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SOME PROPERTIES OF SOLUTIONS OF THE HEAT EQUATION

Abstract. In this paper we investigate some properties of solutions of the heat equation. Their basic properties are established in [3]. Our object is to prove some partial distribution function inequalities for the area integral which can be used to study the local and the global behavior of solutions of the heat equation. Theorem 3 shows that the area integral A and the nontangential maximal function N are remarkably closely related. The method used in this paper is based on the treatment of analogous problems for harmonic functions in [1].

A function $u(t, x)$ defined on a domain $D \subset \mathbf{R}^{n+1}_{+} = \{(t, x) \in \mathbf{R}^{n+1} : t > 0\}$ is called a *solution of the heat equation* on D if $\frac{\partial^2 u}{\partial x_i^2}$, $i = 1, \ldots, n$, and $\frac{\partial u}{\partial t}$ are continuous on D and satisfy the equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = 0 \tag{1}$$

Introduce the following notations:

$$A^2(x) = A_a^2(x) = \iint_{\Gamma(x)} s^{-\frac{n}{2}} |\nabla_y u(s, y)|^2 \, dy \, ds,$$

$$N(x) = N_a(x) = \sup_{(s,y) \in \Gamma(x)} |u(s, y)|,$$

where $\Gamma(x) = \Gamma_a(x) = \{(s, y) \in \mathbf{R}^{n+1}_{+} : |x - y| < a\sqrt{s}\}$. Let R be a measurable subset of $\mathbf{R}^{n+1}_{+}$ and A_R the nonnegative function on $\mathbf{R}^n$ defined by

$$A_R^2(x) = \iint_{\Gamma(x) \cap R} s^{\frac{-n}{2}} |\nabla_y u(s, y)| \, dy \, ds.$$

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If $\Gamma(x) \cap R$ is nonempty, let

$$N_R(x) = \sup_{(s,y) \in \Gamma(x) \cap R} |u(s, y)|, \quad D_R(x) = \sup_{(s,y) \in \Gamma(x) \cap R} \sqrt{s} |\nabla_y u(s, y)|,$$

otherwise, let $N_R(x) = D_R(x) = 0$.

In the following theorem, $m(A_R > \lambda)$ denotes the Lebesgue measure of the set of $x \in \boldsymbol{R}^n$ satisfying $A_R(x) > \lambda$. Similarly $m(N_R > \lambda)$ and $m(D_R > \lambda)$. Throughout the remainder of this paper c denotes a positive constant whose value may change from line to line.

THEOREM 1. *Let G be an open bounded subset of $\boldsymbol{R}^n$ and R the interior of the complement of $\bigcup_{x \notin G} \Gamma(x)$. Let $\alpha > 1$ and $\beta > 1$. Then*

$$m(A_R > \lambda) \leqslant cm(cN_R > \lambda) + cm(cD_R > \lambda) \tag{2}$$

for all $\lambda > 0$ satisfying

$$m(A_R > \lambda) < \alpha m(A_R > \beta\lambda). \tag{3}$$

The choise of c depends only on α, β, n and a.

(The theorem is also true with G unbounded; this follows easily from the bounded case since A_R, N_R and D_R increase as R increases).

Similarly as Theorem 2 in [1] the proof is based on the following lemma:

LEMMA 1. *Suppose that G is an open bounded nonempty subset of $\boldsymbol{R}^n$ and that F is its complement. Let $\alpha > 1$ and suppose that E is a measurable subset of G satisfying $m(G) \leqslant \alpha m(E)$. Then there is a ball $B \subset G$, with at least one of its boundary points in F, such that*

$$m(B) \leqslant c\alpha m(E \cap B). \tag{4}$$

The choise of c depends only on n.

Proof of Theorem 1. We may assume, without loss of generality that $a = 1$. Suppose that λ satisfies (3). Let A_{R_ε} be defined similarly as A_R, where

$$R_\varepsilon = \{(t, y) \in R : t > \varepsilon\} \text{ for } \varepsilon > 0.$$

Then $A_{R_\varepsilon} \to A_R$ as $\varepsilon \to 0$ and the inequality

$$m(A_{R_\varepsilon} > \lambda) < \alpha m(A_{R_\varepsilon} > \beta\lambda) \tag{5}$$

holds for all small $\varepsilon > 0$. Note that A_{R_ε} is a continuous function vanishing outside of G. Therefore, $G_0 = \{A_{R_\varepsilon} > \lambda\}$ is an open set whose closure is contained in G. Let

$$E = \{A_{R_\varepsilon} \geqslant \beta\lambda, N_{R_\varepsilon} \leqslant \gamma\lambda, D_{R_\varepsilon} \leqslant \delta\lambda\}$$

where γ and δ are positive numbers to be chosen later. Then, by (5),

$$m(G_0) \leqslant \alpha m(E) + \alpha m(N_{R_\varepsilon} > \gamma\lambda) + \alpha m(D_{R_\varepsilon} > \delta\lambda).$$

We shall show that

(6) $$\alpha m(E) \leqslant \frac{1}{2} m(G_0)$$

provided γ and δ are suitably chosen, the choice depends only on α, β and n. The desired inequality follows:

$$m(G_0) \leqslant 2\alpha m(N_{R_\varepsilon} > \gamma\lambda) + 2\alpha m(D_{R_\varepsilon} > \delta\lambda).$$

Suppose that (6) does not hold. Then $m(G_0) < 2\alpha m(E)$ and, by Lemma 1, there is a ball $B \subset G_0$, with at least one of its boundary points not in G_0, such that

$$m(B) \leqslant \alpha_0 \, m(E \cap B),$$

where $\alpha_0 = 2\alpha c_{(4)}$ ($c_{(4)}$ denotes the constant c from the condition (4)). We can assume that B is centred at the origin and has unit radius. Let V be the interior of the set $R^{n+1}_{+} - \bigcup_{x \notin B} \Gamma(x)$. Then the closure of $V_\varepsilon = \{(t, x) \in V : t > \varepsilon\}$ is contained in R. Choose $0 < \eta < \frac{1}{2}$ so that the ball B_0 with centre at the origin and radius $1 - 2\eta$ satisfies

$$\frac{m(B_0)}{m(B)} = 1 - (2\alpha_0)^{-1}.$$

Then, letting $E_0 = E \cap B_0$, we have

(7) $$m(B) \leqslant 2\alpha_0 \, m(E_0).$$

Let $W = \bigcup_{x \in E_0} \Gamma(x) \cap V_\varepsilon$, then $W \subset V_\varepsilon$. Observe that

(8) $$|u(s, y)| \leqslant \gamma\lambda, \qquad \sqrt{s}\,|\nabla_y u(s, y)| \leqslant \delta\lambda$$

for all $(s, y) \in W$, hence for all $(s, y) \in \partial W$ (because indeed, these inequalities hold for all (s, y) in $\Gamma(x) \cap R_\varepsilon$ if $x \in E_0$).

Now consider the integral $A^2_W(x)$. For a suitable δ, the choice of which depends only on a, β, and n, we shall prove that

(9) $$A^2_W(x) \geqslant \frac{1}{2}(\beta^2 - 1)\lambda^2, \quad x \in E_0.$$

To prove this, we fix $x \in E_0$ and observe that

$$\beta^2\lambda^2 \leqslant A^2_{R_\varepsilon}(x) \leqslant A^2_W(x) + A^2_{u_1 \cap \Gamma(x_0)}(x) + A^2_{u_1 \cap \Gamma'(x_0)}(x) + A^2_{u_2}(x) + A^2_{u_3}(x),$$

where

$$u_1 = \{(s, y) \in R_\varepsilon : |y| < \sqrt{s-1}, \, s > 1\}$$

$$u_2 = \{(s, y) \in R_\varepsilon : |y| > \sqrt{s-1}, \, s > 1\}$$

$$u_3 = \{(s, y) \in R_\varepsilon : (s, y) \notin W, \, s \leqslant 1\}$$

and x_0 is a boundary point of B, not in G_0. Using the last remark we have

$$A^2_{u_1 \cap \Gamma(x_0)}(x) \leqslant A^2_{\Gamma(x_0) \cap R_\varepsilon}(x) \leqslant A^2_{R_\varepsilon}(x_0) \leqslant \lambda^2.$$

By (8) we prove

$$A^2_{u_1 \cap \Gamma'(x_0)}(x) = \iint\limits_{\Gamma(x) \cap u_1 \cap \Gamma'(x)} s^{\frac{-n}{2}} |\nabla_y u(s, y)|^2 \, ds\, dy$$

$$\leqslant \delta^2 \lambda^2 \int\limits_1^\infty \int\limits_{\substack{|y| < \sqrt{s-1} \\ |y - x_0| \geqslant \sqrt{s}}} s^{\frac{-n}{2} - 1} \, dy\, ds = c\delta^2 \lambda^2,$$

and similarly

$$A^2_{u_2}(x) = \iint\limits_{\Gamma(x) \cap u_2} s^{\frac{-n}{2}} |\nabla_y u(s, y)|^2 \, ds\, dy \leqslant$$

$$\leqslant \delta^2 \lambda^2 \int\limits_1^\infty \int\limits_{\substack{|x-y| < \sqrt{s} \\ |y| > \sqrt{s-1}}} s^{\frac{-n}{2} - 1} \, dy\, ds = c\delta^2 \lambda^2 \int\limits_1^\infty \frac{s^{\frac{n}{2}} - (s-1)^{\frac{n}{2}}}{s^{\frac{n}{2}+1}} \, ds =$$

$$= c\delta^2 \lambda^2 \int\limits_1^\infty \frac{s^n - (s-1)^n}{s^{\frac{n}{2}+1}[s^{\frac{n}{2}} + (s-1)^{\frac{n}{2}}]} \, ds \leqslant c\delta^2 \lambda^2 \int\limits_1^\infty \frac{ds}{s^2} = c\delta^2 \lambda^2.$$

We now use the fact that $x \in B_0$. This means that $|x| < 1 - 2\eta$ and if $(s, y) \in u_3$, then $\eta^2 < s < 1$. Therefore,

$$A^2_{u_3}(x) = \iint\limits_{u_3 \cap \Gamma(x)} s^{\frac{-n}{2}} |\nabla_y u(s, y)|^2 \, ds\, dy \leqslant \delta^2 \lambda^2 \int\limits_{\eta^2}^1 \int\limits_{|y| < r} s^{\frac{-n}{2} - 1} \, dy\, ds = c\delta^2 \lambda^2$$

using the fact that G is a bounded set. The last estimates (after choosing of a suitable δ) imply the inequality (9). Now using (9) and Green's theorem we have

$$(\beta^2 - 1)\lambda^2 m(E_0) \leqslant 2 \int\limits_{R^n} A^2_W(x)\, dx \leqslant c \iint\limits_W |\nabla_y u(s, y)|^2 \, ds\, dy =$$

$$= \frac{c}{2} \iint\limits_W \left[\sum_{i=1}^n \frac{\partial^2 u^2}{\partial y_i^2} - \frac{\partial u^2}{\partial s} \right] ds\, dy =$$

$$= \frac{c}{2} \left[\int\limits_{\partial W} \sum_{i=1}^n \frac{\partial u^2}{\partial y_i} \cos(y_i, \nu)\, d\sigma - \int\limits_{\partial W} u^2 \cos(t, \nu)\, d\sigma \right].$$

The fact that $\sigma(\partial W) \leqslant cm(B)$ (where σ denotes the measure of the surface area of W), (8) and (7) give

$$(\beta^2 - 1)\lambda^2 m(E_0) \leqslant \frac{c}{2} \left[\frac{2\gamma\delta\lambda^2}{\sqrt{\varepsilon}} \sigma(\partial W) + \gamma^2 \lambda^2 \sigma(\partial W) \right] \leqslant$$

$$\leqslant c(\gamma\delta\lambda^2 + \gamma^2\lambda^2)\, m(B) \leqslant 2\alpha_0 c(\gamma\delta\lambda^2 + \gamma^2\lambda^2)\, m(E_0).$$

This gives a contradiction for γ suitably small.

Introduce the following notation:

$$N^0_R(x) = \sup_{(s,y) \in \Gamma(x) \cap R} |u(s, y) - u(s_y, y)|$$

if $\Gamma(x)\cap R$ is not empty, otherwise let $N_R^0(x)=0$, where (s_y, y) is the point on the upper boundary of R directly above $(s, y): s_y = \sup\{s : (s, y)\in R\}$.

THEOREM 2. *Let G be a bounded open subset of R^n and R the interior of the complement of $\bigcup_{x\notin G}\Gamma(x)$. Let $\alpha>1$ and $\beta>1$. Then*

$$m(N_R^0>\lambda)\leqslant cm(cA_R>\lambda)+cm(cD_R>\lambda) \tag{10}$$

for all $\lambda>0$ satisfying

$$m(N_R^0>\lambda)<\alpha m(N_R^0>\beta\lambda). \tag{11}$$

The choice of c depends only on α, β, n and a.
The proof is similar to that of Theorem 3 in [1].

Now let $N_{b,k}(x)$ be defined by

$$N_{b,k}(x)=\sup_{(s,y)\in\Gamma_b{}^k(x)}|u(s, y)|,$$

where $\Gamma_b^k(x)=\{(t, y): |x-y|<b\sqrt{t},\ 0<t<k\}$, b and k are positive real numbers. We define similarly $A_{b,k}$ and $D_{b,k}$.

LEMMA 2. *Let G and R be as in Theorems 1 and 2 and let k be a positive number such that $a\sqrt{k}$ is not less than the diameter of G. Let $b=2a$. Then*

$$D_R\leqslant cN_{b,k}, \tag{12}$$

$$D_R\leqslant cA_{b,k}, \tag{13}$$

and the choice of c depends only on n and a.
Therefore in Theorem 1, (2) can be replaced by

$$m(A_R>\lambda)\leqslant cm(cN_{b,k}>\lambda, G) \tag{14}$$

and in Theorem 2, (10) can be replaced by

$$m(N_R^0>\lambda)\leqslant cm(cA_{b,k}>\lambda, G) \tag{15}$$

(the comma denotes the intersection).

Proof. The height of R does not exceed $h=\frac{k}{4}$; if $(s, x)\in R$, then $a\sqrt{s}\leqslant \leqslant \inf\{|x-y| : y\notin G\}\leqslant a\sqrt{h}$. Therefore $D_R\leqslant D_{a,h}$. By Lemmas 1 and 2 of [3] $D_{a,h}\leqslant cN_{b,k}$ and $D_{a,h}\leqslant cA_{b,k}$. Accordingly, (12) and (13) follow. Because both N_R and D_R vanish off G and are dominated by $cN_{b,k}$ the right side of (2) is dominated by the right side of (14). A similar comparison holds for (10) and (15).

As in Burkholder and Gundy paper [1] we can establish

LEMMA 3. *Let $b>a>0$. Then, for all $\lambda>0$,*

$$m(N_b>\lambda)\leqslant cm(N_a>\lambda). \tag{16}$$

The choice of c depends only on n and ratio $\frac{a}{b}$.

Now let Φ be any function on $[0, \infty]$ such that $0 < \Phi(1) < \infty$ and

$$\Phi(b) = \int_0^b \varphi(\lambda)\,d\lambda, \quad 0 \leqslant b \leqslant \infty$$

for some nonnegative measurable function φ on $(0, \infty)$ satisfying the growth condition

$$\varphi(2\lambda) \leqslant c\varphi(\lambda) \tag{17}$$

(the examples of such functions can be found in the paper [1]).

THEOREM 3. *Under the above conditions*

$$\int_{R^n} \Phi(A)\,dx \leqslant c \int_{R^n} \Phi(N)\,dx. \tag{18}$$

If the left side of (18) *is finite, then* $\lim_{s\to\infty} u(s, y)$ *exists, and is finite and constant, for* $x \in \boldsymbol{R}^n$. *If u is normalized so that this limit is zero, then the converse inequality holds*:

$$\int_{R^n} \Phi(N)\,dx \leqslant c \int_{R^n} \Phi(A)\,dx. \tag{19}$$

The choice of the constants in (18) *and* (19) *depends only on n, a and the growth constant c in* (17).

To prove this we need the following lemma (see [1]).

LEMMA 4. *Let* $f\colon \boldsymbol{R}^n \to [0, \infty]$ *be measurable with compact support. Let* Φ *be as in Theorem* 3 *and suppose that* $\alpha > 1$, $\beta > 1$, $0 < \gamma < \frac{\alpha}{\beta}$, *and*

$$\varphi(\beta\lambda) \leqslant \gamma\varphi(\lambda), \quad \lambda > 0.$$

Then

$$\int_{R^n} \Phi(f)\,dx \leqslant \frac{\alpha\beta\gamma}{\alpha - \beta\gamma} \int_{\Lambda} \varphi(\lambda)\, m(f > \lambda)\,d\lambda,$$

where

$$\Lambda = \{\lambda > 0 : m(f > \lambda) < \alpha m(f > \beta\lambda)\}.$$

Proof of Theorem 3. We apply the lemma to $f = A_R$ for R and G as in previous theorems. Notice that A_R vanishes outside of G. Let $\beta = 2$, $\gamma = c$ (the constant from the growth condition (17)) $\alpha = 4\gamma$. Then, by Lemma 4,

$$\int_{R^n} \Phi(A_R)\,dx \leqslant \alpha \int_{\Lambda} \varphi(\lambda)\, m(A_R > \lambda)\,d\lambda, \tag{20}$$

where Λ is the set of all $\lambda > 0$ satisfying (3). By Theorem 1 and Lemma 2 (inequality (14)) and Lemma 3 (inequality (16)) we get

$$m(A_R > \lambda) \leqslant cm(cN_{b,k} > \lambda, G) \leqslant cm(cN_b > \lambda) \leqslant cm(cN_a > \lambda),$$

hence using Fubini's theorem we infer that the right side of (20) is no greater than

$$\alpha \int_0^\infty \varphi(\lambda)\, cm(cN > \lambda)\,d\lambda = c\alpha \int_{R^n} \Phi(cN)\,dx \leqslant c \int_{R^n} \Phi(N)\,dx.$$

Therefore, (18) holds with A replaced by A_R. Now let $R \nearrow R_+^{n+1}$. By the monotone convergence theorem, (18) follows.

Now we consider the converse inequality and let $b = 2a$. Using the same pattern of reasoning as above, here, in conjunction with Theorem 2, we obtain

$$\int_{R^n} \Phi(N_R^0)\,dx \leqslant c \int_{R^n} \Phi(A_b)\,dx. \tag{21}$$

We assume that the right side of (21) is finite and we show first that $u(t, 0)$ converges as $t \to \infty$. We restrict our attention to the regions $R = R_t$ corresponding to $G = B(0, a\sqrt{t}) = \{x : |x| < a\sqrt{t}\}$. Suppose that x is any point in R^n and that $|x| < a\sqrt{s} < a\sqrt{z} < a\sqrt{t}$. Then $(s, 0)$ and $(z, 0)$ belong to $\Gamma(x) \cap R$ and it follows from the definition of N_R^0 that

$$|u(s, 0) - u(z, 0)| \leqslant |u(s, 0) - u(t, 0)| + |u(z, 0) - u(t, 0)| \leqslant 2N_R^0(x).$$

Therefore,

$$\delta = \frac{1}{2} \limsup_{s,z \to \infty} |u(s, 0) - u(z, 0)| \leqslant \liminf_{t \to \infty} N_R^0(x),$$

and by Fatou's lemma and (21), we have

$$\int_{R^n} \Phi(\delta)\,dx \leqslant c \int_{R^n} \Phi(A_b)\,dx < \infty,$$

which gives $\Phi(\delta) = 0$. Therefore, $\delta = 0$ and this implies that $u(s, 0)$ converges as $s \to \infty$. Using the mean value theorem and (13), we have that

$$|u(s, y) - u(s, 0)| \leqslant s^{-\frac{1}{2}} |y| \sup_{t>0} D_R(x_0) \leqslant c A_b(x_0) s^{-\frac{1}{2}} |y|,$$

provided $|x_0 - y| < a\sqrt{s}$ and $|x_0| < a\sqrt{s}$. Since $A_b(x_0)$ is finite for at least one x_0,

$$\lim_{s \to \infty} |u(s, y) - u(s, 0)| = 0,$$

and the convergence is uniform for $y \in B(0, r)$. This proves the existence, finiteness, and constancy of the limit of $u(s, \cdot)$ as $s \to \infty$. From now on, assume this limit is 0. Let

$$f_{r,R}(x) = \sup\{|u(s, y) - u(s_y, y)| : (s, y) \in \Gamma(x) \cap R,\ |y| < r\},$$
$$f_r(x) = \sup\{|u(s, y)| : (s, y) \in \Gamma(x),\ |y| < r\}.$$

As usual, if the sets are empty, $f_{r,R}(x) = f_r(x) = 0$. Then $f_{r,R} \leqslant N_R^0$ and $\lim_{t \to \infty} f_{r,R} = f_r$, $\lim_{r \to \infty} f_r = N$. Using (21) and Fatou's lemma, we obtain

$$\int_{R^n} \Phi(N)\,dx \leqslant c \int_{R^n} \Phi(A_b)\,dx.$$

Observe that by Lemma 3 we have the complete proof of Theorem 3.

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