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RANDOM FIXED POINTS OF MULTIFUNCTIONS IN GAMES AND DYNAMIC PROGRAMMING

Abstract. Recently several authors demonstrated random fixed point theorems for various classes of multifunctions ([7], [8], [2], [3], [12], [10]). On the other hand we do not know any work on applications of these theorems. In this paper we apply to games and dynamic programming a random analogue of the Fan-Kakutani fixed point theorem. We consider a zero-sum two-person game depending on a random parameter, and present sufficient conditions for the existence of a measurable solution. Then we study the existence of measurable stationary optimal programs in discounted dynamic programming with a random parameter.

1. Preliminaries. Let X, Y be non-empty sets. A *multifunction* φ from X to Y is a function defined on X whose values are non-empty subsets of Y. By the graph of φ we mean

$$\Gamma\varphi := \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

Let X, Y be linear spaces, and Z a convex subset of X. A multifunction φ from Z to Y is called *concave* if for all $x_1, x_2 \in Z$, $\lambda \in [0, 1]$

$$\varphi(\lambda x_1 + (1-\lambda) x_2) \supset \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2).$$

It is easy to see that φ is concave iff its graph $\Gamma\varphi$ is a convex subset of $Z \times Y$. A real-valued function f defined on Z is called *quasiconvex* if for all $x_1, x_2 \in Z$, $\lambda \in [0, 1]$,

$$f(\lambda x_1 + (1-\lambda) x_2) \leqslant \max\{f(x_1), f(x_2)\}.$$

The function f is *quasi-concave* if $-f$ is quasi-convex.

LEMMA 1.1. *Let X, Y be linear spaces, Z a convex subset of X, φ a multifunction from Z to Y, and u a real-valued function defined on $\Gamma\varphi$. If φ is concave, u quasi-concave, and for each $x \in Z$, $u(x, \cdot)$ is bounded from above on $\varphi(x)$, then the function*

$$v(x) := \sup_{y \in \varphi(x)} u(x, y), \quad x \in Z$$

is quasi-concave, and the sets

$$\psi(x) := \{y \in \varphi(x) : v(x) = u(x, y)\}$$

are convex (possibly empty).

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Proof. Let $x_1, x_2 \in Z$ and $\lambda \in [0, 1]$. Then

$$v(\lambda x_1 + (1-\lambda) x_2) \geqslant \sup_{y \in \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)} u(\lambda x_1 + (1-\lambda) x_2, y) =$$

$$= \sup_{y_1 \in \varphi(x_1),\, y_2 \in \varphi(x_2)} u(\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \geqslant$$

$$\geqslant \sup_{y_1 \in \varphi(x_1),\, y_2 \in \varphi(x_2)} \min\{u(x_1, y_1), u(x_2, y_2)\} =$$

$$= \min\{\sup_{y_1 \in \varphi(x_1)} u(x_1, y_1), \sup_{y_2 \in \varphi(x_2)} u(x_2, y_2)\} = \min\{v(x_1), v(x_2)\}.$$

Hence v is quasi-concave.

Now let $x \in Z$, $y_1, y_2 \in \psi(x)$ and $\lambda \in [0, 1]$. It follows from the concavity of φ that $\varphi(x)$ is convex, thus $\lambda y_1 + (1-\lambda) y_2 \in \varphi(x)$. Then

$$u(x, \lambda y_1 + (1-\lambda) y_2) \geqslant \min\{u(x, y_1), u(x, y_2)\} = v(x).$$

Consequently, $\lambda y_1 + (1-\lambda) y_2 \in \psi(x)$.

Throughout the remainder of this section X, Y are metric spaces and $(\Omega, \mathscr{A}, P)$ a probability space. The Borel σ-field of X is denoted by $\mathscr{B}_X$. A function $f: \Omega \to X$ is *measurable* if for each $B \in \mathscr{B}_X, f^{-1}(B) \in \mathscr{A}$. The product $\Omega \times X$ is always considered with the product σ-field $\mathscr{A} \times \mathscr{B}_X$. A function $u: \Omega \times X \to Y$ is called a *Carathéodory map* if for each $\omega \in \Omega$, $u(\omega, \cdot)$ is continuous, and for each $x \in X$, $u(\cdot, x)$ is measurable. If X is separable and u is a Carathéodory map, then u is jointly measurable (see e.g. [6, Theorem 6.1]).

Let φ be a multifunction from X to Y. We call φ *closed* (*compact, convex*) *-valued* if $\varphi(x)$ is closed (compact, convex) for all $x \in X$. For $A \subset Y$ we define

$$\varphi^{-1}(A) := \{x \in X : \varphi(x) \cap A \neq \emptyset\}.$$

The multifunction φ is said to be *upper semicontinuous* (abbreviated to u.s.c.) if for each closed $F \subset Y$, $\varphi^{-1}(F)$ is closed in X. φ is called *lower semicontinuous* if for each open $G \subset Y$, $\varphi^{-1}(G)$ is open. φ is *continuous* if it is upper and lower semicontinuous.

A multifunction φ from Ω to X is measurable if for each open $G \subset X$, $\varphi^{-1}(G) \in \mathscr{A}$ (this is called weakly measurable by Himmelberg [6]). If X is separable, φ closed--valued and measurable, then $\Gamma\varphi \in \mathscr{A} \times \mathscr{B}_X$. The multifunction φ is called *separable* if it is closed-valued, measurable, and X contains a countable subset E such that $E \cap \varphi(\omega)$ is dense in $\varphi(\omega)$ for all $\omega \in \Omega$. If X is separable, φ measurable and $\varphi(\omega) = \overline{\operatorname{int} \varphi(\omega)}$ for all $\omega \in \Omega$, then φ is separable ([3, Proposition 4]).

Let φ be a multifunction from $\Omega \times X$ to Y, and u a real-valued function defined on $\Gamma\varphi$. Define

$$v(\omega, x) := \sup_{y \in \varphi(\omega, x)} u(\omega, x, y),$$

$$\psi(\omega, x) := \{y \in \varphi(\omega, x) : u(\omega, x, y) = v(\omega, x)\}, \quad \omega \in \Omega, \; x \in X.$$

LEMMA 1.2. *Let Y be Polish, $(\Omega, \mathscr{A}, P)$ complete, φ compact-valued, separable in ω and continuous in x. Assume that u is measurable in ω, i.e. for each $(x, y) \in X \times Y$ and each $r \in \boldsymbol{R}$,*

$$\{\omega \in \Omega : y \in \varphi(\omega, x), u(\omega, x, y) > r\} \in \mathscr{A}$$

and continuous in (x, y), i.e. for each $\omega \in \Omega$, $u(\omega, \cdot)$ is continuous on

$$\Gamma\varphi(\omega, \cdot) = \{(x, y) \in X \times Y : y \in \varphi(\omega, x)\}.$$

Then v is a Carathéodory map, and ψ is a compact-valued multifunction measurable in ω and u.s.c. in x.

Proof. It is well known that under our assumptions, v is measurable in ω (cf. [14, Theorem 9.1]; [11, Theorem 1.7]). Because of the continuity assumptions, for each $\omega \in \Omega$, $v(\omega, \cdot)$ is continuous and $\psi(\omega, \cdot)$ is compact-valued and u.s.c. ([1, p. 122]). In order to prove measurability of $\psi(\cdot, x)$ it suffices to show that $\Gamma\psi(\cdot, x) \in \mathscr{A} \times \mathscr{B}_Y$ ([6, Theorem 3.5]). We have

$$\Gamma\psi(\cdot, x) = \{(\omega, y) \in \Gamma\varphi(\cdot, x) : u(\omega, x, y) = v(\omega, x)\}.$$

Since φ is closed-valued and measurable in ω, $\Gamma\varphi(\cdot, x) \in \mathscr{A} \times \mathscr{B}_Y$. The functions u and v are measurable in ω and continuous in (x, y), so they are jointly measurable. Thus $\Gamma\psi(\cdot, x) \in \mathscr{A} \times \mathscr{B}_Y$, as a measurable subset of $\Gamma\varphi(\cdot, x)$.

By $C(X)$ we denote the Banach space of all real-valued bounded continuous functions on X with the sup norm. If X is compact, then $C(X)$ is a Polish space.

A function $F: \Omega \times X \to X$ is a *random contraction* if for each $x \in X$, $F(\cdot, x)$ is measurable, and there is a measurable $k : \Omega \to [0, 1)$ such that for all $\omega \in \Omega$, $x_1, x_2 \in X$,

$$d(F(\omega, x_1), F(\omega, x_2)) \leqslant k(\omega)\, d(x_1, y_2),$$

where d is the metric of X. A mapping $\xi : \Omega \to X$ is called a *random fixed point of F* if it is measurable and for each $\omega \in \Omega$,

$$F(\omega, \xi(\omega)) = \xi(\omega).$$

It holds the following random analogue of the Banach fixed point theorem:

THEOREM 1.3 ([5, Theorem 5]). *If X is a Polish space and $F: \Omega \times X \to X$ is a random contraction, then there exists the unique random fixed point of F.*

Let D be a multifunction from Ω to X with the $\mathscr{A} \times \mathscr{B}_X$-measurable graph ΓD, and let φ be a multifunction from ΓD to X. D is called *stochastic domain* of φ. A function $\xi : \Omega \to X$ is a *random fixed point of φ* if it is measurable, and for each $\omega \in \Omega$,

$$\xi(\omega) \in D(\omega) \cap \varphi(\omega, \xi(\omega)).$$

A multifunction φ with stochastic domain D is said to be *measurable in ω* if for all $x \in X$ and all open $G \subset X$,

$$\{\omega \in \Omega : x \in D(\omega),\ \varphi(\omega, x) \cap G \neq \emptyset\} \in \mathscr{A}.$$

φ is called *u.s.c. (continuous) in x* if for each $\omega \in \Omega$, the multifunction $\varphi(\omega, \cdot)$ is u.s.c. (continuous) on $D(\omega)$.

The main results of this paper are based on the following stochastic version of the Fan-Kakutani fixed point theorem:

THEOREM 1.4 ([3, Theorem 16, Remark 17]; [10, Theorem 6]). *Let X be a Fréchet space (i.e. linear, metric, complete, locally convex), $(\Omega, \mathscr{A}, P)$ a complete probability space, and D a separable multifunction from Ω to X with compact, convex values. Let φ be a closed convex-valued multifunction from ΓD to X. If φ is measurable in ω, u.s.c. in x and for each $(\omega, x) \in \Gamma D$, $\varphi(\omega, x) \subset D(\omega)$, then φ has a random fixed point.*

2. Random minimax theorem. In this section we give a random analogue of the Ky Fan minimax theorem (c.f. [4]).

Let X, Y be non-empty sets and $(\Omega, \mathscr{A}, P)$ a probability space. Let A be a multifunction from Ω to X, B a multifunction from Ω to Y, and f a real-valued function defined in the graph of $A \times B$,

$$\Gamma(A \times B) = \{(\omega, x, y) \in \Omega \times X \times Y : x \in A(\omega), y \in B(\omega)\}.$$

We shall consider a family $\{G_\omega\}_{\omega \in \Omega}$ of *zero-sum two-person games*, where $G_\omega = (A(\omega), B(\omega), f(\omega, \cdot))$; ω is interpreted as *a state of nature*, $A(\omega)$ and $B(\omega)$ are *sets of strategies*, and $f(\omega, \cdot)$ is the *payoff function* in state ω. A pair $(x_0, y_0) \in A(\omega) \times B(\omega)$ is a *solution* of the game G_ω if

$$\max_{x \in A(\omega)} f(\omega, x, y_0) = f(\omega, x_0, y_0) = \min_{y \in B(\omega)} f(\omega, x_0, y).$$

We present sufficient conditions for the existence of a solution depending measurably on ω.

THEOREM 2.1. *Let X, Y be Fréchet spaces and $(\Omega, \mathscr{A}, P)$ a complete probability space. Let A, B be convex compact-valued and separable, f measurable in ω and continuous in (x, y). If for each $(\omega, x, y) \in \Gamma(A \times B)$ the sets*

$$\varphi(\omega, y) := \{x' \in A(\omega) : f(\omega, x', y) = \max_{z \in A(\omega)} f(\omega, z, y)\},$$

$$\psi(\omega, x) := \{y' \in B(\omega) : f(\omega, x, y') = \min_{z \in B(\omega)} f(\omega, x, z)\}$$

are convex, then there exists a measurable $\xi : \Omega \to X \times Y$ such that for each $\omega \in \Omega$, $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega))$ is a solution of the game G_ω.

Proof. Define a new multifunction Φ from $\Gamma(A \times B)$ to $X \times Y$ by

$$\Phi(\omega, x, y) := \varphi(\omega, y) \times \psi(\omega, x).$$

Note that (x_0, y_0) is a solution of G_ω iff $(x_0, y_0) \in A(\omega) \times B(\omega)$ and $(x_0, y_0) \in \Phi(\omega, x_0, y_0)$. We prove that Φ satisfies assumptions of Theorem 1.4. The multifunction $A \times B$ is convex compact-valued and separable. It follows from Lemma 1.2 that φ and ψ are compact-valued, measurable in ω and u.s.c. in the second variable. Hence Φ is convex compact-valued, measurable in ω, and u.s.c. in (x, y). In order to complete the proof we apply Theorem 1.4.

REMARK 2.1. If the function f in Theorem 2.1 is quasi-concave in x and quasi-convex in y, then the sets $\varphi(\omega, y)$ and $\psi(\omega, x)$ are convex (see Lemma 1.1).

REMARK 2.2. We have considered zero-sum two-person games for the sake of simplicity. Our result can be easily generalized for noncooperative n-person games.

3. Measurable stationary optimal programs in discounted dynamic programming. W. Sutherland [13] studied a deterministic model of the economy and presented sufficient conditions for the existence of a stationary optimal program. In this section we present a random analogue of his result.

Let $(\Omega, \mathscr{A}, P)$ be a probability space. We shall consider a family of *dynamic programming models* $M_\omega = (S, \varphi(\omega, \cdot), r(\omega, \cdot), \beta(\omega))$, $\omega \in \Omega$, where S is the *set of states* of some controlled system, the same for all models; φ is a multifunction from $\Omega \times S$ to S, $\varphi(\omega, s)$ is the *set of all states attainable from s in one step*; r is a bounded from above real-valued function defined on $\Gamma\varphi$, $r(\omega, \cdot)$ is the *reward function* in the model M_ω; $\beta : \Omega \to [0, 1)$, $\beta(\omega)$ is the *discount factor* in M_ω.

We assume that the random parameter $\omega \in \Omega$ is known prior to the decision making. Suppose we start to control our system when it is in state $s_0 \in S$. At the first step we choose $s_1 \in \varphi(\omega, s_0)$ and receive a reward $r(\omega, s_0, s_1)$. At the second step we choose $s_2 \in \varphi(\omega, s_1)$, and so on. Such a sequence $\{s_n\}_{n=0}^{\infty}$ is called a *program* starting from s_0 for the model M_ω. Future rewards are discounted with the factor $\beta(\omega)$, so to a program $\{s_n\}$ there corresponds the *total discounted reward*

$$R(\omega, s_0, s_1, \ldots) := \sum_{n=0}^{\infty} \beta^n(\omega)\, r(\omega, s_n, s_{n+1}).$$

A program $\{s_n\}$ is optimal if it maximizes $R(\omega, s_0, s_1, \ldots)$ among all programs starting from the same state s_0. The *decision problem* associated with the model M_ω is following: given s_0 find an optimal program starting from s_0. The *value function* of the model M_ω is defined by

$$V(\omega, s) := \sup R(\omega, s, s_1, s_2, \ldots),$$

where supremum is taken over all programs $\{s_n\}$ such that $s_0 = s$. It is well known that V satisfies the *optimality equation*

$$V(\omega, s) = \sup_{t \in \varphi(\omega, s)} \left(r(\omega, s, t) + \beta(\omega)\, V(\omega, t) \right), \quad \omega \in \Omega, \ s \in S. \tag{3.1}$$

If $r(\omega, \cdot)$ is bounded, then $V(\omega, \cdot)$ is the unique solution of this equation. A program $\{s_n\}$ is optimal in the model M_ω iff

$$V(\omega, s_n) = r(\omega, s_n, s_{n+1}) + \beta(\omega)\, V(\omega, s_{n+1}), \quad n = 0, 1, 2, \ldots \tag{3.2}$$

Throughout the remainder of this section we shall assume that S is a metric space, and φ, r, β depend measurably on ω.

A program $\{s_n\}$ is called *stationary* if $s_n = s_0$ for $n = 0, 1, 2, \ldots$ Such a program is denoted by s_0^∞. We shall give sufficient conditions for the existence of a stationary optimal program which depends measurably on ω. First we examine the existence of stationary programs.

LEMMA 3.1. *If S is a convex compact subset of a Fréchet space, and the multifunction φ is closed convex-valued and u.s.c. in s, then for each $\omega \in \Omega$ there exists a stationary program in the model M_ω.*

Proof. Note that s^∞ is a stationary program in the model M_ω iff $s \in \varphi(\omega, s)$. By the Fan-Kakutani fixed point theorem, for each $\omega \in \Omega$ there is $s \in S$ such that $s \in \varphi(\omega, s)$.

THEOREM 3.2. *Let S be a convex compact subset of a Fréchet space, $(\Omega, \mathscr{A}, P)$ a complete probability space, φ closed convex-valued multifunction from $\Omega \times S$ to S which is separable in ω and continuous in s, r measurable in ω and continuous in (s, t), and β measurable. If for each $\omega \in \Omega$ and each $s \in S$ the set*

$$\psi(\omega, s) := \{t \in \varphi(\omega, s) : V(\omega, s) = r(\omega, s, t) + \beta(\omega) V(\omega, t)\} \tag{3.3}$$

is convex, then there exists a measurable $f: \Omega \to S$ such that for each $\omega \in \Omega$, $f(\omega)^\infty$ is an optimal program in the model M_ω.

Proof. By (3.2), s^∞ is optimal in M_ω iff $s \in \psi(\omega, s)$. We show that ψ satisfies assumptions of Theorem 1.4 with $D(\omega) = S$ for all $\omega \in \Omega$. First we prove that V is a Carathéodory map. For $u \in C(S)$ we define

$$L(\omega, u)(s) := \sup_{t \in \varphi(\omega, s)} \big(r(\omega, s, t) + \beta(\omega) u(t)\big), \quad \omega \in \Omega, \ s \in S. \tag{3.4}$$

L is a random contraction on $C(S)$ (see [11, Lemma 3.1]). By Theorem 1.3, L has the unique random fixed point $\xi: \Omega \to C(S)$. For each $\omega \in \Omega$, $\xi(\omega)$ is a solution of the optimality equation (3.1), thus $V(\omega, s) = \xi(\omega)(s)$. Hence V is a Carathéodory map.

In virtue of Lemma 1.2, ψ is a compact-valued multifunction, measurable in ω and u.s.c. in s. We have assumed that ψ is convex-valued. Because of Theorem 1.4, ψ has a random fixed point $f: \Omega \to S$. Thus for each $\omega \in \Omega$, $f(\omega)^\infty$ is an optimal program in M_ω.

Now we replace rather technical assumption about convexity of $\psi(\omega, s)$ by some additional conditions on φ and r.

THEOREM 3.3. *Let S, $(\Omega, \mathscr{A}, P)$, φ, r and β be as in Theorem 3.2. If φ is concave in s and r is concave in (s, t), then there exists a measurable function $f: \Omega \to S$ such that for each $\omega \in \Omega$, $f(\omega)^\infty$ is an optimal program in M_ω.*

Proof. We show that under our assumptions the multifunction ψ defined by (3.3) is convex-valued, and apply Theorem 3.2. Denote by $CC(S)$ the set of all $u \in C(S)$ which are concave. It is not difficult to see that $CC(S)$ is a closed subset of $C(S)$. Hence $CC(S)$ is a Polish space. Restrict the operator L defined by (3.4) to $\Omega \times CC(S)$. Under our assumptions, $L(\omega, \cdot)$ is an endomorphism of $CC(S)$ for each $\omega \in \Omega$. Then L is a random contraction on $CC(S)$. By the same argument as in the proof of Theorem 3.2, we obtain the concavity of $V(\omega, \cdot)$. Then the function $r + \beta V$ is concave in (s, t). Because of the optimality equation (3.1) and Lemma 1.1, ψ is convex-valued.

REMARK 3.1. We can generalize our model to the case when the state space also varies with ω. Theorems 3.2 and 3.3 hold if we assume that S is a separable multifunction from Ω to a Fréchet space X with compact convex values.

REMARK 3.2. In [9] we studied similar problems in a stochastic dynamic programming model.

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