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A CHARACTERIZATION OF FUNCTIONS WITH DENSE GRAPH IN THE PLANE OR HALF-PLANE

Abstract. Let $\boldsymbol{R}$ be the set of all real numbers. In the present paper we shall characterize functions $f: \boldsymbol{R} \to \boldsymbol{R}$ which are either linear or have graph contained and dense in the plane or half-plane determined by a linear function. For this purpose we consider functions satisfying certain limitary conditions which are related to the additivity equation but considerably weaker than that.

Let us introduce the following

DEFINITION. A function f: $\boldsymbol{R} \to \boldsymbol{R}$ is called *limit-additive* iff the following conditions are fulfilled:

$$\bigwedge_{x,\, y \in \boldsymbol{R}} \; \bigvee_{\substack{(z_n)_{n \in N} \\ z_n \in \boldsymbol{R},\, n \in N}} \left[z_n \xrightarrow[n \to \infty]{} x+y,\ f(z_n) \xrightarrow[n \to \infty]{} f(x)+f(y)\right], \tag{1}$$

$$\bigwedge_{x,\, y \in \boldsymbol{R}} \; \bigvee_{\substack{(x_n)_{n \in N},\, (y_n)_{n \in N} \\ x_n,\, y_n \in \boldsymbol{R},\, n \in N}} \left[x_n \xrightarrow[n \to \infty]{} x,\ y_n \xrightarrow[n \to \infty]{} y,\ f(x_n)+f(y_n) \xrightarrow[n \to \infty]{} f(x+y)\right], \tag{2}$$

$$\bigwedge_{x \in \boldsymbol{R}} \; \bigvee_{\substack{(x_n)_{n \in N} \\ x_n \in \boldsymbol{R},\, n \in N}} \left[x_n \xrightarrow[n \to \infty]{} x,\ 2f(x_n) \xrightarrow[n \to \infty]{} f(2x)\right]. \tag{3}$$

Conditions (1) and (2) are, in a sense, mutually symmetric. Condition (3) can not be obtained from (2) by setting $x = y$, since even then sequences $(x_n)_{n \in N}$ and $(y_n)_{n \in N}$ occuring in (2) may not coincide. Adding condition (3) we obtain the possibility of the choice of a common sequence in the case where $x = y$.

Clearly, every additive function is limit-additive (it suffices to take constant sequences). There exist, however, limit-additive functions which are not additive. Indeed, one can easily check that an arbitrary function f: $\boldsymbol{R} \to \boldsymbol{R}$ with the graph being dense on the plane $\boldsymbol{R}^2$ is limit-additive. Let us note that if a function f: $\boldsymbol{R} \to \boldsymbol{R}$ is limit-additive and continuous then it is additive and consequently has the form

$$f(x) = ax, \quad x \in \boldsymbol{R},$$

where a is a constant.

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Now, we are going to give some necessary and sufficient conditions for a limit--additive function to be continuous.

LEMMA 1. *Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be a limit-additive function. Then, for any $k \in \boldsymbol{N}$, $k \geqslant 2$, and each points $x_1, \ldots, x_k \in \boldsymbol{R}$, there exists a sequence $(z_n)_{n\in N}$ of real numbers such that*

$$z_n \xrightarrow[n\to\infty]{} x_1 + \ldots + x_k \text{ and } f(z_n) \xrightarrow[n\to\infty]{} f(x_1) + \ldots + f(x_k).$$

Proof. For $k = 2$ the assertion of the lemma coincides with condition (1). Suppose that our lemma holds true for some $k \in \boldsymbol{N}$, $k \geqslant 2$ and for any system of k points $x_1, \ldots, x_k \in \boldsymbol{R}$. Fix $k+1$ points $x_1, \ldots, x_{k+1} \in \boldsymbol{R}$. On account of our assumption, there exists a sequence $(u_n)_{n\in N}$ such that

$$u_n \xrightarrow[n\to\infty]{} x_1 + \ldots + x_k, \quad f(u_n) \xrightarrow[n\to\infty]{} f(x_1) + \ldots + f(x_k).$$

In view of (1), for each $n \in \boldsymbol{N}$ one can find a sequence $(w_{n,m})_{m\in N}$ such that

$$w_{n,m} \xrightarrow[m\to\infty]{} u_n + x_{k+1} \text{ and } f(w_{n,m}) \xrightarrow[m\to\infty]{} f(u_n) + f(x_{k+1}).$$

Hence

$$\bigwedge_{n\in N} \bigvee_{m_n'\in N} \bigwedge_{m\geqslant m_n'} |w_{n,m} - u_n - x_{k+1}| < \frac{1}{n},$$

$$\bigwedge_{n\in N} \bigvee_{m_n''\in N} \bigwedge_{m\geqslant m_n''} |f(w_{n,m}) - f(u_n) - f(x_{k+1})| < \frac{1}{n}.$$

We put $m_n := \max(m_n', m_n'')$, $n \in \boldsymbol{N}$ and $z_n := w_{n,m_n}$, $n \in \boldsymbol{N}$. Then we have

$$|z_n - x_1 - \ldots - x_{k+1}| \leqslant$$
$$\leqslant |w_{n,m_n} - u_n - x_{k+1}| + |u_n - x_1 - \ldots - x_k| < \frac{1}{n} + |u_n - x_1 - \ldots - x_k| \xrightarrow[n\to\infty]{} 0,$$

$$|f(z_n) - f(x_1) - \ldots - f(x_{k+1})| \leqslant$$
$$\leqslant |f(w_{n,m_n}) - f(u_n) - f(x_{k+1})| + |f(u_n) - f(x_1) - \ldots - f(x_k)| <$$
$$< \frac{1}{n} + |f(u_n) - f(x_1) - \ldots - f(x_k)| \xrightarrow[n\to\infty]{} 0,$$

whence

$$z_n \xrightarrow[n\to\infty]{} x_1 + \ldots + x_{k+1} \text{ and } f(z_n) \xrightarrow[n\to\infty]{} f(x_1) + \ldots + f(x_{k+1})$$

which, by induction, completes the proof.

THEOREM 1. *If a limit-additive function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is continuous at a point then it is continuous everywhere.*

Proof. Assume that f is continuous at the point $x_0 \in \boldsymbol{R}$.

(a) Let $(x_n)_{n\in N}$ be an arbitrary sequence of real numbers convergent to zero. Since

$$x_0 = (x_0 - x_n) + x_n, \quad n \in \boldsymbol{N}$$

from (1) it follows that, for each $n \in N$, there exists a sequence $(z_{n,m})_{m \in N}$ such that

$$z_{n,m} \xrightarrow[m\to\infty]{} x_0, \; f(z_{n,m}) \xrightarrow[m\to\infty]{} f(x_0-x_n)+f(x_n).$$

Hence

$$\bigwedge_{n \in N} \bigvee_{m'_n \in N} \bigwedge_{m \geq m'_n} |z_{n,m}-x_0| < \frac{1}{n},$$

$$\bigwedge_{n \in N} \bigvee_{m''_n \in N} \bigwedge_{m \geq m''_n} |f(z_{n,m})-f(x_0-x_n)-f(x_n)| < \frac{1}{n}.$$

Put $m_n := \max(m'_n, m''_n)$, $z_n := z_{n,m_n}$, $n \in N$. Then

$$|z_n - x_0| < \frac{1}{n}, \; n \in N$$

and

$$|f(z_n)-f(x_0-x_n)-f(x_n)| < \frac{1}{n}, \; n \in N,$$

whence

(4) $$z_n \xrightarrow[n\to\infty]{} x_0 \text{ and } f(z_n)-f(x_0-x_n)-f(x_n) \xrightarrow[n\to\infty]{} 0.$$

By the continuity of f at x_0 we have

$$f(z_n) \xrightarrow[n\to\infty]{} f(x_0) \text{ and } f(x_0-x_n) \xrightarrow[n\to\infty]{} f(x_0)$$

which, together with (4), gives

(5) $$f(x_n) \xrightarrow[n\to\infty]{} 0, \text{ for any sequence } (x_n)_{n \in N} \text{ such that } x_n \xrightarrow[n\to\infty]{} 0.$$

(b) Fix an $x \in \boldsymbol{R}$ and write $0 = x+(-x)$. Using condition (1) again we find a sequence $(z_n)_{n \in N}$, $z_n \xrightarrow[n\to\infty]{} 0$, for which

$$f(z_n) \xrightarrow[n\to\infty]{} f(x)+f(-x).$$

Hence and from (5) it follows that

$$f(-x) = -f(x), \; x \in \boldsymbol{R}.$$

(c) Now, choose an arbitrary $x \in \boldsymbol{R}$ and a sequence $(x_n)_{n \in N}$, $x_n \xrightarrow[n\to\infty]{} x$. On account of Lemma 1, for each $n \in N$ one can find a sequence $(z_{n,m})_{m \in N}$ such that

(6) $$z_{n,m} \xrightarrow[m\to\infty]{} x_n - x + x_0$$

and

(7) $$f(z_{n,m}) \xrightarrow[m\to\infty]{} .\,(x_n)+f(-x)+f(x_0) = f(x_n)-f(x)+f(x_0).$$

In view of (6) and (7) we have

$$\bigwedge_{n \in N} \bigvee_{m'_n \in N} \bigwedge_{m \geq m'_n} |z_{n,m}-(x_n-x+x_0)| < \frac{1}{n},$$

$$\bigwedge_{n \in N} \bigvee_{m''_n \in N} \bigwedge_{m \geq m''_n} |f(z_{n,m})-(f(x_n)-f(x)+f(x_0))| < \frac{1}{n}.$$

We put $m_n := \max(m_n', m_n)$, $z_n := z_{n,m_n}$, $n \in N$. With the aid of this notion we get

$$|z_n - x_0| \leqslant |z_{n,m_n} - (x_n - x + x_0)| + |x_n - x| < \frac{1}{n} + |x_n - x| \xrightarrow[n\to\infty]{} 0,$$

$$|f(z_n) - (f(x_n) - f(x) + f(x_0))| < \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

Consequently,

$$z_n \xrightarrow[n\to\infty]{} x_0 \text{ and } f(z_n) - f(x_n) + f(x) - f(x_0) \xrightarrow[n\to\infty]{} 0.$$

Hence and from the continuity of f at the point x_0 it follows that

$$f(x_n) \xrightarrow[n\to\infty]{} f(x),$$

which implies that f is continuous at x.

LEMMA 2. *If a function $f: \boldsymbol{R} \to \boldsymbol{R}$ is limit-additive and bounded in a neighbourhood of a point $x_0 \in \boldsymbol{R}$ then it is bounded in a neighbourhood of zero.*

Proof. Suppose that there exist $M > 0$ and $\delta > 0$ such that

$$|f(y)| \leqslant M, \text{ for } y \in (x_0 - \delta, x_0 + \delta).$$

Take an $x \in (-\delta, \delta)$. Then $x + x_0 \in (x_0 - \delta, x_0 + \delta)$ and there exists a sequence $(z_n)_{n\in N}$, $z_n \xrightarrow[n\to\infty]{} x + x_0$ such that $f(z_n) \xrightarrow[n\to\infty]{} f(x) + f(x_0)$.
For almost every $n \in \boldsymbol{N}$ we have

$$z_n \in (x_0 - \delta, x_0 + \delta) \text{ and } |f(z_n)| \leqslant M$$

whence

$$|f(x) + f(x_0)| \leqslant M.$$

Thus

$$|f(x)| \leqslant M + |f(x_0)|, \text{ for } x \in (-\delta, \delta).$$

THEOREM 2. *If $f: \boldsymbol{R} \to \boldsymbol{R}$ is a limit-additive function bounded (in absolute value) on a set $A \subset \boldsymbol{R}$ such that* $\operatorname{int} A \neq \varnothing$ *then f is continuous.*

Proof. Taking Lemma 2 into account, we may suppose that there exist $M > 0$ and $\delta > 0$ such that

$$|f(x)| \leqslant M \text{ for } x \in (-\delta, \delta). \tag{8}$$

Assume that there exists a sequence of real numbers $(x_n)_{n\in N}$, $x_n \xrightarrow[n\to\infty]{} 0$ such that the sequence $(f(x_n))_{n\in N}$ is not convergent to zero. Then there exist an $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k\in N}$ of the sequence $(x_n)_{n\in N}$ with the property $|f(x_{n_k})| \geqslant \varepsilon$, $k \in \boldsymbol{N}$. From the sequence $(x_{n_k})_{k\in N}$ one can still choose either a subsequence $\left(x_{n_{k_p}}\right)_{p\in N}$ such that $f\left(x_{n_{k_p}}\right) \geqslant \varepsilon$, $p \in N$ or a subsequence $\left(x_{n_{k_s}}\right)_{s\in N}$ such that $f\left(x_{n_{k_s}}\right) \leqslant -\varepsilon$, $s \in \boldsymbol{N}$. Suppose, for instance, that we have a sequence $(y_n)_{n\in N}$, $y_n \xrightarrow[n\to\infty]{} 0$ such that $f(y_n) \geqslant \varepsilon$, $n \in N$. Let us choose numbers $N \in \boldsymbol{N}$ and $n_0 \in \boldsymbol{N}$ so that $N\varepsilon > M$ and $Ny_{n_0} \in (-\delta, \delta)$. According to Lemma 1, there exists a sequence $(z_m)_{m\in N}$ such that $z_m \xrightarrow[m\to\infty]{} N \cdot y_{n_0}$ and $f(z_m) \xrightarrow[m\to\infty]{} Nf(y_{n_0}) \geqslant N\varepsilon > M$.

Hence

$$(9) \qquad \bigvee_{m_1 \in N} \bigwedge_{m \geqslant m_1} z_m \in (-\delta, \delta),$$

$$(10) \qquad \bigvee_{m_2 \in N} \bigwedge_{m \geqslant m_2} f(z_m) > M.$$

For $m \geqslant \max(m_1, m_2)$ conditions (9) and (10) are incompatible with (8). If we have a sequence $(y_n)_{n \in N}$, $y_n \xrightarrow[n \to \infty]{} 0$ such that $f(y_n) \leqslant -\varepsilon, n \in N$, we obtain the contradiction in a similar manner, using the boundedness of f from below. So we have

$$(11) \qquad f(x_n) \xrightarrow[n \to \infty]{} 0, \text{ for any sequence } (x_n)_{n \in N} \text{ such that } x_n \xrightarrow[n \to \infty]{} 0.$$

Putting $x = y = 0$ in (1), we obtain the existence of a sequence $(z_n)_{n \in N}$, $z_n \xrightarrow[n \to \infty]{} 0$, for which $f(z_n) \xrightarrow[n \to \infty]{} 2f(0)$. This, jointly with (11), implies $f(0) = 0$. Consequently we obtain the continuity of f at zero. In virtue of Theorem 1, f is continuous everywhere on $\boldsymbol{R}$.

Now, we are going to investigate some properties of discontinuous limit-additive functions. It follows from Theorem 2 that such functions can not be bounded in absolute value on any non-degenerate interval. In the sequel, the word "interval" will always mean a bounded non-degenerate interval. The example of an arbitrary function $f: \boldsymbol{R} \to \boldsymbol{R}$ which has the graph contained and dense in one of the half-planes $\{(x, y) \in \boldsymbol{R}^2 : y \geqslant 0\}$ or $\{(x, y) \in \boldsymbol{R}^2 : y \leqslant 0\}$ shows that a discontinuous limit-additive function may be bounded from one side. In the same way as in the proof of Lemma 2 one can show that any limit-additive function bounded below (above) on some interval is bounded below (above) on every interval.

For any function $f: \boldsymbol{R} \to \boldsymbol{R}$ bounded below on every interval, the function $\varphi_f: \boldsymbol{R} \to \boldsymbol{R}$

$$(12) \qquad \varphi_f(x) := \sup_{\delta > 0} \inf \{f(z) : z \in (x-\delta, x+\delta)\}, \quad x \in \boldsymbol{R}$$

is well defined.

Analogously, for any function $f: \boldsymbol{R} \to \boldsymbol{R}$ bounded above on every interval we define the function $\psi_f: \boldsymbol{R} \to \boldsymbol{R}$ by the formula

$$(13) \qquad \psi_f(x) := \inf_{\delta > 0} \sup \{f(z) : z \in (x-\delta, x+\delta)\}, \quad x \in \boldsymbol{R}.$$

LEMMA 3. *If $f: \boldsymbol{R} \to \boldsymbol{R}$ is bounded below (above) on every interval, then the function φ_f (function ψ_f) is lower (upper) semi-continuous.*

For the proof see e.g. [2] or [3].

Up to now, we have only made use of property (1) from the definition of limit-additive functions. From now on, we shall be applying properties (2) and (3), too.

LEMMA 4. *If $f: \boldsymbol{R} \to \boldsymbol{R}$ is a limit-additive function bounded below (above) on every interval, then the function φ_f (function ψ_f) is additive.*

Proof. We proceed only with the proof for the function φ_f. Fix numbers $x, y \in \boldsymbol{R}$, $\varepsilon > 0$, $\delta > 0$, $\eta > 0$, arbitrarily. We have

$$(14) \qquad \bigvee_{u_0 \in (x-\delta, x+\delta)} f(u_0) < \inf \{f(u) : u \in (x-\delta, x+\delta)\} + \frac{\varepsilon}{3}$$

and

$$(15)\qquad \bigvee_{w_0\in(y-\eta,\,y+\eta)} f(w_0)<\inf\{f(w): w\in(y-\eta,\,y+\eta)\}+\frac{\varepsilon}{3}.$$

Observe that $u_0+w_0\in(x+y-\delta-\eta,\,x+y+\delta+\eta)$. It follows from (1) that there exists a sequence $(z_n)_{n\in N}$, $z_n \xrightarrow[n\to\infty]{} u_0+w_0$ such that

$$f(z_n)\xrightarrow[n\to\infty]{} f(u_0)+f(w_0).$$

Hence

$$(16)\qquad \bigvee_{n_0\in N}\bigwedge_{n\geqslant n_0}\left[z_n\in(x+y-\delta-\eta,\,x+y+\delta+\eta),\, f(z_n)<f(u_0)+f(w_0)+\frac{\varepsilon}{3}\right].$$

(14), (15) and (16) yield

$$\inf\{f(z): z\in(x+y-\delta-\eta,\,x+y+\delta+\eta)\}\leqslant f(u_0)+f(w_0)+\frac{\varepsilon}{3}\leqslant$$

$$\leqslant\inf\{f(u): u\in(x-\delta,\,x+\delta)\}+\inf\{f(w): w\in(y-\eta,\,y+\eta)\}+\varepsilon.$$

Since $\varepsilon>0$ has been chosen arbitrarily, we have

$$(17)\quad \inf\{f(z): z\in(x+y-\delta-\eta,\,x+y+\delta+\eta)\}\leqslant\inf\{f(u): u\in(x-\delta,\,x+\delta)\}+$$
$$+\inf\{f(w): w\in(y-\eta,\,y+\eta)\}\leqslant\varphi_f(x)+\varphi_f(y).$$

As inequality (17) holds for all $\delta>0$, $\eta>0$, we obtain the subadditivity of φ_f:

$$(18)\qquad \varphi_f(x+y)\leqslant\varphi_f(x)+\varphi_f(y),\quad x,y\in\boldsymbol{R}.$$

Fix again numbers $x,y\in\boldsymbol{R}$, $\varepsilon>0$, $\delta>0$ arbitrarily. We have

$$(19)\qquad \bigvee_{z_0\in(x+y-\delta,\,x+y+\delta)} f(z_0)<\inf\{f(z): z\in(x+y-\delta,\,x+y+\delta)\}+\frac{\varepsilon}{2}.$$

One can choose $u_0\in\left(x-\frac{\delta}{2},\,x+\frac{\delta}{2}\right)$ and $w_0\in\left(y-\frac{\delta}{2},\,y+\frac{\delta}{2}\right)$ so that $z_0=u_0+w_0$. In view of (2) there exist sequences $(u_n)_{n\in N}$ and $(w_n)_{n\in N}$, $u_n\xrightarrow[n\to\infty]{}u_0$, $w_n\xrightarrow[n\to\infty]{}w_0$ such that $f(u_n)+f(w_n)\xrightarrow[n\to\infty]{}f(z_0)$. Hence

$$(20)\quad \bigvee_{n_0\in N}\bigwedge_{n\geqslant n_0}\left[u_n\in\left(x-\frac{\delta}{2},\,x+\frac{\delta}{2}\right),\, w_n\in\left(y-\frac{\delta}{2},\,y+\frac{\delta}{2}\right),\, f(u_n)+f(w_n)<f(z_0)+\frac{\varepsilon}{2}\right].$$

From (19) and (20) we obtain

$$\inf\left\{f(u): u\in\left(x-\frac{\delta}{2},\,x+\frac{\delta}{2}\right)\right\}+\inf\left\{f(w): w\in\left(y-\frac{\delta}{2},\,y+\frac{\delta}{2}\right)\right\}\leqslant$$

$$\leqslant f(z_0)+\frac{\varepsilon}{2}<\inf\{f(z): z\in(x+y-\delta,\,x+y+\delta)\}+\varepsilon.$$

Letting ε tend to zero we get

$$(21)\qquad \inf\left\{f(u): u\in\left(x-\frac{\delta}{2}, x+\frac{\delta}{2}\right)\right\}+\inf\left\{f(w): w\in\left(y-\frac{\delta}{2}, y+\frac{\delta}{2}\right)\right\} \leqslant$$
$$\leqslant \inf\{f(z): z\in(x+y-\delta, x+y+\delta)\} \leqslant \varphi_f(x+y).$$

Since inequality (21) holds true for all $\delta>0$, the function φ_f is superadditive:

$$(22)\qquad \varphi_f(x)+\varphi_f(y) \leqslant \varphi_f(x+y), \quad x, y\in \boldsymbol{R}.$$

Conjunction of conditions (18) and (22) gives the additivity of φ_f. In the same manner one may prove that condition (1) leads to superadditivity of ψ_f and condition (2) to its subadditivity.

As is well known any lower (upper) semi-continuous function is bounded below (above) on every compact interval. Hence and from Lemmas 3 and 4 as well as from the properties of the additive functions we obtain immediately the following

THEOREM 3. *If $f: \boldsymbol{R}\to\boldsymbol{R}$ is a limit-additive function bounded below (above) on every interval, then the function φ_f (function ψ_f) is additive and continuous.*

LEMMA 5. *Let $f: \boldsymbol{R}\to\boldsymbol{R}$ be a limit-additive function. For any $x\in\boldsymbol{R}$ and each $k\in\boldsymbol{N}$ there exists a sequence of real numbers $(x_n)_{n\in N}$ such that*

$$x_n \xrightarrow[n\to\infty]{} x \text{ and } 2^k f(x_n) \xrightarrow[n\to\infty]{} f(2^k x).$$

Proof. For $k=1$ the assertion of our lemma coincides with condition (3). Suppose that this assertion holds true for any $x\in\boldsymbol{R}$ and some $k\in N$. Therefore, for arbitrarily fixed $x\in\boldsymbol{R}$ there exists a sequence $(y_n)_{n\in N}$ such that

$$y_n \xrightarrow[n\to\infty]{} 2x \text{ and } 2^k f(y_n) \xrightarrow[n\to\infty]{} f(2^{k+1}x).$$

From (3) it follows that to each $n\in\boldsymbol{N}$ there corresponds a sequence $(x_{n,m})_{m\in N}$ such that

$$x_{n,m} \xrightarrow[m\to\infty]{} \frac{y_n}{2} \text{ and } 2f(x_{n,m}) \xrightarrow[m\to\infty]{} f(y_n).$$

Hence

$$\bigwedge_{n\in N}\ \bigvee_{m_n'\in N}\ \bigwedge_{m\geqslant m_n'} \left|x_{n,m}-\frac{y_n}{2}\right| < \frac{1}{n},$$

$$\bigwedge_{n\in N}\ \bigvee_{m_n''\in N}\ \bigwedge_{m\geqslant m_n''} |2f(x_{n,m})-f(y_n)| < \frac{1}{n}.$$

Put $m_n := \max(m_n', m_n'')$, $x_n := x_{n,m_n}$, for $n\in N$. Then we get

$$|x_n-x| \leqslant \left|x_{n,m_n}-\frac{y_n}{2}\right| + \left|\frac{y_n}{2}-x\right| \leqslant \frac{1}{n}+\frac{1}{2}|y_n-2x| \xrightarrow[n\to\infty]{} 0$$

and

$$\left|2^{k+1}f(x_n)-f(2^{k+1}x)\right| \leqslant \left|2^{k+1}f(x_{n,m_n})-2^k f(y_n)\right|+\left|2^k f(y_n)-f(2^{k+1}x)\right| \leqslant$$

$$\leqslant 2^k \frac{1}{n}+\left|2^k f(y_n)-f(2^{k+1}x)\right| \xrightarrow[n\to\infty]{} 0.$$

Consequently

$$x_n \xrightarrow[n\to\infty]{} x \text{ and } 2^{k+1}f(x_n) \xrightarrow[n\to\infty]{} f(2^{k+1}x).$$

By induction, the assertion of our lemma holds true for any $k \in \boldsymbol{N}$.

LEMMA 6. *Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be a limit-additive function. For any $x \in \boldsymbol{R}$, $l, k \in \boldsymbol{N}$, $r := \frac{l}{2^k}$ there exists a sequence of real numbers $(x_n)_{n\in N}$ such that $x_n \xrightarrow[n\to\infty]{} rx$ and $f(x_n) \xrightarrow[n\to\infty]{} rf(x)$.*

Proof. Fix $x \in \boldsymbol{R}$, $l, k \in \boldsymbol{N}$, $r := \frac{l}{2^k}$. On account of Lemma 5 there exists a sequence $(y_n)_{n\in N}$ with the property

$$y_n \xrightarrow[n\to\infty]{} \frac{x}{2^k} \text{ and } f(y_n) \xrightarrow[n\to\infty]{} \frac{1}{2^k} f(x).$$

Hence

$$ly_n \xrightarrow[n\to\infty]{} rx \text{ and } lf(y_n) \xrightarrow[n\to\infty]{} rf(x).$$

In view of (1), for each $n \in \boldsymbol{N}$ one can find a sequence $(x_{n,m})_{m\in N}$ such that

$$x_{n,m} \xrightarrow[m\to\infty]{} ly_n \text{ and } f(x_{n,m}) \xrightarrow[m\to\infty]{} lf(y_n)$$

which implies that

$$\bigwedge_{n\in N} \bigvee_{m'_n \in N} \bigwedge_{m \geqslant m'_n} \left|x_{n,m}-ly_n\right| < \frac{1}{n},$$

$$\bigwedge_{n\in N} \bigvee_{m''_n \in N} \bigwedge_{m \geqslant m''_n} \left|f(x_{n,m})-lf(y_n)\right| < \frac{1}{n}.$$

Setting $m_n := \max(m'_n, m''_n)$, $x_n := x_{n,m_n}$, $n \in \boldsymbol{N}$ we obtain

$$\left|x_n-rx\right| \leqslant \left|x_{n,m_n}-ly_n\right|+\left|ly_n-rx\right| \leqslant \frac{1}{n}+\left|ly_n-rx\right| \xrightarrow[n\to\infty]{} 0,$$

$$\left|f(x_n)-rf(x)\right| \leqslant \left|f(x_{n,m_n})-lf(y_n)\right|+\left|lf(y_n)-rf(x)\right| \leqslant \frac{1}{n}+\left|lf(y_n)-rf(x)\right| \xrightarrow[n\to\infty]{} 0$$

which ends the proof.

LEMMA 7. *Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be a limit-additive function. For any $x, y \in \boldsymbol{R}$, $l, k \in \boldsymbol{N}$, $l < 2^k$, $r := \frac{l}{2^k}$ there exists a sequence $(z_n)_{n\in N}$ such that*

$$z_n \xrightarrow[n\to\infty]{} rx+(1-r)y \text{ and } f(z_n) \xrightarrow[n\to\infty]{} rf(x)+(1-r)f(y).$$

Proof. According to Lemma 6 there exist sequences $(x_n)_{n\in N}$, $(y_n)_{n\in N}$ such that

$$x_n \xrightarrow[n\to\infty]{} rx, \quad f(x_n) \xrightarrow[n\to\infty]{} rf(x),$$

$$y_n \xrightarrow[n\to\infty]{} (1-r)y, \quad f(y_n) \xrightarrow[n\to\infty]{} (1-r)f(y).$$

From (1) it follows that for each $n \in N$ there exists a sequence $(z_{n,m})_{m \in N}$ such that

$$z_{n,m} \xrightarrow[m \to \infty]{} x_n + y_n, \quad f(z_{n,m}) \xrightarrow[m \to \infty]{} f(x_n) + f(y_n).$$

Hence

$$\bigwedge_{n \in N} \bigvee_{m'_n \in N} \bigwedge_{m \geqslant m'_n} |z_{n,m} - x_n - y_n| < \frac{1}{n},$$

$$\bigwedge_{n \in N} \bigvee_{m''_n \in N} \bigwedge_{m \geqslant m''_n} |f(z_{n,m}) - f(x_n) - f(y_n)| < \frac{1}{n}.$$

Putting $m_n := \max(m'_n, m''_n)$, $z_n := z_{n,m_n}$, for $n \in N$ we get

$$|z_n - rx - (1-r)y| \leqslant |z_{n,m_n} - x_n - y_n| + |x_n - rx| + |y_n - (1-r)y| \leqslant$$

$$\leqslant \frac{1}{n} + |x_n - rx| + |y_n - (1-r)y| \xrightarrow[n \to \infty]{} 0,$$

$$|f(z_n) - rf(x) - (1-r)f(y)| \leqslant |f(z_{n,m_n}) - f(x_n) - f(y_n)| + |f(x_n) - rf(x)| +$$

$$+ |f(y_n) - (1-r)f(y)| \leqslant \frac{1}{n} + |f(x_n) - rf(x)| + |f(y_n) - (1-r)f(y)| \xrightarrow[n \to \infty]{} 0$$

which completes the proof.

Recall that by the *graph* of a function $f: \boldsymbol{R} \to \boldsymbol{R}$ we mean the set $\{(x, y) \in \boldsymbol{R}^2 : y = f(x)\}$. We consider the plane $\boldsymbol{R}^2$ with its natural topology.

THEOREM 4. *If $f: \boldsymbol{R} \to \boldsymbol{R}$ is a limit-additive function, then the following four cases are the only possible*:

(i) *f is an additive and continuous function*;

(ii) *f is a function with the dense graph in $\boldsymbol{R}^2$*;

(iii) *there exists an additive and continuous function $\varphi_f: \boldsymbol{R} \to \boldsymbol{R}$ such that the graph of f is contained and dense in the half-plane $\{(x, y) \in \boldsymbol{R}^2 : y \geqslant \varphi_f(x)\}$*;

(iv) *there exists an additive and continous function $\psi_f: \boldsymbol{R} \to \boldsymbol{R}$ such that graph of f is contained and dense in the half-plane $\{(x, y) \in \boldsymbol{R}^2 : y \leqslant \psi_f(x)\}$.*
Conversely, every function fulfilling one of the conditions (i)—(iv) *is limit-additive.*

Proof. Suppose $f: \boldsymbol{R} \to \boldsymbol{R}$ to be limit-additive. In virtue of the previous theorems and lemmas the following cases are the only possible:

(i) f is an additive and continuous function;

(ii′) the restriction of f to any interval is unbounded from above and from below;

(iii′) f is a function bounded from below and unbounded from above on every interval;

(iv′) f is a function bounded from above and unbounded from below on every interval.

Suppose that (ii′) holds and choose an arbitrary rectangle $(a, b) \times (c, d)$. Since the set $A := \left\{r = \frac{l}{2^k} : l, k \in N, l < 2^k\right\}$ is dense in the interval $(0, 1)$, we deduce that

$$\bigvee_{r \in A} rf(x) + (1-r) f(y) \in (c, d)$$

provided $f(x) < c$, $f(y) > d$; the existence of such a pair $(x, y) \in (a, b)^2$ results from our assumption. Let $(z_n)_{n \in N}$ be such a sequence that

$$z_n \xrightarrow[n \to \infty]{} rx + (1-r)y \text{ and } f(z_n) \xrightarrow[n \to \infty]{} rf(x) + (1-r)f(y).$$

Hence, for sufficiently large $n \in N$, we have $(z_n, f(z_n)) \in (a, b) \times (c, d)$. Now, supppose that (iii′) holds and let $\varphi_f : \boldsymbol{R} \to \boldsymbol{R}$ denote the function defined by (12); φ_f is additive and continuous. Moreover, the definition of φ_f yields $f(x) \geqslant \varphi_f(x)$, for $x \in \boldsymbol{R}$. Suppose that $(a, b) \times (c, d) \subset \{(x, y) \in \boldsymbol{R}^2 : y > \varphi_f(x)\}$. Then

$$c > \varphi_f\left(\frac{a+b}{2}\right) \geqslant \inf\{f(x) : x \in (a, b)\}$$

whence

$$\bigvee_{x \in (a,b)} f(x) < c.$$

Since f is not upper-bounded on (a, b), one can find a $y \in (a, b)$ such that $f(y) > d$. Proceeding further in the same way as in case (ii′) we prove that there exists a $z \in (a, b)$ such that $f(z) \in (c, d)$. Consequently, condition (iii) holds true. Using the properties of the function ψ_f defined by (13) one can show that (iv′) implies (iv). It is easy to check the converse: every function $f: \boldsymbol{R} \to \boldsymbol{R}$ fulfilling one of the conditions (i)—(iv) is limit-additive.

Our last theorem gives full description of the class of limit-additive functions.

REFERENCES

[1] J. ACZÉL, *Lectures on functional equations and their applications*, Academic Press, New York and London, 1966.

[2] S. ŁOJASIEWICZ, *Wstęp do teorii funkcji rzeczywistych*, PWN, Warszawa, 1976.

[3] B. SZ-NAGY, *Introduction to real functions and orthogonal expansions*, Akadémiai Kiado, Budapest, 1964.