $d = \det\psi$, then $\varphi = \langle\langle -w, (-1)^m d\rangle\rangle + \langle\langle -w\rangle\rangle\psi'$. It follows that $\langle\langle -w, (-1)^m d\rangle\rangle \in I^3(R)$ thus $\langle\langle -w, (-1)^m d\rangle\rangle = 0$ in $W(R)$ by Lemma 2.3. As a result, φ is a $\mathbf{Z}$-linear combination of torsion 3-fold Pfister form. By the hypothesis, $\varphi = 0$ in $W(R)$.

To prove the following lemma we employ the technique of [11, Theorem 2.1].

LEMMA 2.5. *Let $d \in U(R)$, $S = R(\sqrt{d})$ and suppose φ is an anisotropic 3-fold Pfister form over S with $s_*(\varphi) = 0$ in $W(R)$. There exists a form ψ in $I^3(R)$ such that $i^*(\psi) = \varphi$ in $W(S)$.*

Proof. By [11, Proposition 2.1], there is an anisotropic form ψ' in $W(R)$ with $i^*(\psi') = \varphi$. If $\dim\psi' > 8$ then $i^*(\psi')$ is isotropic. By [1, Satz 2.3] we can write $\psi' = \varrho_1 + \langle\langle -d\rangle\rangle\varrho_2$ with $\dim\varrho_1 = 8$ and $\dim\varrho_2 \geqslant 1$. But then $i^*(\psi') = i^*(\varrho_1)$ in $W(S)$ hence we may assume $\dim\psi' = 8$. Write $\psi' = \langle u_1, \ldots, u_8\rangle$ in $W(R)$ with $u_1, \ldots, u_8 \in U(R)$. Since $\varphi \in I^3(S)$, $\det\psi' \in [U(S)]^2$ thus $i^*(\langle u_1 \det\psi', u_2, \ldots, u_8\rangle) = \varphi$ as well. Consequently we may assume $\det(\langle u_1, \ldots, u_8\rangle) = 1$ and thus $\psi' \in I^2(R)$. By Proposition 1.5, $\psi' = \sum_{i=1}^{3} \langle c_i\rangle \langle\langle a_i, b_i\rangle\rangle$ in $W(R)$ for some $c_i, a_i, b_i \in U(R)$. Now, $\gamma(i^*(\psi')) = \prod_{i=1}^{3} \mu(-a_i, -b_i) = 1$ in $\mathrm{Br}(S)$ since $\varphi \in I^3(S)$. Let $\psi_0 = \langle\langle a_1, b_1\rangle\rangle + \langle -d\rangle \langle\langle a_2, b_2\rangle\rangle + \langle e\rangle \langle\langle a_3, b_3\rangle\rangle$ where $e \in U(R)$ is yet to be determined. Clearly $\psi' \equiv \psi_0 \pmod{I^3(R)}$. By [6, Theorem 1.1 (1)], $\langle a_1, b_1, a_1 b_1\rangle + \langle -d\rangle \langle a_2, b_2, a_2 b_2\rangle$

is isotropic over S and by [6, Lemma 2.1] there exist $f, g, h, k \in U(R)$ such that $\langle\langle a_1, b_1\rangle\rangle + \langle -d\rangle \langle\langle a_2, b_2\rangle\rangle \simeq \langle\langle -d, f\rangle\rangle + \langle g\rangle \langle\langle h, k\rangle\rangle$ over R. Consequently,

$$\psi_0 \simeq \langle\langle -d, f\rangle\rangle + \langle g\rangle \langle\langle h, k\rangle\rangle + \langle e\rangle \langle\langle a_3, b_3\rangle\rangle.$$

Let $e = -dg$. Since $i^*(\psi') - i^*(\psi_0) \in I^3(S)$, $\gamma(i^*(\psi')) = \gamma(i^*(\psi_0)) = 1$ in $\mathrm{Br}(S)$. As a result, $\mu(d, -f)\,\mu(-h, -k)\,\mu(-a_3, -b_3) = 1$ in $\mathrm{Br}(S)$. As above there exist $l, m, n, p \in U(R)$ such that $\langle\langle h, k\rangle\rangle + \langle -d\rangle \langle\langle a_3, b_3\rangle\rangle \simeq \langle\langle -d, l\rangle\rangle + \langle m\rangle \langle\langle n, p\rangle\rangle$. But then

$$\psi_0 \simeq \langle\langle -d, f\rangle\rangle + \langle g\rangle \langle\langle -d, l\rangle\rangle + \langle gm\rangle \langle\langle n, p\rangle\rangle$$

and $i^*(\psi_0) = \langle gm\rangle \langle\langle n, p\rangle\rangle = 0$ in $W(S)$ by Lemma 2.3. If we take $\psi = \psi' - \psi_0$ then $\psi \in I^3(R)$ and $i^*(\psi) = i^*(\psi') = \varphi$.

PROPOSITION 2.6. *Let* $w \in D_R(\infty)$ *and* $S = R(\sqrt{w})$. *If there are no anisotropic 3-fold torsion Pfister forms over R then there are no anisotropic 3-fold torsion Pfister forms over S.*

Proof. Let φ be an anisotropic 3-fold torsion Pfister form over S. By Proposition 1.3 we may assume $2 \cdot \varphi = 0$ in $W(S)$. By Proposition 1.2 we can write $\varphi = \langle\langle x, y, -z\rangle\rangle$ where $x, y \in U(S)$ and $z \in D_S(\langle 1, 1\rangle)$. Now by Proposition 2.2, $s_*(\varphi) = 0$ and by Lemma 2.5 there exists a form ψ in $I^3(R)$ such that $i^*(\psi) = \varphi$. It suffices to show $\psi = 0$ in $W(R)$. Assume not. As in [7, Lemma 3, step 2] we may assume $\psi_A = \langle b_1\rangle \langle\langle b_2, b_3, b_4\rangle\rangle$ for some $b_i \in U(R)$. But by Proposition 2.4, i^* is injective hence $\psi_A \in W_t(R)$. By the hypothesis, $\psi_A = 0$ in $W(R)$, a contradiction.

PROPOSITION 2.7. *Suppose there are no anisotropic torsion 3-fold Pfister forms over R. Then* $I^3(R)$ *is torsion free.*

Proof. The proof given in [7, (A) ⇒ 2, p. 337] will work here. Replace their Proposition 2, Lemma 3 and Proposition 1 (3) by our Proposition 1.1, Proposition 2.6 and Proposition 2.4, respectively.

COROLLARY 2.8. *Let* $w \in D_R(\infty)$ *and* $S = R(\sqrt{w})$. *If* $I^3(R)$ *is torsion free then* $I^3(S)$ *is torsion free.*

Proof. This follows from Proposition 2.6 and Proposition 2.7.

LEMMA 2.9. *Suppose* $I^3(R)$ *is torsion free,* $\varphi \in I^2(R)$, $2\varphi = 0$ *in* $W(R)$ *and* $\gamma(\varphi) = 1$ *in* $\mathrm{Br}(R)$. *Then* $\varphi = 0$ *in* $W(R)$.

Proof. Assume $\varphi \neq 0$ in $W(R)$. By Proposition 1.1 and Proposition 1.4 $\varphi = \Sigma \langle\langle b_j, -c_j\rangle\rangle$ in $W(R)$ for some $b_j \in U(R)$ and $c_j \in D_R(\infty)$. We may assume R and φ are chosen so that n is minimal. Let $S = R(\sqrt{c_1})$ and let $\psi = \sum_{j=2}^{n} \langle\langle b_j, -c_j\rangle\rangle$. Now $2\psi = 0$ in $W(S)$, $\psi \in I^2(S)$, $\gamma(\psi) = 1$ in $\mathrm{Br}(S)$ and $I^3(S)$ is torsion free by Corollary 2.8. Consequently $\psi = 0$ in $W(S)$ by the choice of n. By [2, Korollar 2.9], $\varphi = \langle\langle -c_1\rangle\rangle \langle d_1, \ldots, d_{2r}\rangle$ in $W(R)$ for some $d_1, \ldots, d_{2r} \in U(R)$. In $W(R)$,

$$\begin{aligned}\varphi &= \langle d_1\rangle \langle\langle -c_1, d_1 d_{r+1}\rangle\rangle + \ldots + \langle d_r\rangle \langle\langle -c_1, d_r d_{2r}\rangle\rangle \\ &\equiv \langle\langle -c_1, d_1 d_{r+1}\rangle\rangle + \ldots + \langle\langle -c_1, d_r d_{2r}\rangle\rangle \\ &\equiv \langle\langle -c_1, (-1)^{r+1} d_1 \ldots d_{2r}\rangle\rangle \pmod{I^3(R)}.\end{aligned}$$

It follows that $\varphi - \langle\langle -c_1, (-1)^{r+1} d_1 \ldots d_{2r}\rangle\rangle \in I^3(R) \cap W_t(R) = 0$. Consequently

$\varphi = \langle\langle -c_1, (-1)^{r+1}d_1 \dots d_{2r}\rangle\rangle$ in $W(R)$. Since $1 = \gamma(\varphi) = \gamma(\langle\langle -c_1, (-1)^{r+1}d_1 \dots \dots d_{2r}\rangle\rangle) = \mu(c_1, (-1)^r d_1 \dots d_{2r})$ we must have $\varphi = 0$ in $W(R)$, a contradiction.

THEOREM 2.10. *Forms over R are classified by dimension, determinant, Hasse invariant and total signature if and only if $I^3(R)$ is torsion free.*

Proof. ($\Rightarrow$) By Proposition 2.7 and Proposition 1.3 it suffices to show that there is no anisotropic 3-fold Pfister form φ with $2 \cdot \varphi = 0$ in $W(R)$. Suppose there is such a form φ. By Corollary 1.2 we may write $\varphi \simeq \langle\langle x, y, -w\rangle\rangle$ with $x, y \in U(R)$ and $w \in D_R(\langle 1, 1\rangle)$. The two forms $\langle\langle x, y, -w\rangle\rangle$ and $\langle\langle 1, 1, -1\rangle\rangle$ both have dimension 8, determinant 1, Hasse invariant 1 and total signature 0. Consequently $\varphi = 0$ in $W(R)$, a contradiction.

($\Leftarrow$) Suppose $I^3(R)$ is torsion free and suppose φ_1 and φ_2 are forms over R with the same dimension, determinant, Hasse invariant, and total signature. $\varphi_1 - \varphi_2$ has total signature 0, thus $\varphi_1 - \varphi_2 \in W_t(R)$. Since $\det \varphi_1 = \det \varphi_2$ and $\dim \varphi_1 = \dim \varphi_2$, $\varphi_1 - \varphi_2 \in I^2(R)$. Consequently, $2(\varphi_1 - \varphi_2) \in I^3(R) \cap W_t(R) = 0$. Since $\gamma(\varphi_1 - \varphi_2) = 1$, $\varphi_1 - \varphi_2 = 0$ in $W(R)$ by Lemma 2.9. By Witt's cancellation theorem $\varphi_1 \simeq \varphi_2$ as desired.

Finally, we show that Theorem 2.10 remains valid if we remove the restriction that R is connected. Suppose R is a semi-local ring with $2 \in U(R)$ and whose residue class fields contain more than three elements. We can write $R \cong R_1 \times \dots \times R_t$ with $U(R) \simeq U(R_1) \times \dots \times U(R_t)$ where each R_i is connected and where $R_i = Re_i$, $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$ and $e_1 + \dots + e_t = 1$. Moreover $W(R) \cong W(R_1) \times \dots \times W(R_t)$ with $I^n(R) \cong I^n(R_1) \times \dots \times I^n(R_t)$ (see [8] and [9]). Consequently two forms φ and ψ are equivalent over R if and only if $e_i\varphi \simeq e_i\psi$ for $i = 1, 2, \dots, t$ and $I^n(R)$ is torsion free if and only if $I^n(R_i)$ is torsion free for $i = 1, 2, \dots, t$.

LEMMA 2.11. *Forms are classified by dimension, determinant, Hasse invariant ond total signature over R if and only if forms are classified by the same invariants over each R_i.*

Proof. ($\Leftarrow$) is clear. ($\Rightarrow$) Let φ and ψ be forms over R_i with the same invariants. Write $\varphi = \langle a_1, \dots, a_n\rangle$ and $\psi = \langle b_1, \dots, b_n\rangle$. Let $u_i = e_1 + e_2 + \dots + a_i + \dots + e_t$ and $v_i = e_1 + e_2 + \dots + b_i + \dots + e_t$. Write $\varphi_1 = \langle u_1, \dots, u_n\rangle$ and $\psi_1 = \langle v_1, \dots, v_n\rangle$. $e_j\varphi_1 = e_j\psi_1 = \langle e_j, \dots, e_j\rangle$ if $i \neq j$ and $e_i\varphi_1 = \varphi$, $e_i\psi_1 = \psi$. Clearly $\dim \varphi_1 = \dim \psi_1$ and $\det \varphi_1 = \det \psi_1$. A signature on R gives rise to a signature σ_j on some R_j [8, Corollary 2.8] hence φ_1 and ψ_1 have the same total signature. Since $\mu(u_k, u_j) = \mu(e_i u_k, e_i u_j)$ we see also that Hasse $(\varphi_1) = \text{Hasse}(\psi_1)$. Consequently $\varphi_1 \simeq \psi_1$ over R and thus $\varphi \simeq \psi$ over R_i.

In view of the above remarks it is now easy to see that Theorem 2.10 remains valid if we remove the connected condition.

We conclude this paper with the example which motivated this work.

Let R be a valuation ring whose field of fractions K is a finite extension of Q (e. g. $Z_{(p)}$ for all primes $p \neq 2, 3$). And suppose φ is a totally indefinite form of dimension 5 over R. A signature on K restricts to a signature on R thus φ is totally indefinite over K. By the Hasse-Minkowski Theorem, φ is isotropic over K hence φ must be isotropic over R. Consequently $I^3(R)$ is torsion free and forms over R are classified by dimension, determinant, Hasse invariant and total signature.

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