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AN EASY PROOF THAT $\beta N - N - \{p\}$ IS NOT NORMAL

Abstract. We give a simple proof that, under CH, $\beta N - N - \{p\}$ is not normal for any $p \in \beta N - N$.

One of the most outstanding open problems in general topology is the question whether $N^* - \{p\}$ (for any space X we write $X^* = \beta X - X$) is not normal for any $p \in N^*$. Under CH the question has been answered in the affirmative: for non P-points by Gillman (see [1]) and for P-points independently by Rajagopalan [8] and Warren [10]. The proof of the non P-point case uses Parovičenko's [7] characterization of N^* and the known proofs of the P-point case do not make use of this characterization. The aim of this note is to give a simple proof that $N^* - \{p\}$ is not normal under CH. Our proof is different since we use Parovičenko's characterization in the P-point case and use the P-point case to solve the non P-point case.

It will be convenient to call a space X a *Parovičenko space* if

(a) X is a zero-dimensional compact space without isolated points of weight 2^ω,

(b) every two disjoint open F_σ's in X have disjoint closures,

and

(c) every nonempty G_δ in X has nonempty interior.

Notice that (b) implies that every countable subspace is C^*-embedded.

It is known, [7], [3], that CH is equivalent to the statement that every Parovičenko space is homeomorphic to N^*.

The following lemma is known. It follows directly from the proof of Gillman's [5] result that, under CH, $N^* - \{p\}$ is not C^*-embedded in N^* for any p. Since I do not know a reference for it I will give the easy proof.

LEMMA (CH). *If $p \in N^*$, then there is a Parovičenko space $X \subset N^*$ containing p such that p is a P-point of X.*

Proof. W.l.o.g, p is not a P-point, so take an open F_σ $U \subset N^*$ with $p \in U^- - U$. Let, by CH, $\{C_\alpha : \alpha < \omega_1\}$ enumerate all nonempty clopen subsets of N^* containing p. By (b) and (c) we can find for each $\alpha < \omega_1$ a nonempty clopen $E_\alpha \subset N^*$ such that

$$E_\alpha \subset \bigcap_{\beta < \alpha} C_\beta - (U \cup \bigcup_{\beta < \alpha} E_\beta)^- .$$

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If $Y=\left(\bigcup_{\alpha<\omega_1}E_\alpha\right)^-$ and if $y\in Y\cap(U^- - \{p\})$ then for some $\mu<\omega_1$,

$$y\in\left(\bigcup_{\varkappa<\mu}E_\varkappa\right)^-,$$

which condraticts (b). Hence $Y\cap U^-=\{p\}$. This implies that p is a P-point of Y for if $F\subset Y-\{p\}$ is any F_σ, then $F\cap U^-=\emptyset$ and consequently, by (b), $F^-\cap U^-=$ $=\emptyset$, i.e. $p\notin F^-$.

For each $\alpha<\omega_1$ take $p_\alpha\in E_\alpha$ and put $X=\{p_\alpha:\alpha<\omega_1\}^- - \{p_\alpha:\alpha<\omega_1\}$. Since p is a P-point of X and since $\{p_\varkappa:\varkappa<\mu\}^- - \{p_\varkappa:\varkappa<\mu\}\approx N^*$ for any $\omega\leqslant\mu<\omega_1$, it easily follows that X satisfies (c). That X satisfies (b) is clear since (b) is closed hereditary in normal spaces. This implies that X is a Parovičenko space, since X clearly satisfies (a).

By the above Lemma we only need to show that $N^*-\{p\}$ is not normal for any P-point $p\in N^*$. Since W. Rudin [9] showed that $N^*-\{p\}\approx N^*-\{q\}$ if $p,q\in N^*$ are P-points (under CH), the proof is completed by the following

EXAMPLE 1. *There is a Parovičenko space X having a P-point p such that $X-\{p\}$ is not normal.*

Let $Z_1=\{\langle\varkappa,\mu\rangle\in(\omega_1+1)\times(\omega_1+1):\mu\leqslant\varkappa\}$ and let $Y=(\omega\times Z)^*$. We claim that Y is a Parovičenko space. $\omega\times Z$ is strongly zero-dimensional, hence so is $\beta(\omega\times Z)$, [6, 16.11]. Also, $\omega\times Z$ is a Lindelöf space with weight ω_1, hence $\omega\times Z$ has $\omega_1^\omega=2^\omega$ clopen subsets, hence $\beta(\omega\times Z)$ has weight 2^ω. It is clear that Y has no isolated points. Y satisfies (b), since $\omega\times Z$ is σ-compact and locally compact, [6, 14.27]. Finally, Y satisfies (c), since $\omega\times Z$ is real compact and locally compact, [4, 3.1].

Let $\pi:\omega\times Z\to Z$ be the projection and $\beta\pi$ its Stone extension.

CLAIM. $\beta\pi^{-1}(\langle\omega_1,\omega_1\rangle)=(\omega\times\{\langle\omega_1,\omega_1\rangle\})^-$.

Clearly $(\omega\times\{\langle\omega_1,\omega_1\rangle\})^-\subset\beta\pi^{-1}(\langle\omega_1,\omega_1\rangle)$. Take

$$x\in\beta\pi^{-1}(\langle\omega_1,\omega_1\rangle)-(\omega\times\{\langle\omega_1,\omega_1\rangle\})^-.$$

Let C be a clopen neighborhood of x in $\beta(\omega\times Z)$ which misses $\omega\times\{\langle\omega_1,\omega_1\rangle\}$. Then $x\in(C\cap(\omega\times Z))^-$ and consequently

$$\beta\pi(x)\in\beta\pi\big((C\cap(\omega\times Z))^-\big)\subset\big(\pi(C\cap(\omega\times Z))\big)^-.$$

$\pi(C\cap(\omega\times Z))$ is σ-compact because $C\cap(\omega\times Z)$ is σ-compact, and $\langle\omega_1,\omega_1\rangle\notin$ $\notin\pi(C\cap(\omega\times Z))$, and $\langle\omega_1,\omega_1\rangle\notin(\pi(C\cap(\omega\times Z)))^-$, which is a contradiction since $\beta\pi(x)=\langle\omega_1,\omega_1\rangle$.

We conclude that $Y-(\omega\times\{\langle\omega_1,\omega_1\rangle\})^*$ admits a perfect map onto $Z-\{\langle\omega_1,\omega_1\rangle\}$, and hence is not normal.

Let $X=Y/(\omega\times\{\langle\omega_1,\omega_1\rangle\})^*$ be the quotient space obtained from Y by collapsing $(\omega\times\{\langle\omega_1,\omega_1\rangle\})^*$ to a point. We claim that X and $p=\{(\omega\times\{\langle\omega_1,\omega_1\rangle\})^*\}$ are as required. First, since $\langle\omega_1,\omega_1\rangle$ is a P-point of Z, it easily follows that $(\omega\times$ $\times\{\langle\omega_1,\omega_1\rangle\})^-$ is a P-set of $\beta(\omega\times Z)$ (a subset A of a space S is called a P-set whenever the intersection of countably many neighborhoods of A is again a neighborhood of A), hence $(\omega\times\{\langle\omega_1,\omega_1\rangle\})^*$ is a P-set of $(\omega\times Z)^*$ and consequently, p is a P-point of X. Second, X is a Parovičenko space. This follows from the fact that Y is a Parovičenko space and that $(\omega\times\{\langle\omega_1,\omega_1\rangle\})^*$ is a nowhere dense closed P-set of Y.

The above example suggests the question whether it can be proved in ZFC that for any Parovičenko space X and for any $p\in X$ it is true that $X-\{p\}$ is not normal. Unfortunately, this is not possible, as the following example shows.

EXAMPLE 2. *There is a compact zero-dimensional space X without isolated points of weight $\omega_2\cdot 2^\omega$ which satisfies* (b) *and* (c), *having a P-point p such that $X-\{p\}$ is both normal and C^*-embedded in X.*

Let $P=\{\alpha\leqslant\omega_2:\mathrm{cf}(\alpha)\geqslant\omega_1\}$. Put $Y=\beta P$ and $X=Y-\{y\in Y: y$ is isolated$\}$. Van Douwen [2] showed that $Y-\{\omega_2\}$ is almost compact, i.e. if A and B are disjoint closed subsets of $Y-\{\omega_2\}$ then one of them is compact, and that Y has weight $\omega_2\cdot 2^\omega$. That implies that $X-\{\omega_2\}$ is almost compact, and hence is normal and C^*-embedded in X. Since X has clearly weight $\omega_2\cdot 2^\omega$ it remains to be shown that X satisfies (b) and (c). That X satisfies (b) is trivial since Y satisfies (b) ([2]). Let G be any nonempty closed G_δ of X. If $G\cap P\neq\emptyset$, then int $G\neq\emptyset$, since $P\cap X$ consists of P-points of X. Hence $G\subset\{\xi\in P:\xi\leqslant\alpha\}^-$ for certain $\alpha<\omega_2$. By transfinite induction it is easy to show that $X\cap\{\xi\in P:\xi\leqslant\alpha\}^-$ has the property that each nonempty G_δ has nonempty interior. Since all these sets are clopen in X it follows that G has nonempty interior.

Since the space of the above example is a Parovičenko space if CH fails our claim follows. It is interesting that such a space exists since it shows that the properties (a), (b) and (c) of N^* are not enough to prove that $N^*-\{p\}$ is not normal for any $p\in N^*$ in ZFC alone.

Since, as remarked earlier, under CH, $N^*-\{p\}$ is neither normal nor C^*--embedded in N^*, we have also obtained the following result.

THEOREM. *Each of the following statements is equivalent to* CH:

(a) *if X is any Parovičenko space and $p\in X$ then $X-\{p\}$ is not normal*;

(b) *if X is any Parovičenko space and $p\in X$ then $X-\{p\}$ is not C^*-embedded in X.*

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