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WITT RINGS OF FIELDS OF FORMAL POWER SERIES IN TWO VARIABLES

Abstract. Let k be any field of characteristic different from 2, F will denote the ring of formal power series in two variables with coefficients from k and K its field of quotients. The aim of the paper is to investigate the structure of the Witt ring of $K - W(K)$. We shall construct certain exact and split sequences of additive homomorphism. We count special the cases of complex and real number fields. The results are also valid for rings of Nash series.

Let k be any field of characteristic different from 2. Throughout this paper F will denote the ring of formal power series in two variables with coefficients from k , i.e., $F = k[[X, Y]]$ and K its field of quotients, $K = \bar{F} = k((X, Y))$. Our aim is to investigate the structure of the Witt ring $W(K)$ of K . We shall construct certain exact and split sequences of additive homomorphisms. First exact sequence has the following form:

$$0 \rightarrow W(k((X))) \rightarrow W(K) \rightarrow \bigoplus_f W(\bar{F}/f) \rightarrow 0$$

where the direct sum extends over all distinguished irreducible polynomials $f \in k[[X]][Y]$. Second one is more complicated:

$$0 \rightarrow W(k) \rightarrow W(K) \rightarrow \bigoplus_p W(\bar{F}/p) \rightarrow W(k) \rightarrow 0$$

where the direct sum extends over all prime ideals of F such that their height equals one. The second exact sequence does not depend on the choice of the system of parameters $\{X, Y\}$ of the ring F like the first one, hence it is more natural.

In the last chapters we work out two special cases: $k = \mathbf{C}$ (the field of complex numbers) and $k = \mathbf{R}$ (the field of real numbers). We shall try to give more plain and geometric description of homomorphisms which occur in both exact sequences. The results of this paper are valid also for rings of Nash series i.e. the henselizations of $k[X, Y]$ in the ideal (X, Y) or convergent power series (if k is complete in an absolute value, e.g. \mathbf{R}, \mathbf{C}).

1. Notation. $W(k)$ denotes the ring of equivalence classes of nonsingular anisotropic quadratic forms over k , and $\langle a_1, \dots, a_n \rangle$ the class containing the form

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$a_1 T_1^2 + \dots + a_n^2 T^2$. The reader can find the definition and basic facts in [7, Ch. I] or [4, Ch. II]. The structure of Witt ring for an arbitrary field is rather complicated and not always known. Below we give some easier examples (see [4, Ch. II §3]).

EXAMPLE 1. If k is algebraically closed field, e.g., $k = \mathbf{C}$, then $W(k) = \mathbf{Z}_2$ and it is generated by $\langle 1 \rangle$.

EXAMPLE 2. If k is real-closed, e.g., $k = \mathbf{R}$, then $W(k) = \mathbf{Z}$ and it is generated by $\langle 1 \rangle$.

Sometimes the structure of the Witt ring of one field can be described in terms of Witt rings of other fields, e.g. the field of normal power series in one variable (Laurent series).

EXAMPLE 3. $W(k((X))) \approx W(k) \oplus W(k)$, where the first component is generated by forms $\langle a \rangle$, $a \in k \setminus \{0\}$ and the second one by $\langle Xa \rangle$, $a \in k \setminus \{0\}$. The isomorphism is given by so called *first and second residue homomorphisms*: f_1, f_2 , which can be described as follows: If $a \in k((X)) \setminus \{0\}$, then a is of form $a = X^i \cdot u$, where $u \in k[[X]]$ and $u(0) \neq 0$, $i \in \mathbf{Z}$,

$$f_1 \langle a \rangle = \begin{cases} \langle u(0) \rangle, & \text{iff } i \text{ is even,} \\ 0, & \text{iff } i \text{ is odd,} \end{cases}$$

$$f_2 \langle a \rangle = \begin{cases} \langle u(0) \rangle, & \text{iff } i \text{ is odd,} \\ 0, & \text{iff } i \text{ is even.} \end{cases}$$

More details and the proofs can be found in [4, Ch. VI, §1].

In the next chapter we shall use a certain generalization of the residue homomorphisms (see [7, Ch. IV, § 1]).

EXAMPLE 4. Let D be a discrete valuation ring, m its maximal ideal and \bar{D} its field of quotients. Let p be the uniformizer of the valuation, i.e., $m = p \cdot D$. Then there exist additive homomorphisms (called *residue homomorphisms*):

$$f_i : W(\bar{D}), \longrightarrow W(D/m), \quad i = 1, 2.$$

Which act as follows: if $a \in \bar{D} \setminus \{0\}$, then $a = p^i \cdot u$, where $u \in D \setminus m$, and

$$f_1 \langle a \rangle = \begin{cases} \langle \bar{u} \rangle, & \text{iff } i \text{ is even,} \\ 0, & \text{iff } i \text{ is odd,} \end{cases}$$

$$f_2 \langle a \rangle = \begin{cases} \langle \bar{u} \rangle, & \text{iff } i \text{ is odd,} \\ 0, & \text{iff } i \text{ is even,} \end{cases}$$

where \bar{u} denotes the image of u in D/m . We shall often write ∂_π instead of f_2 where π is a chosen uniformizer.

2. The first exact sequence. The first exact sequence is an analogue of Milnor's one: see [4, Ch. IX, § 3]. Our proof is an adaptation of the one given by Lam.

THEOREM 1. *The following sequence is split and exact:*

$$0 \rightarrow W(k((X))) \xrightarrow{i} W(K) \xrightarrow{\oplus \partial_f} \oplus_f W(\bar{F}/f) \rightarrow 0,$$

where:

1) i is induced by inclusion $k((X)) \subset K$,

2) the direct sum extends over all distinguished irreducible polynomials $f \in k[[X]][Y]$

(f is distinguished) iff $f = Y^p + \sum_{i=0}^{p-1} a_i(X) \cdot Y^i$ where a_i belong to the maximal ideal of $k[[X]]$, i.e. $a_i(0) = 0$.

3) ∂_f are induced by f -adic valuation on K ($F_{(f)}$ is a valuation ring).

Before we shall start the proof we need some information about the ring $F = k[[X, Y]]$.

LEMMA 1. Every element $a \in F \setminus \{0\}$ can be decomposed in the following way:

$$a = X^i f \cdot q \cdot Q^2,$$

where

1) $i \in \mathbb{N}$,

2) f is a distinguished polynomial from $k[[X]][Y]$,

3) $q \in k$,

4) Q is an invertible element of F .

Proof. F is Noetherian so we can decompose a ,

$$a = X^i \cdot b, \text{ where } b \in F \text{ and } X \nmid b.$$

Hence b is regular in Y , i.e. $b(0, Y) = Y^l \cdot c(Y)$, $c(0) \neq 0$ and we are able to apply the preparation theorem ([1, Ch. VII, § 3.8], [9, Ch. II. 1.3], [6, Cor. III 3.7] or [12, Ch. VII, § 1, Cor. 1]). Thus $b = f \cdot P$ where f is a distinguished polynomial in Y and P is invertible, i.e. $P(0, 0) \neq 0$. Let $P(0, 0) = q$. A routine argument shows that P/q is a square of an invertible element in F .

Now we are able to begin the proof of Theorem 1. The proof is based on a certain filtration of the ring $W(K)$. We put L_i as a subring of $W(K)$ generated by a where a are distinguished polynomials from $k[[X]][Y]$ of degree less than or equal to i and elements from $k[[X]]$. Then $L_0 \subset L_1 \subset L_2 \subset \dots \subset L_n \subset \dots \subset W(K)$.

Step 1. $L_i, i \in \mathbb{N}$, forms a filtration of $W(K)$. It is enough to show that $\bigcup_{i=0}^{\infty} L_i \supset W(K)$. $W(K)$ is generated additively by elements of the form $\langle a \rangle$, $a \in K \setminus \{0\}$, so $a = \frac{f}{g}$, $f, g \in F \setminus \{0\}$. Hence $\langle a \rangle = \langle f/g \rangle = \langle f \cdot g \rangle = \langle X^i \cdot h \cdot q \rangle$ (as in Lemma 1).

So $\langle a \rangle \in L_{\deg h}$.

Step 2. $L_i/L_{i-1} \xrightarrow{\oplus \partial_f} \bigoplus_{\deg f=i} W(\overline{F/f})$ is an isomorphism, $i = 1, 2, \dots$, where f 's are distinguished and irreducible polynomials from $k[[X]][Y]$.

REMARK. If $\deg f = i$, then $L_{i-1} \subset \ker \partial_f$, hence ∂_f is well defined on L_i/L_{i-1} . Our aim is to construct the inverse homomorphism v . We define v by the following rule: for $\langle g \rangle \in W(\overline{F/f})$ we put $v \langle g \rangle_f = \langle f \cdot \bar{g} \rangle + L_{i-1}$ where \bar{g} is the unique polynomial in $k[[X]][Y]$ of degree less than i such that g is its image in $\overline{F/f}$ (we use the division theorem (see [9, Th. II, 1.2], [6, Cor. III. 3.7] or [12, Ch. VII, § 1, Th. 5]). We must show that v is a group homomorphism:

$$v : \bigoplus_{\deg f=1} W(\overline{F/f}) \rightarrow L_i/L_{i-1}.$$

This can be done by checking the additive relations among the generators $\langle g \rangle \in \in W(\overline{F/f})$.

- (i) $v\langle ab^2 \rangle = v\langle a \rangle$,
- (ii) $v\langle a \rangle + v\langle b \rangle - v\langle (a+b)ab \rangle - v\langle a+b \rangle = 0$,
- (iii) $v\langle 1 \rangle + v\langle -1 \rangle = 0$,

where $a, b, a+b \in \overline{F/f} \setminus \{0\}$ (see [4, Ch. II, Th. 4.3]). But first we shall prove the following lemma.

LEMMA 2. Let $f, a, a_1, a_2, \dots, a_s$ be such polynomials from $k[[X]][Y]$, that

- (i) f is monic of degree i ,
- (ii) $\text{dega} < i$ and $\text{dega}_j < i, j = 1, 2, \dots, s$.

If $a = a_1 a_2 \dots a_s \pmod f$, then $\langle fa \rangle = \langle fa_1 a_2 \dots a_s \rangle \pmod{L_{i-1}}$.

Proof. We shall show how to shrink s . Our proof will be based on the isometry

$$\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle.$$

Say $a_1 a_2 = fk + h$, where $\text{deg}k < i$ and $\text{deg}h < i$, so $fa_1 a_2 = f^2 k + fh$ and

$$\langle k \rangle + \langle fh \rangle = \langle fa_1 a_2 \rangle + \langle kha_1 a_2 \rangle,$$

hence

$$\langle fa_1 a_2 \rangle = \langle fh \rangle \pmod{L_{i-1}},$$

and

$$\langle fa_1 a_2 \dots a_s \rangle = \langle fha_3 \dots a_s \rangle \pmod{L_{i-1}}.$$

Using the last procedure inductively we obtain the assertion of the lemma.

Now with the help of Lemma 2 we show that v respects relations (i), (ii), and (iii).

(i) Suppose that $\bar{a}\bar{b}^2 = c \pmod f$ and $\text{deg}c < i$. Then according to Lemma 2 we obtain

$$\langle f\bar{a} \rangle = \langle f\bar{a}\bar{b}^2 \rangle = \langle fc \rangle \pmod{L_{i-1}}.$$

(ii) Let $\bar{a}\bar{b}(\bar{a}+\bar{b}) = c \pmod f$; $\text{deg}c < i$. Then

$$\langle f\bar{a} \rangle + \langle f\bar{b} \rangle = \langle f(\bar{a}+\bar{b}) \rangle + \langle f\bar{a}\bar{b}(\bar{a}+\bar{b}) \rangle = \langle f(\bar{a}+\bar{b}) \rangle + \langle fc \rangle \pmod{L_{i-1}}.$$

(iii) $v\langle 1 \rangle + v\langle -1 \rangle = \langle f \rangle + \langle -f \rangle = 0$.

Next we shall show that $\oplus \partial_f \circ v = \text{id}$. It will be enough if we check the above equality on generators. Let f, f' be any distinguished irreducible polynomials of degree i , and $a \in \overline{F/f} \setminus \{0\}$.

$$v\langle a \rangle_f = \langle \bar{a}f \rangle,$$

$$\partial_f \langle \bar{a}f \rangle = \langle a \rangle,$$

$$\partial_{f'} \langle \bar{a}f \rangle = 0, \text{ if } f' \neq f.$$

To finish the step we need to show that $\text{Im}v \supset L_i/L_{i-1}$. L_i/L_{i-1} is generated by $\langle f_1 \dots f_1 g_1 \dots g_n \rangle + L_{i-1}$, where f_j are monic irreducible and $\text{deg}f_j = i, j = 1, \dots, l$; $\text{deg}g_j < i, j = 1, \dots, n$.

We shall show how to reduce l . We use once more the isometry:

$$\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle.$$

From the obvious equality $f_1^2 = f_1 f_2 + f_1(f_1 - f_2)$ we obtain

$$\langle f_1 f_2 \rangle + \langle f_1(f_1 - f_2) \rangle = \langle 1 \rangle + \langle f_2(f_1 - f_2) \rangle.$$

Moreover f_1, f_2 are both monic, hence $\deg f_1 - f_2 < i$. Multiplying by $\langle f_3 \dots f_1 g_1 \dots g_n \rangle$ we get

$$\begin{aligned} \langle f_1 \dots f_1 g_1 \dots g_n \rangle &= \langle f_3 \dots f_1 g_1 \dots g_n \rangle + \langle f_2 \dots f_1 g_1 \dots g_n(f_1 - f_2) \rangle - \\ &\quad - \langle f_1 f_3 \dots f_1 g_1 \dots g_n(f_1 - f_2) \rangle. \end{aligned}$$

Using this procedure inductively we obtain that L_i/L_{i-1} is generated by $\langle f g_1 \dots g_n \rangle + L_{i-1}$, where $\deg f = i$, and $\deg g_j < i, j = 1, 2, \dots, n$. But according to Lemma 2, $\langle f g_1 \dots g_n \rangle = \langle f g \rangle \bmod L_{i-1}$, where $g = g_1 \dots g_n \bmod f$, and $\deg g < i$.

Step 3. $\bigoplus W(\overline{F}/f)$ is isomorphic to $W(K)/L_0$. We make use of the five isomorphism of Lemma 2 and induction. Obviously

$$L_1/L_0 = \bigoplus_{\deg f = 1} W(\overline{F}/f).$$

We assume that

$$L_{j-1}/L_0 = \bigoplus_{\deg f \leq j-1} W(\overline{F}/f).$$

Hence we obtain the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & L_{j-1}/L_0 & \rightarrow & L_j/L_0 & \rightarrow & L_j/L_{j-1} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \bigoplus_{\deg f < j} W(\overline{F}/f) & \rightarrow & \bigoplus_{\deg f \leq j} W(\overline{F}/f) & \rightarrow & \bigoplus_{\deg f = j} W(\overline{F}/f) \rightarrow 0. \end{array}$$

The both horizontal sequences are exact, the extreme vertical arrows are isomorphisms, hence the five isomorphism lemma yields that the middle arrow is an isomorphism, too. So we have proved that $L_j/L_0 = \bigoplus_{\deg f \leq j} W(\overline{F}/f)$, for every j .

Now passing to the direct limit, we see that the required groups are isomorphic.

Step 4. Let $i: W(k((X))) \rightarrow W(K)$ be the canonical mapping induced by inclusion $k((X)) \subset K$. Then

- (i) i is split,
- (ii) the image of i equals to L_0 .

(i) Let $f: W(K) \rightarrow W(k((X)))$ be the first residue homomorphism associated with Y (see Example 3). Then $f \circ i = \text{id}$. For every $a \in k((X)) \setminus \{0\} \subset K$, we have $i\langle a \rangle = \langle a \rangle_K$ and $f\langle a \rangle_K = \langle a \rangle$, since $a(x, 0) = a(x)$.

(ii) Both L_0 and $i(W(k((X))))$ are spanned by all forms $\langle a \rangle$, where a does not depend on Y hence $i(W(k((X)))) = L_0$.

3. The second exact sequence. The second exact sequence is an analogue of Scharlau exact sequence for $k(X)$ [4, Ch. IX, § 4]. The first exact sequence is based on the choice of a special coordinate system X, Y for F . Usually it is more convenient not to choose any such system. This inconvenience is removed in the second exact sequence.

THEOREM 2. *There exists an additive homomorphism s such that the following sequence is split exact:*

$$0 \rightarrow W(k) \xrightarrow{i} W(K) \xrightarrow{\bigoplus_p} \bigoplus W(\overline{F}/p) \xrightarrow{s} W(k) \rightarrow 0,$$

where i is induced by inclusion $k \subset K$, the direct sum extends over all prime ideals of ht 1 (we shall denote the set of such ideals by P).

REMARK. The homomorphisms ∂_p are defined up to choice of the uniformizer f_p of F_p . We choose them in the following way:

(i) if $p = (\underline{X})$ then we put $f_p = X$,

(ii) if $p \neq (\underline{X})$ then we put as f_p the distinguished polynomial in Y of the smallest possible degree which generates the ideal p in $k[[X; Y]] = F$.

NOTE. The change of uniformizer of $F_p, f \rightarrow f' = uf$, can be described as composition with an automorphism of $W(\overline{F/p}) : \partial_{f'} = \langle \bar{u} \rangle \partial_f$. Hence the choice of uniformizers is not essential.

Proof. Step 1. i is split. We shall construct j such that $j \circ i = \text{id}$. Let f be the first residue homomorphism associated to (Y) ,

$$f: W(K) \rightarrow W(k((X))),$$

$$f\langle Y^i u \rangle = \begin{cases} \langle u(X, 0) \rangle, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd} \end{cases}$$

(u does not belong to (Y)), and g the first residue homomorphism on $W(k((X)))$ (see Ex. 3). Then $g \circ f \circ i = \text{id}$, since if $a \in k \setminus \{0\} \subset K$, then $i\langle a \rangle = \langle a \rangle$ (a does not depend on Y), hence $f\langle a \rangle = \langle a \rangle$ (a does not depend on X), hence $g\langle a \rangle = \langle a \rangle$.

Step 2.

$$\bigoplus_{p \in P} \partial_p \circ i = 0.$$

For every $p \in P$ $\partial_p \circ i\langle a \rangle = \partial_p \langle a \rangle = 0$ since a is invertible in F_p .

Step 3.

$$\ker \bigoplus_{p \in P} \partial_p \subset \text{Im } i.$$

Let A be any element of $\ker \bigoplus_{p \in P} \partial_p$. Hence from the first exact sequence we obtain that $A \in W(k((X)))$. The rest follows from the following fact: The following diagram commutes:

$$\begin{array}{ccc} W(k((X))) & \xrightarrow{\partial} & W(k) \\ \downarrow i & & \downarrow i' \\ W(K) & \xrightarrow{\partial_x} & W(k((Y))), \end{array}$$

where ∂, ∂_x are the second residue homomorphisms and i, i' are induced by inclusion $k((X)) \subset K$ and $k \subset k((Y))$.

$W(k((X)))$ is generated by elements of the form $\langle X^j u \rangle$, where u is invertible, i.e. $u(0) \neq 0$ and $j = 1, 0$.

$$\partial \langle u \rangle = 0,$$

and $\partial_x \circ i \langle u \rangle = \partial_x \langle u \rangle = 0$,

$$\partial \langle Xu \rangle = \langle u(0) \rangle,$$

and $\partial_X \circ i \langle Xu \rangle = \partial_X \langle Xu \rangle = \langle u(0) \rangle$ (since u does not depend on Y). Now the following notice finishes step 3. $\partial_X A = 0$ yields $\partial A = 0$ and $A \in \ker \partial = W(k)$.

Step 4. The construction of $s \oplus_{p \in P} W(\overline{F/p})$ is generated by forms $\langle a \rangle_p$, where $p \in P$ and $a \in \overline{F/p} \setminus \{0\}$. We define s as follows:

$$s \langle a \rangle_p = \partial \langle a \rangle, \text{ if } p = (X),$$

$$s \langle a \rangle_p = -\partial \circ \partial_X \circ v \langle a \rangle, \text{ otherwise,}$$

where ∂ is the second residue homomorphism on $W(k((X)))$, v is any homomorphism which splits $\oplus \partial_f$ in the first exact sequence.

We are going to show that:

- (i) $s \circ \oplus_{p \in P} \partial_p = 0$,
 - (ii) $\ker s \subset \text{Im} \oplus_{p \in P} \partial_p$ and $\oplus_{p \in P} \partial_p$ splits,
 - (iii) s is split.
- (i) Let $a \in K \setminus \{0\}$, then

$$\begin{aligned} s \circ \oplus_{p \in P} \partial_p \langle a \rangle &= s \circ \partial_X \langle a \rangle + s \circ \oplus_{p \in P \setminus (X)} \partial_p \langle a \rangle = \\ &= \partial \circ \partial_X \langle a \rangle - \partial \partial_X v \circ \oplus_{p \in P \setminus (X)} \partial_p \langle a \rangle = \\ &= \partial \circ \partial_X (\langle a \rangle - v \circ \oplus_{p \in P \setminus (X)} \partial_p \langle a \rangle). \end{aligned}$$

But from the first exact sequence we obtain that

$$B = \langle a \rangle - v \circ \oplus_{p \in P \setminus (X)} \partial_p \langle a \rangle \in \ker \oplus_{p \in P \setminus (X)} \partial_p = W(k((X))).$$

Hence $\partial \partial_X B = 0$.

REMARK. s does not depend on the choice of v .

Let v, v' be two splitting homomorphisms. Then

$$\text{Im}(v-v') \subset \ker \oplus_{p \in P \setminus (X)} \partial_p = W(k((X))), \text{ and } \partial \partial_X (v-v') = 0.$$

(ii) Let $A = A_1 + A_2$ belong to $\ker s$, where A_1 is an image of A under projection on $W(\overline{F/X})$, and A_2 on $\oplus_{p \in P \setminus (X)} W(\overline{F/p})$. We shall show that $\oplus_{p \in P} \partial_p (vA_2 + B \langle X \rangle) = A$ for some $B \in W(k)$.

$$\oplus_{p \in P} \partial_p (vA_2) = \oplus_{p \in P \setminus (X)} \partial_p (vA_2) + \partial_X vA_2 = A_2 + \partial_X vA_2.$$

Now we must determine B . Both A and $\oplus_{p \in P} \partial_p (vA_2)$ belong to $\ker s$, hence $\ker s \ni A - \oplus_{p \in P} \partial_p (vA_2) = A_1 - \partial_X vA_2$. So $\partial(A_1 - \partial_X vA_2) = 0$ and $A_1 - \partial_X vA_2 \in W(k)$.

We put $B = A_1 - \partial_X vA_2$.

(iii) $s|_{W(k((X)))}$ equals to ∂ and it is on to $W(k)$.

If we take $j: W(k) \rightarrow W(k((X)))$, $j \langle a \rangle = \langle aX \rangle$, then $s \circ j = \text{id}$.

4. The special case $k = \mathbb{C}$. As an application of our exact sequences we shall investigate more exactly the cases of $k = \mathbb{C}$ and $k = \mathbb{R}$. We start with the complex case. First we recall some facts.

FACT 1. $\overline{C[[X, Y]]/p} = C((t))$, for all $p \in P$.

Proof. If $p = (X)$ then $C[[X, Y]]/p = C[[Y]]$ and $t = Y$. If $p \neq (X)$ then $\overline{C[[X, Y]]/p}$ is a simple algebraic extension of $C((X))$ ([12, Cor., p. 149]) and it is complete under discrete valuation ([10, Ch. XVIII, § 144] or [3]). So it is isomorphic to $C((t))$.

FACT 2. $W(C((t))) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ (see Example 3) and it is generated by $\langle 1 \rangle$ and $\langle t \rangle$. Moreover, if $a \in C((t)) \setminus \{0\}$ then $\langle a \rangle = \langle 1 \rangle$ or $\langle a \rangle = \langle t \rangle$.

As an immediate corollary of the above facts and Theorems 1 and 2 we obtain:

THEOREM 3. The following sequences are split and exact:

$$0 \rightarrow \mathbf{Z}_2 \oplus \mathbf{Z}_2 \rightarrow W(K) \rightarrow \bigoplus_f \mathbf{Z}_2 \oplus \mathbf{Z}_2 \rightarrow 0,$$

where the direct sum extends over all distinguished irreducible polynomials $f \in C[[X]][Y]$;

$$0 \rightarrow \mathbf{Z}_2 \rightarrow W(K) \rightarrow \bigoplus_p \mathbf{Z}_2 \oplus \mathbf{Z}_2 \rightarrow \mathbf{Z}_2 \rightarrow 0,$$

where the direct sum extends over all prime ideals of ht 1 ($p \in P$).

Using Theorem 3 we may obtain some interesting results concerning Pfister forms. We recall that by an n -fold Pfister form we mean

$$\langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \dots \otimes \langle 1, a_n \rangle, \text{ where } a_i \in K \setminus \{0\},$$

$i = 1, \dots, n$ (see [4, Ch. X]).

THEOREM 4. Every 3-fold Pfister form over $K = C((X, Y))$ is hyperbolic, i.e. $I^3 K = 0$.

Proof. Step 1. As shown in [4, p. 316, Example 7], the u -invariant of $C((t))$ is 2. Hence every 2-fold Pfister form is isotropic, hence hyperbolic.

Step 2. The image of 3-fold Pfister form under the second residue homomorphism is a linear combination of two-fold Pfister forms. This is a special case of more general fact: the image of the ideal generated by n -fold Pfister forms is the ideal generated by $(n-1)$ -fold forms (see [7, Ch. IV, § 1, Lemma 1.4]).

Step 3. Every 3-fold Pfister form A over K is split. From the second exact sequence and steps 1 and 2 we obtain that A belongs to $W(C) = \mathbf{Z}_2$. But every even dimensional form over C is split.

COROLLARY. $W(K)$ is generated additively by the following forms $\langle 1 \rangle$, $\langle a \rangle$, $\langle ab \rangle$, where a, b are irreducible elements of F .

Proof. Let G be a subgroup of $W(K)$ generated by $\langle 1 \rangle$, $\langle a \rangle$, $\langle ab \rangle$, where a, b are irreducible elements of F . It is enough to show that every form $\langle abc \rangle$, where $a, b, c \in F$ and are irreducible, belongs to G . From the Theorem 4 we obtain that every 3-fold Pfister form splits. So; $\langle 1, a \rangle \otimes \langle 1, b \rangle \otimes \langle 1, c \rangle = 0$, hence $\langle abc \rangle = \langle 1, a, b, c, ab, ac, bc \rangle$.

In the next part of this chapter we shall give the plain description of s and ∂p . We shall use the so called intersection numbers.

DEFINITION 1. Let a, b be two elements of $F = C[[X, Y]]$ and a be irreducible, then by the intersection number of a and b we mean

$$i(a, b) = \text{ord } \varphi_a(b),$$

where $\varphi_a: F \rightarrow \overline{F/a} = C((t))$ is the canonical mapping.

NOTE. If $X \rightarrow f(t)$ and $Y \rightarrow g(t)$, where $f, g \in \mathbb{C}[[T]]$, then

$$\varphi_a(b(X, Y)) = b(f(t), g(t)).$$

FACT 1. $i(a, b)$ does not depend on parametrization of $\mathbb{C}((t))$.

FACT 2. If both a and b are irreducible then $i(a, b) = i(b, a)$.

FACT 3. If, moreover, a and b are polynomials in Y then $i(a, b) = \text{ord } r(a, b)$ ($r(a, b)$ denotes the resultant of a and b).

For the proofs of the above facts the reader is referred to [11, Ch. IV, 5]. Although the theorems in this work concern only the algebraic case but the proofs are valid also in the formal one.

Now we are able to count $\partial_p(p \in P)$. Let f be a generator of $p \in P$. In the following part of the chapter we shall denote by ∂_p the unique residue homomorphism ∂_f . ∂_p does not depend on the choice of the generator f of p . If $a \in F \setminus p$ then

$$\partial_p \langle f \cdot a \rangle = \begin{cases} \langle 1 \rangle, & \text{if } i(f, a) \text{ is even,} \\ \langle t \rangle, & \text{if } i(f, a) \text{ is odd.} \end{cases}$$

Next we define $s: \bigoplus_{p \in P} W(\overline{F/p}) \rightarrow \mathbb{Z}_2$ such that

(i) s is linear,

(ii) for each $p \in P$, $s \langle 1 \rangle_p = 0$, $s \langle t \rangle_p = 1$.

We must check whether the new defined homomorphisms is the same as the one defined in Chapter 3. It follows from the lemma below.

LEMMA 3. $s \circ \bigoplus_{p \in P} \partial_p = 0$.

Proof. $W(K)$ is generated by the following elements:

(i) $\langle 1 \rangle$,

(ii) $\langle a \rangle$, where $a \in F$ is irreducible,

(iii) $\langle bc \rangle$, where $b, c \in F$ are irreducible and not associate.

So it will be enough to check the assertion of the lemma on generators of these types.

(i) For every $p \in P$, $\partial_p \langle 1 \rangle = 0$.

(ii)

$$\partial_p \langle a \rangle = \begin{cases} \langle 1 \rangle, & \text{iff } p = (a), \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Case A. The intersection number $i(b, c)$ is even.

$$\partial_p \langle bc \rangle = \begin{cases} \langle 1 \rangle, & \text{if } p = (b) \text{ or } p = (c), \\ 0, & \text{otherwise.} \end{cases}$$

Case B. The intersection number $i(b, c)$ is odd.

$$\partial_p \langle bc \rangle = \begin{cases} \langle t \rangle, & \text{if } p = (b), \\ \langle t \rangle, & \text{if } p = (c), \\ 0, & \text{otherwise.} \end{cases}$$

In all the cases $s \circ \bigoplus_{p \in P} \partial_p = 0$. We obtain that the kernel of new-defined homomorphism contains the kernel of the other one, moreover the both are equal on (X) 's component of the direct sum, hence they are equal.

5. The classifying circle. The aim of this chapter is to construct a classifying circle which will turn out in the next chapter to be a very important tool in the investigating of the ring $R[[X, Y]]$. The field of Laurent series over the field of real numbers $R((X))$ has just two orders, one in which X is positive — σ_1 , and second in which $-X$ is positive — σ_2 . Hence it has two real closures one L_1 according to σ_1 and second L_2 according to σ_2 .

$$L_1 = \bigcup_{k \in \mathbb{N}} R((X^{1/k})), \quad L_2 = \bigcup_{k \in \mathbb{N}} R(((-X)^{1/k}))$$

(compare [11, Ch. IV, § 3, Th. 3.1] and [5, Ch. XI, § 2, Pr. 3]).

Let $I_1(I_2)$ be the open interval in $L_1(L_2)$ of infinitely small elements (i.e. of such elements a that are smaller than any positive real number; $I_i = \{a : \forall e \in R^+ |a| < e\}$). We close both intervals, i.e. we add to each one the upper bound u_i and lower bound l_i , $i = 1, 2$. Next we identify u_1 and u_2 , l_1 and l_2 and we obtain a "circle" S (we denote these points of identification as u and l). With the help of the so defined circle we shall classify all formal curves $R_0^+ \rightarrow R_0^2$, so we shall call S the classifying circle.

First we introduce some notation. We generalize the notion of arc and connection.

DEFINITION 2. By an arc $\overline{a, b}$ (where $a, b \in S$) we mean the following subset of S :

if $a, b \in I_1 \cup \{u, l\}$ then

$$\overline{a, b} = \{c : a \leq c \leq b\} \text{ for } a \leq b,$$

$$\overline{a, b} = S \setminus \{c : a > c > b\} \text{ for } a > b,$$

if $a, b \in I_2 \cup \{u, l\}$ then

$$\overline{a, b} = \begin{cases} \{c : a \geq c \geq b\}, & \text{for } a \geq b \\ S \setminus \{c : a < c < b\}, & \text{for } a < b, \end{cases}$$

if $a \in I_1, b \in I_2$ then $\overline{a, b} = \overline{a, u} \cup \overline{u, b}$,

if $a \in I_2, b \in I_1$ then $\overline{a, b} = \overline{a, l} \cup \overline{l, b}$.

DEFINITION 3. The subset A of S is *connected* if for every two, $a, b \in A$ $\overline{a, b} \subset A$ or $\overline{b, a} \subset A$.

REMARK. S is oriented, I_1 is oriented compatibly with the order, and I_2 oppositely.

Next we will prove the classifying theorem.

THEOREM 5. *There is one to one correspondence F between formal curves $R_0^+ \rightarrow R_0^2$ and points of S .*

Proof. Any curve $(f(t), g(t))$, $f(0) = g(0) = 0$, can be transformed by changes of coordinate system $t \rightarrow t' = t \cdot h(t)$; $h(0) > 0$ and $t \rightarrow t^{1/k}$, $k \in \mathbb{N}$ to the following cases:

- (i) $(t^k, g(t))$, $k \in \mathbb{N}$,
- (ii) $(-t^k, g(t))$, $k \in \mathbb{N}$,
- (iii) $(0, t)$,
- (iv) $(0, -t)$

(for the complex case compare [11, Ch. IV. 2.1]).

We define F as follows:

- (i) $F(t^k, g(t)) = g(X^{1/k}) \in I_1$,
- (ii) $F(-t^k, g(t)) = g((-X)^{1/k}) \in I_2$,
- (iii) $F(0, t) = u$,
- (iv) $F(0, -t) = l$.

F is defined on all curves. We must show that F has an inverse G . Every point of I_1 has the following form $a = g(X^{1/k})$, where $g \in \mathbf{R}[[T]]$, $g(0) = 0$, $k \in \mathbf{N}$. Hence we put $G(a) = (t^k, g(t))$. Similarly for I_2 $a = g((-X)^{1/k})$, where $g \in \mathbf{R}[[T]]$, $g(0) = 0$, $k \in \mathbf{N}$. We put $G(a) = (-t^k, g(t))$. For u we put $G(u) = (0, t)$, and, for l , $G(l) = (0, -t)$. It is easy to see that $F \circ G = \text{id}$ and $G \circ F = \text{id}$.

Next we apply S to investigation the rings of series from $F = \mathbf{R}[[X, Y]]$. We start with the distinguished polynomials in Y .

REMARK. f is distinguished when

$$f(X; Y) = Y^k + \sum_{i=0}^{k-1} c_i(X) Y^i; \quad c_i \in \mathbf{R}[[X]], \quad c_i(0) = 0$$

For f distinguished, $\deg f = k \geq 1$ and $a \in S$ we put:

$$\text{sgn}f(a) = \begin{cases} \text{sgn}f(a) \text{ in } L_1, & \text{if } a \in I_1, \\ \text{sgn}f(a) \text{ in } L_2, & \text{if } a \in I_2, \\ 1, & \text{if } a = u, \\ (-1)^k, & \text{if } a = l. \end{cases}$$

Next we extend this notation for all elements from F . We put

$$\text{sgn}X(a) = \begin{cases} 1, & \text{if } a \in I_1, \\ -1, & \text{if } a \in I_2, \\ 0, & \text{if } a = u \text{ or } a = l, \end{cases}$$

and in general $f = X^j \cdot g \cdot f'$, where f' is distinguished in Y and $g(0, 0) \neq 0$, $\text{sgn}f(a) = \text{sgn}g(0, 0) \cdot (\text{sgn}X(a))^j \cdot \text{sgn}f'(a)$. We will use notation $f(a) = 0$ instead of $\text{sgn}f(a) = 0$.

In the theory of orders on $\mathbf{R}((X, Y))$ an important role plays the sign of $h \in \mathbf{R}[[X, Y]]$ on the curve $(f(t), g(t))$ ($\text{sgn}h(f(t), g(t))$), where t is assumed to be positive. If $h(f(t), g(t)) = \sum_{i=j}^{\infty} a_i t^i$, $a_j \neq 0$, then $\text{sgn}h(f, g) = \text{sgn}a_j$. The following lemma establishes connection between both notions of sign.

LEMMA 4. $\text{sgn}f(a) = \text{sgn}f(G(a))$ for every $f \in F$ and $a \in S$.

Proof. The cases $a = u$ and $a = l$ are obvious. If $a \in I_1$ then $a = g(X^{1/k})$ and $G(a) = (t^k, g(t))$. It is enough to consider only distinguished polynomials.

$$\begin{aligned} f &= Y^n + \sum_{i=0}^{n-1} c_i Y^i, \\ f(a) &= g(X^{1/k})^n + \sum_{i=0}^{n-1} c_i(X) g(X^{1/k})^i = \tilde{f}(X^{1/k}), \\ f(G(a)) &= g(t)^n + \sum_{i=0}^{n-1} c_i(t^k) g(t)^i = \tilde{f}(t), \quad \tilde{f} \in \mathbf{R}[[T]]. \end{aligned}$$

In L_1 $X^{1/k}$ is positive hence signs of both expressions are equal. Similarly for $a \in I_2$.

The following lemma will be very important for our next purposes.

LEMMA 5. *Let f be a distinguished polynomial from $\mathbf{R}[[X]][Y]$ irreducible in $\mathbf{C}[[X, Y]]$. Then f vanishes in two points of S . S without these points has two connected components and $\text{sgn } f$ is constant on these components.*

Proof. f is irreducible in $\mathbf{C}[[X, Y]]$, hence we may assume that f is of the form $\sum_{i=1}^k (Y - h(e^i(nX))^{1/k})$, where $k = \deg f$, e is a primitive k -root of 1, $n = 1$ or $n = -1$. f has a real root, hence we may assume that $h \in \mathbf{R}[[T]]$. Now if k is odd then

$$f(a) = 0 \text{ iff } \begin{cases} a \in I_1 \text{ and } a = h(X^{1/k}), \\ a \in I_2 \text{ and } a = h((-X)^{1/k}) \end{cases}$$

(n does not play any role in this case). If k is even and $n = 1$,

$$f(a) = 0 \text{ iff } a \in I_1 \text{ and } a = h(X^{1/k}) \text{ or } a = h(-X^{1/k}).$$

If k is even and $n = -1$,

$$f(a) = 0 \text{ iff } a \in I_2 \text{ and } a = h((-X)^{1/k}) \text{ or } a = h(-(-X)^{1/k}).$$

The constancy of the sign follows from the fact that in a real closed field if a polynomial has different signs in two points it has a zero between them ([5,] Ch. XI, § 2). The same is true for $\text{arc } R = \overline{a, b}$ of our circle S . If $R \subset I_1$ or $R \subset I_2$ then $R \subset L_1$ or $R \subset L_2$ and the assertion follows from the above fact. If $a \in I_1$ and $b = u$ then f is positive in some neighbourhood of u , i.e. in $\{c \in I_1 : c \geq d\}$ for some $d \in I_1$, since f has only a finite number of roots in L_1 . Hence f has a root in $\overline{a, d} \subset \overline{a, u} = R$. The same arguments are valid if we consider arcs $\overline{l, b}, \overline{a, l}, \overline{u, b}$. If $u \in R$ then f has a root in $\overline{a, u}$ or in $\overline{u, b}$. The same for l . Now if f has no root in an arc then it has a constant sign. Analogically if f has no root in a connected subset of S then it has a constant sign on it.

6. Special case $k = \mathbf{R}$. The real case is more complicated than the complex one. In spite of this we are able to give more explicit forms of both exact sequences and to describe the homomorphisms ∂_p and s in more geometric way.

First we recall some facts. There are two types of prime ideals of ht1 in $\mathbf{R}[[X, Y]]$: a real type and nonreal. We recall that the ideal p is of a real type when

$$\sum a_i^2 \in p \Rightarrow a_i \in p, \text{ for all } i.$$

The set of the ideals of ht1 and of the real type will be denoted by P_1 , of the non-real type by P_2 .

FACT 1. *For every $p \in P_1$, $\overline{F/p} = \mathbf{R}((t))$; for every $p \in P_2$, $\overline{F/p} = \mathbf{C}((t))$.*

Proof. Fact 1 can be proved in the same way as in the complex case. If $p = (X)$ then $\mathbf{R}[[X, Y]]/p = \mathbf{R}[[Y]]$ and $t = Y$. If $p \neq (X)$ then $\overline{\mathbf{R}[[X, Y]]/p}$ is a simple algebraic extension of $\mathbf{R}((X))$ (see [12, Cor. p. 149]) and it is complete under discrete valuation ([10, Ch. XVIII, § 144] or [3, Th. 2.7, 13.15]). So it is isomorphic to $\mathbf{C}((t))$ or $\mathbf{R}((t))$ ([3, Th. 5.7, 5.9]). Now if p is of the real type then $\overline{\mathbf{R}[[X, Y]]/p}$ is formally

real, so it is isomorphic to $R((t))$. If p is not of the real type then $\overline{R[[X, Y]]/p}$ is not formally real so it is isomorphic to $C((t))$.

FACT 2. $W(C((t))) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, $W(R((t))) = \mathbf{Z} \oplus \mathbf{Z}$.

Proof. The both cases are generated by $\langle 1 \rangle$ and $\langle t \rangle$. Moreover, for every $a \neq 0$ we have $\langle a \rangle = \pm \langle 1 \rangle$ or $\langle a \rangle = \pm \langle t \rangle$ (see Example 3).

As an immediate corollary of the above facts and Theorems 1, 2 we obtain

THEOREM 6. *The following sequences are split exact:*

$$0 \rightarrow \mathbf{Z} \oplus \mathbf{Z} \rightarrow W(K) \rightarrow \left(\bigoplus_{p \in P_1 \setminus \{X\}} \mathbf{Z} \oplus \mathbf{Z} \right) \oplus \left(\bigoplus_{p \in P_2} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \right) \rightarrow 0,$$

$$0 \rightarrow \mathbf{Z} \rightarrow W(K) \rightarrow \left(\bigoplus_{p \in P_1} \mathbf{Z} \oplus \mathbf{Z} \right) \oplus \left(\bigoplus_{p \in P_2} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \right) \rightarrow \mathbf{Z} \rightarrow 0.$$

We may obtain from Theorem 6 the new proof of the following fact.

If w is a sum of squares in $K = R((X, Y))$ (i.e. w is totally positive) then w is a square itself or a sum of two squares.

Proof. We put $L = \langle 1, -w \rangle$, so L has a finite order. From Theorem 6 we have $2L = 0$. So there are such a, b, c, d from K not all equal 0, that $a^2 + b^2 - c^2w - d^2w = 0$, i.e.

$$w = \frac{(a^2 + b^2) \cdot (c^2 + d^2)}{(c^2 + d^2)^2} = \left(\frac{ac + bd}{c^2 + d^2} \right)^2 + \left(\frac{ad - bc}{c^2 + d^2} \right)^2.$$

Next using Theorem 6 we prove some interesting results concerning the structure of $W(K)$ which are analogues of Theorem 4.

THEOREM 7. *Let a, b, c be any elements of $K \setminus \{0\}$. Then the following conditions are equivalent:*

- (i) *In any order on K at least one of a, b, c is negative.*
- (ii) *The three fold Pfister form $L = \langle 1, a \rangle \otimes \langle 1, b \rangle \otimes \langle 1, c \rangle$ is hyperbolic.*

REMARK. The condition (i) may be formulated in the following way: L belongs to the torsion subgroup of $W(K)$ (see [4, Ch. VIII, Th. 4.1]).

Proof. (i) \Rightarrow (ii). We know from Theorem 4 that every 2-fold Pfister form over $C((X, Y)) = R((X, Y))(i)$ is universal hence every 2-fold Pfister form over $R((X, Y))$ represents all sums of squares (see [4, Ch. XI, Th. 1.8]). On the other hand from Theorem 6 we have that $2L = 0$, therefore $L = \langle 1, -W \rangle \otimes L_1$, where W is a sum of two squares and L_1 is a 2-fold Pfister form (see [2, Cor. 1]). Since L_1 represents W we obtain that L is hyperbolic.

(ii) \Rightarrow (i). If in an order σ on K $a > 0$, $b > 0$, $c > 0$ then for every $x_1, \dots, x_8 \in K$ not all equal zero

$$x_1^2 + ax_2^2 + bx_3^2 + cx_4^2 + abx_5^2 + acx_6^2 + bcx_7^2 + abcx_8^2 > 0 \quad (\text{in } \sigma).$$

Hence the form L is anisotropic.

COROLLARY. $W(K)$ is generated as a group by elements of the form $\langle 1 \rangle$, $\langle a \rangle$, $\langle bc \rangle$, where a, b, c are irreducible elements of F .

Proof. Let W be a subgroup of $W(K)$ generated by $\langle 1 \rangle$, $\langle a \rangle$, $\langle bc \rangle$, where a, b, c are irreducible in F . It is enough to show that for any three irreducible,

nonassociated $a, b, c \in F$, $\langle abc \rangle$ is an element of W . If any of a, b, c is a sum of squares, let us say a , then the Pfister form $\langle 1, -a \rangle \otimes \langle 1, b \rangle \otimes \langle 1, c \rangle$ is hyperbolic since $-a$ is negative in every order on K (Theorem 7). Hence

$$\langle abc \rangle = \langle 1, -a, b, c, -ab, -ac, bc \rangle.$$

Similarly, in the case when any of $-a, -b, -c$ is a sum of squares we obtain the analogous result.

The case when all three ideals $(a), (b), (c)$ are of the real type is more complicated. We refer to the classifying circle S (Ch. 5).

Every one of a, b, c has two roots in S (Lemma 6). Hence they divide S into six components. The signs of a, b and c are constant on each component. But there are eight combination of signs of three elements. Hence two of them do not occur: Let us assume e_1, e_2, e_3 is just one of them, $e_i = \pm 1$. Then $e_1 a, e_2 b$ and $e_3 c$ have no common positive point in S . Hence in every order on K at least one of $e_1 a, e_2 b, e_3 c$ is negative ([8, Th. 2.8]). Then the Pfister form $\langle 1, e_1 a \rangle \otimes \langle 1, e_2 b \rangle \otimes \langle 1, e_3 c \rangle$ is hyperbolic (Th. 7) and

$$\langle abc \rangle = -e_1 e_2 e_3 \langle 1, e_1 a, e_2 b, e_3 c, e_1 e_2 ab, e_1 e_3 ac, e_2 e_3 bc \rangle.$$

Our next aim is to give a plain description of ∂_p , $p \in P$ and s . We shall need the intersection numbers once more. But there are some differences between the real and complex cases by (i_c will be denoted the intersection number in the complex case).

Let a, b be two elements of $F = \mathbf{R}[[X, Y]]$, an irreducible and $b \in (a)$. If the ideal (a) is of the real type, then a is irreducible in $\mathbf{C}[[X, Y]]$, and $\mathbf{C}[[X, Y]]/a$ is a complexification of $\mathbf{R}[[X, Y]]/a$, hence $\text{ord } \varphi_a b$ in both cases are equal, so $i(a, b) = i_c(a, b)$ ($\varphi: F \rightarrow F/a$). If the ideal (a) is of the non-real type, then $a = c \cdot \bar{c}$ in $\mathbf{C}[[X, Y]]$, hence the order of b in $\mathbf{R}[[X, Y]]/a$ and the order of b in $\mathbf{C}[[X, Y]]/c$ are equal. Hence

$$i(a, b) = i_c(c, b) = i_c(\bar{c}, b).$$

(Note that $\overline{\varphi_c b} = \varphi_{\bar{c}} b$, since $b \in \mathbf{R}[[X, Y]]$).

FACT 1. $i(a, b)$ does not depend on parametrizations of $\mathbf{C}((t))$ or $\mathbf{R}((t))$.

FACT 2. If a, b are both irreducible and the ideals $(a), (b)$ are both of the real type or both of the non-real then $i(a, b) = i(b, a)$.

Proof. If the both ideals are of the real type then the assertion follows from the complex case: $i(a, b) = i_c(a, b) = i_c(b, a) = i(b, a)$. The second part is more complicated. Let $a = c \cdot \bar{c}, b = d \cdot \bar{d}$ in $\mathbf{C}[[X, Y]]$ (c and d are irreducible). From [11, Ch. IV, § 5.1] (the proof can be adapted for formal case) we obtain that

$$i_c(c, b) + i_c(\bar{c}, b) = i_c(d, a) + i_c(\bar{d}, a),$$

but $i_c(c, b) = i_c(\bar{c}, b)$ and $i_c(d, a) = i_c(\bar{d}, a)$. Hence $i_c(c, b) = i_c(d, a)$ and $i(a, b) = i(b, a)$.

The second difference concerns the fixing of ∂_p .

In the complex case ∂_f does not depend on the choice of a generator f of p .

The same takes place in the real case when $p \in P_2$ (since then -1 is a square in $\overline{F/p}$). But it is not true when $p \in P_1$. For any two generators f, f' of p we have $f = qf'$ and

- (1) $\partial_f = \partial_{f'}$, when $q(0) > 0$,
- (2) $\partial_f = -\partial_{f'}$, when $q(0) < 0$.

Hence we must choose a class of generators for every $p \in P_1$. We do it as follows: we choose these which contain f_p , where

$$f_p = \begin{cases} X, & \text{if } p = (X), \\ \text{the distinguished polynomial from } R[[X]][Y] \text{ which generates } p, & \text{otherwise.} \end{cases}$$

Such a class will be called proper.

REMARK. In the terms of the classifying circle S it can be expressed as follows: $p = (f) \neq (X)$, f belongs to proper class iff $\text{sgn} f(u) = 1$.

In the next parts of the chapter we shall refer to ∂_p as to one fixed as follows:

$$\partial_p = \partial_f, \text{ where } \begin{cases} f \text{ is any generator, if } p \in P_2, \\ f \text{ belongs to the proper class, if } p \in P_1. \end{cases}$$

The third thing we are going to do is to choose the uniformizer of the field $\overline{F/p}$.

REMARK. In the complex case all uniformizers are quadratically equivalent. In the real case there are two classes of uniformizers (e.g. t and $-t$ are not quadratically equivalent), and each class corresponds to one root in S of any generator of p .

We choose the proper classes of uniformizers accordingly to the proper classes of generators. Let f be a generator of $p \in P_1$ belonging to the proper class. Let a, b be roots of f in S . If f is positive on the arc $\overline{a, b}$ then we choose the class of uniformizers which corresponds to a , i.e. the class which contains t such that

$$G(a) = (f(t), g(t)) \text{ for } t > 0.$$

To conclude this consideration we define the sign index.

DEFINITION 4. Let $f, g \in F$, f irreducible. Let a, b be roots of f in S , and $\text{sgn} f = 1$ on the arc $\overline{a, b}$. Then the sign index $e(f, g)$ equals to $\text{sgn} g(a)$.

There are some connections between intersection number and sign index.

LEMMA 6. Let $f, g \in F$, f irreducible, $g \notin (f) \in P_1$, a, b roots of f in S . Then $i(f, g)$ is even iff $\text{sgn} g(a) = \text{sgn} g(b)$.

Proof. We may assume that f is positive on $\overline{a, b}$. There are just two orders on $R((t))$ one in which $t > 0$ and second in which $t < 0$. The image \bar{g} of any $g \in R[[X, Y]]$ is positive in the first order iff $\text{sgn} g(a) = 1$ (t belongs to the proper class), and in second iff $\text{sgn} g(b) = 1$. If g is positive in both points a, b (or negative) then \bar{g} is positive (or negative) in both orders on $R((t))$, hence \bar{g} (or $-\bar{g}$) is a square. So the ord \bar{g} and the intersection number $i(f, g)$ is even. If the intersection number is even then \bar{g} or $-\bar{g}$ is a square in $R((t))$, hence g does not change sign.

LEMMA 7. Let $f, g \in F$, be irreducible and nonassociate, and $(f), (g)$ belong to P_1 . If $i(f, g)$ is odd then $-e(f, g) = e(g, f)$.

Proof. Let $a, b(c, d)$ be roots of $f(g)$ in S , and $f(g)$ be positive on $\overline{a, b}$ ($\overline{c, d}$). We obtain from Lemma 6 that one of c, d belongs to $\overline{a, b}$ and the second to $\overline{b, a}$.

Hence if $c \in \overline{a, b}$ then $a \in \overline{c, d}$, and if $c \in \overline{a, b}$ then $a \in \overline{c, d}$. So

$$e(g, f) = \text{sgn}f(c) = -\text{sgn}g(a) = -e(f, g).$$

Now we are able to describe ∂_p and s in terms of proper classes. If $p \in P_1$, f belongs to the proper class of generators of p and $g \in F \setminus p$, then

$$\partial_p \langle fg \rangle = \begin{cases} \langle 1 \rangle, & \text{if } i(f, g) \text{ is even and } e(f, g) = 1, \\ -\langle 1 \rangle, & \text{if } i(f, g) \text{ is even and } e(f, g) = -1, \\ \langle t \rangle, & \text{if } i(f, g) \text{ is odd and } e(f, g) = 1, \\ -\langle t \rangle, & \text{if } i(f, g) \text{ is odd and } e(f, g) = -1. \end{cases}$$

If $p \in P_2$, f is a generator of p and $g \in F \setminus p$, then

$$\partial_p \langle fg \rangle = \begin{cases} \langle 1 \rangle, & \text{if } i(f, g) \text{ is even,} \\ \langle t \rangle, & \text{if } i(f, g) \text{ is odd.} \end{cases}$$

We define s as follows:

$$s \langle 1 \rangle_p = 0 \text{ for all } p \in P,$$

$$s \langle t \rangle_p = 0 \text{ for all } p \in P_2,$$

$$s \langle t \rangle_p = 1 \text{ for all } p \in P_1.$$

We assume that t belongs to the proper class of uniformizers. We must check whether the new defined homomorphism s is the same as the one defined in Chapter 3. But it follows from the lemma below.

LEMMA 8. $s \circ \bigoplus_{p \in P} \partial_p = 0$.

Proof. $W(K)$ is generated by the following elements:

- (i) $\langle 1 \rangle$,
- (ii) $\langle a \rangle$, $a \in F$ irreducible,
- (iii) $\langle ab \rangle$, $a, b \in F$ irreducible and nonassociate.

We check the thesis on these generators.

- (i) for every p from P , $\partial_p \langle 1 \rangle = 0$.
- (ii)

$$\partial_p \langle a \rangle = \begin{cases} \pm \langle 1 \rangle, & \text{if } p = (a), \\ 0, & \text{otherwise.} \end{cases}$$

- (iii) Case A. $(a), (b)$ are nonreal.

$$\partial_p \langle ab \rangle = \begin{cases} \langle 1 \rangle \text{ or } \langle t \rangle, & \text{if } p = (a) \text{ or } p = (b), \\ 0, & \text{otherwise.} \end{cases}$$

Case B. (a) is nonreal and (b) is real. In this case $i(b, a)$ is always even.

$$\partial_p \langle ab \rangle = \begin{cases} \pm \langle 1 \rangle, & \text{if } p = (b), \\ \langle 1 \rangle \text{ or } \langle t \rangle, & \text{if } p = (a), \\ 0, & \text{otherwise.} \end{cases}$$

Case C. Both (a) , (b) are of real type, and $i(a, b)$ is even.

$$\partial_p \langle ab \rangle = \begin{cases} \pm \langle 1 \rangle, & \text{if } p = (a) \text{ or } p = (b), \\ 0, & \text{otherwise.} \end{cases}$$

Case D. Both (a) , (b) are real, $i(a, b)$ is odd. We put $e = e(a, b) = -e(b, a)$ (Lemma 7). (We assume a, b to be proper).

$$\partial_p \langle ab \rangle = \begin{cases} e \langle t \rangle, & \text{if } p = (a), \\ -e \langle t \rangle, & \text{if } p = (b), \\ 0, & \text{otherwise.} \end{cases}$$

In all the cases except (iii) D, $s = 0$ from the definition. In the last case $s \circ \bigoplus_{p \in P} \partial_p \langle ab \rangle = s(e \langle t \rangle_a - e \langle t \rangle_b) = e - e = 0$. We obtain that the kernel of new-defined homomorphism contains the kernel of the other one, moreover the both are equal on the (X) 's component of the direct sum, hence they are equal.

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