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OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH ADVANCING ARGUMENT

Abstract. The literature devoted to differential-functional equations with advancing argument is rather scarce. In the papers [1], [2], [4], [7] the existence of solutions of Cauchy problem and in [5], [6] and [8] — of Nicoletti problem for differential-functional equations with bounded advancement of argument are investigated. The differential-functional equations with unbounded advancement of argument are considered in the articles [3] and [9]. Namely in [3] the existence and uniqueness of solution of the Nicoletti problem is proved, and in [9] an existence theorem for the Cauchy problem is given. The purpose of this paper is to formulate an existence theorem for the Nicoletti problem in the case where the advancement of the argument is unbounded. The proof of this theorem is based on Schauder's fixed point theorem.

1. Notations and definitions. Let $(\boldsymbol{R}^n, |\cdot|)$ be an n-dimensional euclidean space, and let $C = C(\boldsymbol{R}^+)$ denote the space of all continuous functions $u\colon \boldsymbol{R}^+ \to \boldsymbol{R}^n$ (where $\boldsymbol{R}^+ = [0, +\infty))$ with the topology of almost uniform convergence. We assume that there are given: the mapping $F\colon \boldsymbol{R}^+ \times C \to \boldsymbol{R}^n$, the Nicoletti operator $N\colon C \to \boldsymbol{R}^n$ and the element $\eta \in \boldsymbol{R}^n$. Consider the problem of existence of solution of the differential-functional equation

$$u'(t) = F(t, u) \text{ for } t \in \boldsymbol{R}^+ \tag{1}$$

together with a generalized Nicoletti condition

$$N(u) = \eta . \tag{2}$$

Here N is a linear operator; the classical Nicoletti operator is given by $N(u) = \big(u_1(t_1), \ldots, u_n(t_n)\big)$ with given $t_1, \ldots, t_n$. The solution of the problem (1)—(2) will be understood as the function $u\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that

1° u is differentiable in $\boldsymbol{R}$;

2° u satisfies the conditions (1) and (2).

In reference to the function in (1), (2) we make the following assumptions:

(A 1) For each fixed $u \in C$ the function $F(\cdot, u) \in C$.

(A 2) The function $F(t, \cdot)$ is continuous uniformly in $t \in [0, \varkappa]$ for each $\varkappa > 0$.

(A 3) The Nicoletti operator N is linear, bounded and such that for each $\xi \in \boldsymbol{R}^n$, $N(\xi \cdot \alpha(t)) = \xi$, where $\alpha(t) = 1$ for $t \in \boldsymbol{R}^+$.

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(A 4) There exist continuous functions $P: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ and $Q: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ and positive constants $\alpha, \beta, \gamma, \lambda$, such that

(i) $\alpha \in (0, 1], \beta \in (0, \alpha \cdot \gamma)$.

(ii) $|F(t, u)| \leqslant P(t) + Q(t) \{\sup_s [|u(s)| \exp(-\gamma \int_t^s L(\tau)\mathrm{d}\tau - \lambda s)]\}^\alpha$

where $t \leqslant s$ and $L(t) = P(t) + Q(t)$ for $t \in \boldsymbol{R}^+$.

(iii) For each $u \in C^1$, the condition

$$|u'(t)| \leqslant \alpha\gamma L(t) \exp(\alpha\gamma \int_0^t L(\tau)\mathrm{d}\tau), \ t \in \boldsymbol{R}^+$$

implies the inequality

$$|u(t) - N(u)| \leqslant \beta \exp(\alpha\gamma \int_0^t L(\tau)\mathrm{d}\tau), \ t \in \boldsymbol{R}^+,$$

where C^1 denotes the class of all continuously differentiable functions u: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$.

Let Φ_0 be the set of all functions $u \in C$, for which the condition

$$\|u\| = \sup_{t \geqslant 0} \{|u(t)| \exp(-\gamma \int_0^t L(\tau)\mathrm{d}\tau - \lambda t)\} \leqslant A \tag{3}$$

holds true for some positive constant A, where

$$A \geqslant \max\left\{1, \frac{\alpha\gamma|\eta|}{\alpha\gamma - \beta}\right\}. \tag{4}$$

It is easy to check that Φ_0 is nonempty, bounded and closed set.

2. The existence theorem.

THEOREM. *If the assumptions* (A1)—(A4) *are satisfied, then there exists a solution of the problem* (1)—(2).

Proof. Notice that the problem (1)—(2) is equivalent to the problem

$$u(t) = \eta + (Tu)(t) - N(Tu) \text{ for } t \in \boldsymbol{R}^+, \tag{5}$$

where

$$(Tu)(t) = \int_0^t F(\tau, u)\mathrm{d}\tau, \ t \in \boldsymbol{R}^+. \tag{6}$$

We define an operator $S: \Phi_0 \to C$, as follows:

$$(Su)(t) = \eta + (Tu)(t) - N(Tu), \ t \in \boldsymbol{R}^+. \tag{7}$$

We shall show that the operator S maps Φ_0 into itself. Let $u \in \Phi_0$; in virtue of (7) we have for $t \in \boldsymbol{R}^+$:

$$|(Su)(t)| \leqslant |\eta| + |(Tu)(t) - N(Tu)|.$$

Using (6), (7) and the assumption (A 4) (ii) we obtain

$$|(Tu)'(t)| = |F(t,u)| \leqslant P(t)+Q(t)\,\{\sup_s[|u(s)|\exp(-\gamma\int_0^t L(\tau)\,d\tau-\lambda s)];\ s\geqslant t\}^\alpha$$

$$= P(t)+Q(t)\,\{\sup_s[|u(s)|\exp(-\gamma\int_0^s L(\tau)\,d\tau-\lambda s+\gamma\int_0^t L(\tau)\,d\tau);\ s\geqslant t\}^\alpha$$

$$\leqslant P(t)+Q(t)\,[A\exp(\gamma\int_0^t L(\tau)\,d\tau)]^\alpha \leqslant [P(t)+Q(t)]\,A^\alpha\exp(\alpha\gamma\int_0^t L(\tau)\,d\tau)$$

$$\leqslant A^\alpha L(t)\exp(\alpha\gamma\int_0^t L(\tau)\,d\tau).$$

Hence,

$$|\alpha\gamma A^{-\alpha}(Tu)'(t)| \leqslant \alpha\gamma L(t)\exp(\alpha\gamma\int_0^t L(\tau)\,d\tau) \tag{8}$$

and by the assumption (A 4) (iii) the following inequality results

$$|\alpha\gamma A^{-\alpha}(Tu)(t)-N(\alpha\gamma A^{-\alpha}Tu)| \leqslant \beta\exp(\alpha\gamma\int_0^t L(\tau)\,d\tau).$$

From the above we obtain the estimation

$$|(Tu)(t)-N(Tu)| \leqslant A^\alpha(\alpha\gamma)^{-1}\beta\exp(\alpha\gamma\int_0^t L(\tau)\,d\tau),\quad t\in \boldsymbol{R}^+, \tag{9}$$

also

$$|(Tu)(t)-N(Tu)|\exp(-\gamma\int_0^t L(\tau)\,d\tau-\lambda t) \leqslant A^\alpha(\alpha\gamma)^{-1}\beta\exp(\gamma\int_0^t L(\tau)\,d\tau)^{\alpha-1}\exp(-\lambda t).$$

Using the definition (3) and the condition (4) we have

$$\|Su\| \leqslant |\eta|+A^\alpha\beta(\alpha\gamma)^{-1} \leqslant A.$$

From these inequalities it follows that $Su\in\Phi_0$. We will show the continuity of the mapping S. In this aim we consider two cases:

1° Let $t\in(\mu(\varepsilon),+\infty)$, where

$$\mu(\varepsilon) = \lambda^{-1}\ln(2A^\alpha\beta(\alpha\gamma\varepsilon)^{-1}) \tag{10}$$

for some constant $\varepsilon>0$. Then for $u_1,u_2\in\Phi_0$, by (7) and (9) we get

$$|(Su_1)(t)-(Su_2)(t)| \leqslant |(Tu_1)(t)-N(Tu_1)|+|(Tu_2)(t)-N(Tu_2)| \leqslant$$

$$\leqslant 2A^\alpha\beta(\alpha\gamma)^{-1}\exp(\alpha\gamma\int_0^t L(\tau)\,d\tau),\quad t\in\boldsymbol{R}^+.$$

Therefore, using (3) and (10) we have the estimation

$$\|(Su_1)(t)-(Su_2)(t)\| \leqslant \varepsilon.$$

2° When $t\in[0,\mu(\varepsilon)]$ we obtain by (7) and (A3)

$$\|(Su_1)(t)-(Su_2)(t)\| = |(Tu_1)(t)-(Tu_2)(t)-N(Tu_1-Tu_2)|.$$

But from (6) and (A1) it follows

$$(11)\qquad \left|(Tu_1)(t)-(Tu_2)(t)\right| \leqslant \int_0^t \left|F(\tau, u_1)-F(\tau, u_2)\right| d\tau \quad \frac{\varepsilon}{2\mu(\varepsilon)} t \leqslant \frac{\varepsilon}{2},$$

if only the norm $\|u_1-u_2\|$ is sufficiently small. Hence by (11) and (A3) the following inequalities are hold

$$\left|(Tu_1)(t)-(Tu_2)(t)-N(Tu_1-Tu_2)\right| \leqslant \left|(Tu_1)(t)-(Tu_2)(t)\right|+\left|N(Tu_1-Tu_2)\right| \leqslant \varepsilon,$$

and the continuity of S is proved.

Now we shall show that the set $S(\Phi_0)$ is compact. Put $B>0$ fixed, and let $0 \leqslant t_1 \leqslant t_2 \leqslant B$. In view of (6), (7), (A4) (ii) and (3) for each $u \in \Phi_0$ we get

$$\left|(Su)(t_2)-(Su)(t_1)\right| \leqslant \int_{t_1}^{t_2} F(\tau, u)\, d\tau \leqslant$$

$$\leqslant \int_{t_1}^{t_2} \Big(P(t)+Q(t) \{\sup_s [|u(s)| \exp(-\gamma \int_t^s L(\tau)\, d\tau - \lambda s)]; s \geqslant t\}^\alpha\Big) dt$$

$$\leqslant \int_{t_1}^{t_2} (P(t)+Q(t) A^\alpha)\, dt = K(B)(t_2-t_1),$$

for some positive constant $K(B)$, where

$$K(B)=\max \{P(t)+Q(t)\, A^\alpha\};\ t \in [0, B].$$

It follows that the restriction $S(\Phi_0)|_{[0,B]}$ of the family $S(\Phi_0)$ to the interval $[0, B]$ is equicontinuous. Moreover, using (7) and (9) we obtain for $t \in [0, B]$

$$\left|(Su)(t)\right| \leqslant |\eta|+\left|(Tu)(t)-N(Tu)\right| \leqslant |\eta|+A^\alpha \beta (\alpha\gamma)^{-1} \exp\Big(\alpha\gamma \int_0^t L(\tau)\, d\tau\Big) \leqslant$$

$$\leqslant |\eta|+A^\alpha \beta(\alpha\gamma)^{-1} \exp\Big(\alpha\gamma \int_0^B L(\tau)\, d\tau\Big) = M(B) = \text{const}.$$

Also, we conclude, that $S(\Phi_0)|_{[0,B]}$ is equibounded. In view of the Arzela theorem for right-hand open intervals (see [1]) we obtain the compactness of the image $S(\Phi_0)$. Now the existence of solution of the problem (1)—(2) follows from the Schauder's fixed point theorem.

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