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EXAMPLES OF CONVOLUTION PRODUCTS

Abstract. In this work we introduce new types of convolution products on the set of the complex-valued continuous functions defined in $[0, \infty]$ and we see that there is a geometrical interpretation. Then we show that the corresponding fields of fractions are isomorphic to the classical field of the Mikusiński operators. The idea of transport of structure is essential in this work.

1. Transport of structure. Let R be an algebra over the field k and let S be a vector space over k, furthermore let ψ be a linear 1-1 map between R and S: $R \xrightarrow{\psi} S$. By setting

$$x \cdot y := \psi\left(\psi^{-1}(x) \cdot \psi^{-1}(y)\right), \quad x, y \in S \tag{1}$$

a product is defined on S which "enlarges" S to a k-algebra which is isomorphic to R.

Now let R be a k-vector space and let S be a k-algebra and let ψ be, like above, a linear 1-1 map between R and S. By

$$\mu \cdot \nu := \psi^{-1}\left(\psi(\mu) \cdot \psi(\nu)\right), \quad \mu, \nu \in R$$

a product is defined on R and makes R isomorphic to S.

In both cases "structure" is transported via 1-1 map from one set to another. In the first case the domain, in the second the image has more structure.

2. A generalization of sinus and cosinus.** Before it is possible to define the new convolution products we must introduce two functions — a generalization of sinus and cosinus.

Let us consider the following system of differential equations

$$\begin{aligned} \sigma_p' &= \gamma_p^{p-1}, \\ \gamma_p' &= -\sigma_p^{p-1} \quad \text{with } 0 < p, \end{aligned} \tag{2}$$

and the initial conditions

$$\sigma_p(0) = 0, \quad \gamma_p(0) = 1\,. \tag{3}$$

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THEOREM 2.1. *For σ_p and γ_p the following conditions hold*:

(i) *σ_p and γ_p exist in the interval $[0, L_p]$, where*

$$L_p = \int_0^1 (1-\mu^p)^{1/p}\, d\mu .$$

(ii) $\sigma_p^p(r)+\gamma_p^p(r) = 1$, $r\in[0, L_p]$.

(iii) *For the denotations introduced in fig.* 1

$$r = 2F, \quad r\in[0, L_p].$$

(iv) $\sigma_p(L_p-r) = \gamma_p(r)$, $r\in[0, L_p]$.

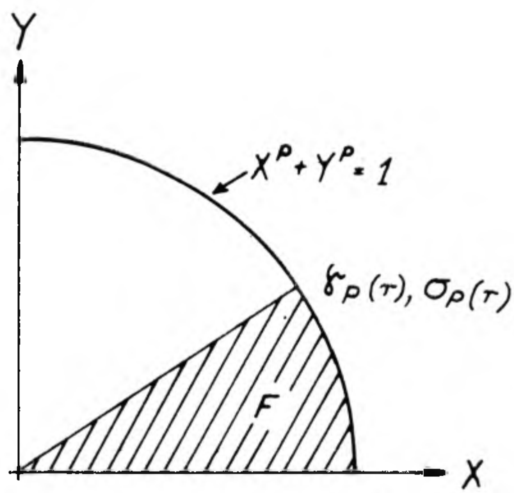


fig. 1

Proof: We begin with (ii). (2) yields

$$\sigma_p^{p-1}\, d\sigma_p = -\gamma_p^{p-1}\, d\gamma_p$$

and with (3) follows (ii). (iii) follows with help of Leibnitz's sector rule

$$2F = \int_0^r \left(\gamma_p(\xi)\sigma_p'(\xi)-\sigma_p(\xi)\gamma_p'(\xi)\right) d\xi = r.$$

(i) can be seen "geometrically" with help of fig. 1, r is two times the area under the function

$$(1-\mu^p)^{1/p}, \quad \mu\in[0, 1]$$

that means

$$L_p = \int_0^1 (1-\mu^p)^{1/p}\, d\mu .$$

(iv) follows from the symmetry of the curve $\gamma_p+\sigma_p = 1$

REMARK. The properties (ii) and (iii) determine the equations (2) and (3). For example

$$\sigma_2(r) = \sin(r), \quad \gamma_2(r) = \cos(r), \quad L_2 = \pi/2,$$

$$\sigma_1(r) = r, \quad \gamma_1(r) = 1-r, \quad L_1 = 1 .$$

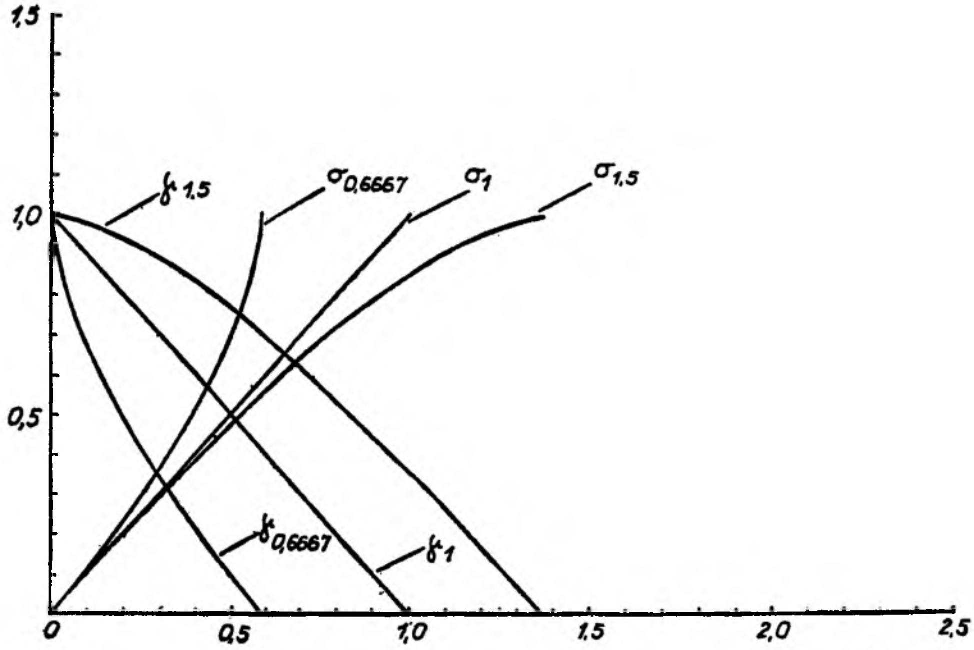
DEFINITION 2.2. The functions σ_p, γ_p, defined in the interval $[0, L_p]$ are called p-sinus resp. p-cosinus.

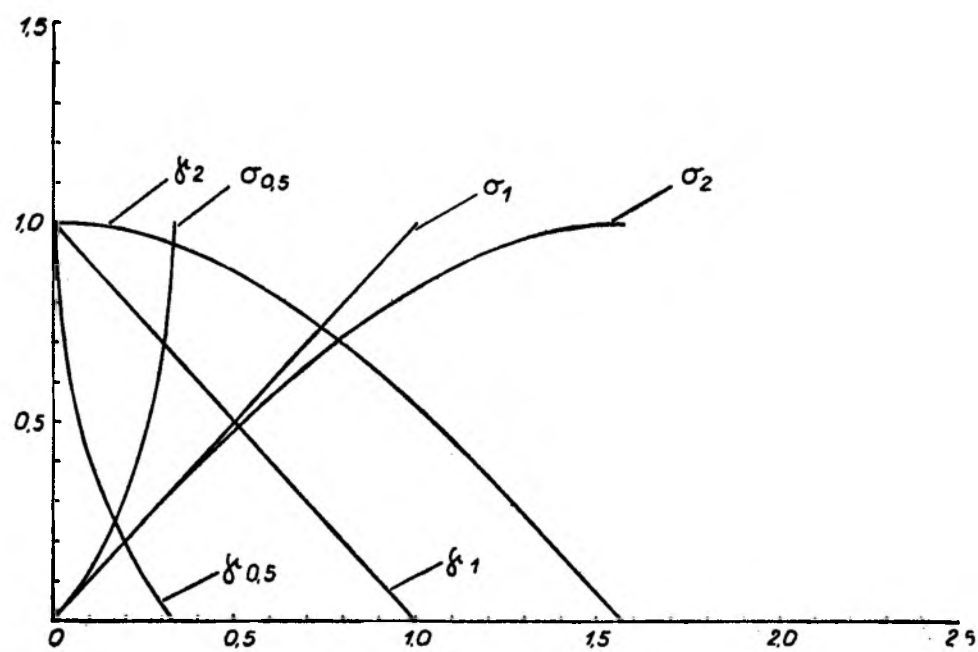
Figure 2 shows σ^p and γ^p for some $0<p$.

3. The new convolution products. With help of σ_p and γ_p we are able to define for every $0<p$ a convolution product. Already now we call it convolution "product" although we have not showed that the convolution product is really a product.

Let C be the set

$$C := \{f\colon [0, \infty) \to \mathbf{C} \mid f \text{continuous}\} .$$


1,5
1,0
0,5
0
0,5
1,0
1,5
2,0
2,5
γ 1,5
σ 0,6667
σ 1
σ 1,5
γ 0,6667
γ 1
fig. 2 a


1,5
1,0
0,5
0
0,5
1,0
1,5
2,0
γ 2
σ 0,5
σ 1
σ 2
γ 0,5
γ 1
fig. 2b

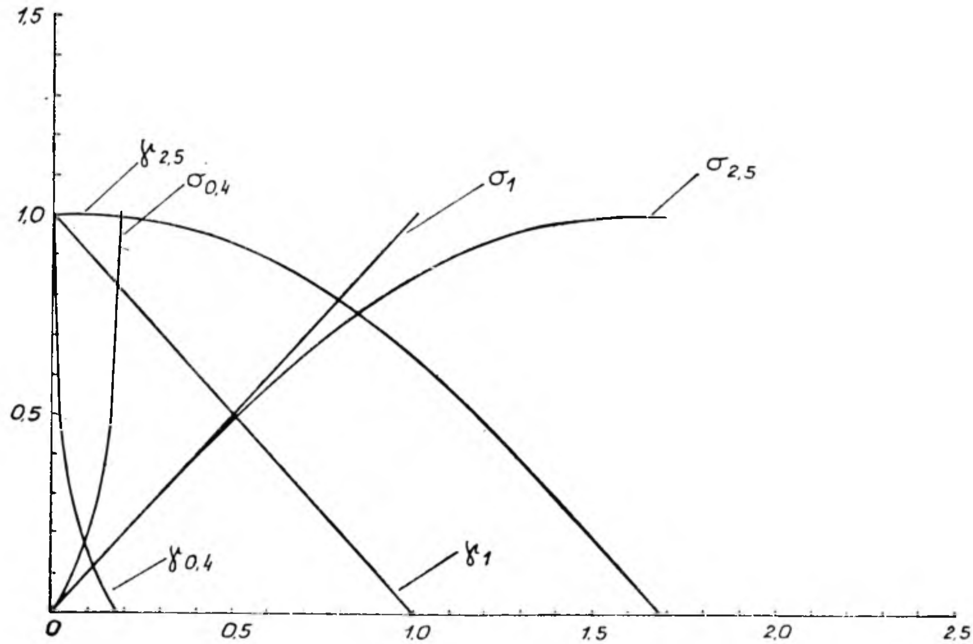

1,5
1,0
0,5
0
0,5
1,0
1,5
2,0
2,5
$\gamma_{2,5}$
$\sigma_{0,4}$
σ_1
$\sigma_{2,5}$
$\gamma_{0,4}$
γ_1

fig. 2c

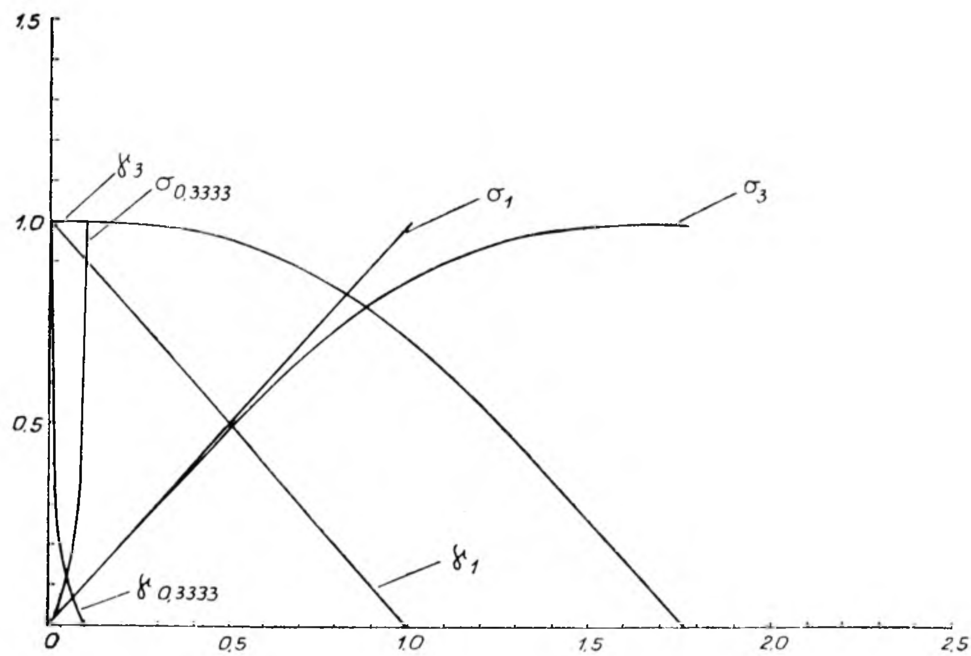
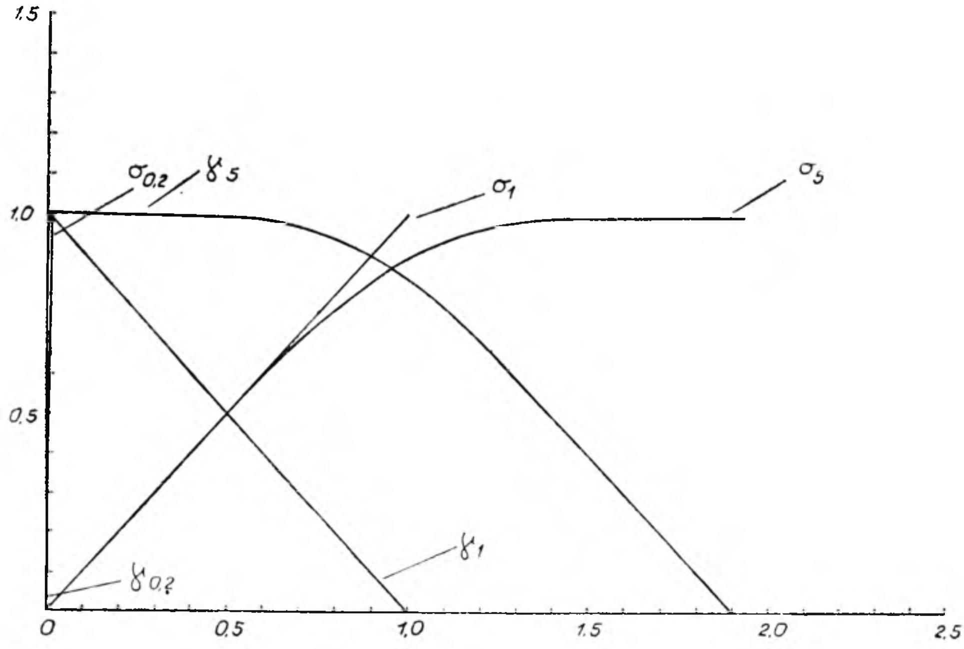
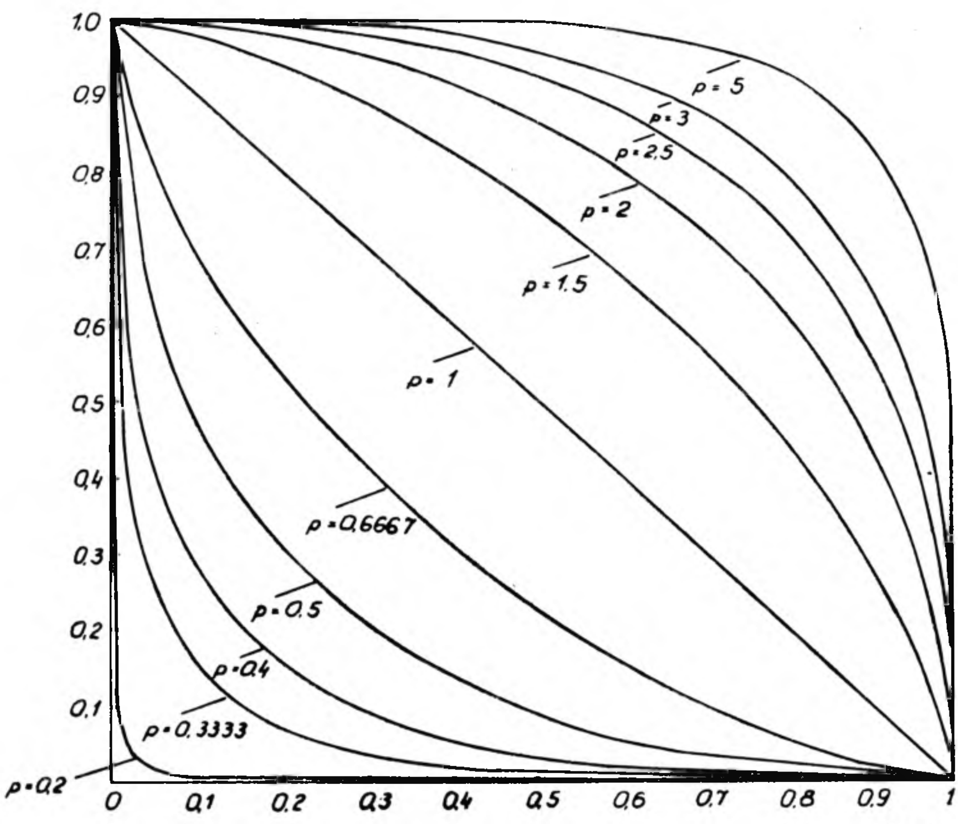

1,5
1,0
0,5
0
0,5
1,0
1,5
2,0
2,5
γ_3
$\sigma_{0,3333}$
σ_1
σ_3
γ_1
$\gamma_{0,3333}$

fig. 2d



1,5
1,0
0,5
0
0,5
1,0
1,5
2,0
2,5

fig. 2e



p = 5
p = 3
p = 2,5
p = 2
p = 1,5
p = 1
p = 0,6667
p = 0,5
p = 0,4
p = 0,3333
p = 0,2
1,0
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1

fig 3

DEFINITION 3.1. Let $f, g \in C$ and let $p > 0$ then the function

$$f *_p g(t) := \int_0^t f\left(t\gamma_p\left(L_p \frac{r}{t}\right)\right) \cdot g\left(t\sigma_p\left(L_p \frac{r}{t}\right)\right) \mathrm{d}r$$

is called *p-convolution product* or *p-convolution* of f and g. Naturally $f *_p g \in C$. The function

$$(r, t) \mapsto \left(\gamma_p\left(L_p \frac{r}{t}\right), t\sigma_p\left(L_p \frac{r}{t}\right)\right), \quad 0 \leqslant r \leqslant t, \; 0 < p$$

determines the values of the p-convolution. Because t only plays the role of a "factor of similarity" it is possible to say that the function

$$[0, L_p] \mapsto R^2, \; r \mapsto (\gamma_p(r), \sigma_p(r)), \; 0 < p,$$

determines the values of the p-convolution. In fig. 3 we show some of such functions.

Now we introduce the following notation: C^p denotes the vector space of the continuous in $[0, \infty)$ complex-valued functions with the p-convolution as product. In this notation C^1 denotes the ring (algebra) of the continuous in $[0, \infty)$ complex-valued functions with the classical convolution (1-convolution)

$$f *_1 g(t) = \int_0^t f(t-r)\, g(r) \mathrm{d}r$$

as product. In [1] it is proved that C^1 has no zero divisors.

4. Transport of structure in the case of the convolution products. In this section we will show that the p-convolution is really a product that has non zero divisors and that the corresponding field of fractions is isomorphic to the field of the Mikusiński operators. We introduce the following notation: C_q^p is defined like C^p in 3. but it holds also

$$C_q^p = \left\{ f \in C^p \,\middle|\, \lim_{t \to 0} \left| \frac{f(t)}{t^{q-1}} \right| < \infty \right\}, \quad q \geqslant 1, \; 0 < p.$$

A computation yields that C_q^p is closed under the p-convolution.

Two cases must be distinguished in the following calculations. Let $p \geqslant 1$. Then the linear map

$$\Gamma_p : C_1^1 \to C_p^p, \; \Gamma_p(f)(t) = L_p t^{p-1} f\left(\frac{t^p}{p}\right), \quad p \geqslant 1,$$

is an isomorphism between the vector spaces C_1^1 and C_p^p. In the case $0 < p < 1$ the map

$$\Gamma_p : C_{1/p}^1 \to C_1^p, \; \Gamma_p(f)(t) = L_p t^{p-1} f\left(\frac{t^p}{p}\right), \quad 0 < p < 1,$$

is an isomorphism botween the voctor spaces $C_{1/p}^1$ and C_1^p.

C_1^1 is the ring (algebra) of the continuous in $[0, \infty)$ complex-valued functions with the 1-convolution (the classical one) as product, furthermore for $0 < p < 1$, $C_{1/p}^1$ is a subring (subalgebra) of C_1^1.

With formula (1) we define by transport of structure for $p \geqslant 1$ on C_p^p resp. for $0 < p < 1$ on C_1^p a product

$$f \circ_p g = \Gamma_p(\Gamma_p^{-1}(f) *_1 \Gamma_p^{-1}(g))$$

with $f, g \in C_p^p$ for $p \geqslant 1$ and $f, g \in C_1^p$ for $0 < p < 1$.

THEOREM 4.1. *$f \circ_p g = f *_p g$; $f, g \in C_p^p$ for $p \geqslant 1$ resp. $f, g \in C_1^p$ for $0 < p < 1$.*

Proof: It is enough to show that

$$\Gamma_p(f *_1 g) = \Gamma_p(f) *_p \Gamma_p(g).$$

Assume that $p \geqslant 1$. Then we have

$$\Gamma_p(f *_1 g)(t) = L_p t^{p-1} \int\limits_0^{t^p/p} f\left(\frac{t^p}{p} - r\right) g(r) \, \mathrm{d}r.$$

The substitution $r = \dfrac{u^p}{p}$ yields

$$\Gamma_p(f *_1 g)(t) = L_p t^{p-1} \int\limits_0^t f\left(\frac{t^p}{p} - \frac{u^p}{p}\right) g\left(\frac{u^p}{p}\right) u^{p-1} \, \mathrm{d}u,$$

and the substitution $u = t\sigma_p\left(L_p \dfrac{v}{t}\right)$,

by (2) and Theorem 2.1 (ii), yields

$$\Gamma_p(f *_1 g)(t) =$$

$$= L_p t^{p-1} \int\limits_0^t f\left(\frac{t^p}{p} \gamma_p^p\left(L_p \frac{v}{t}\right)\right) g\left(\frac{t^p}{p} \sigma_p^p\left(L_p \frac{v}{t}\right)\right) t^{p-1} \sigma_p^{p-1}\left(L_p \frac{v}{t}\right) L_p \gamma_p^{p-1}\left(L_p \frac{v}{t}\right) \mathrm{d}v.$$

A rearrangement of the right side yields

$$\Gamma_p(f *_1 g)(t) = \int\limits_0^t \Gamma_p(f)\left(t\gamma_p\left(L_p \frac{v}{t}\right)\right) \Gamma_p(g)\left(t\sigma_p\left(L_p \frac{v}{t}\right)\right) \mathrm{d}v,$$

hence from Definition 3.1

$$\Gamma_p(f *_1 g)(t) = (\Gamma_p(f) *_p \Gamma_p(g))(t).$$

The case $0 < p < 1$ can be proved similar with the same substitutions.

Now we have proved that, in the case $p \geqslant 1$ the rings (algebras) C_1^1 and C_p^p, and in the case $0 < p < 1$ the rings (algebras) $C_{1/p}^1$ and C_1^p are isomorphic, that means the p-convolution is in $C_p^p (p \geqslant 1)$ and $C_1^p (0 < p < 1)$ a product without zero divisors.

In the case $p \geqslant 1$, C_1^1 can be embedded in the field of fractions generated by C_p^p by the map

$$f \mapsto \frac{\{t^{p-1}\} *_p f}{\{t^{p-1}\}},$$

that means that the p-convolution has no zero divisors in C_1^p and that the field of fractions of C_1^p is equal to the field of fractions of C_p^p and so isomorphic to the Mikusiński operators.

In the case $0 < p < 1$, C_1^1 can be embedded in the field of fractions of $C_{1/p}^1$ by the map

$$f \mapsto \frac{\{t^{1/p-1}\} *_1 f}{\{t^{1/p-1}\}}$$

that means that the field of fractions generated by $C_{1/p}^1$ is equal to the field of the Mikusiński operators.

REFERENCES

1] J. MIKUSIŃSKI, *Operational Calculus*, PWN and Pergamon Press 1959.