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NOWHERE DENSE CHOICES AND π-WEIGHT

Abstract. The paper is devoted to inequalities between $\pi_0(X)$ and $\pi_d(X)$ where

$$\pi_0(X) := \min\{\pi(U) : U \text{ open and non-empty subset of } X\},$$

$$\pi_d(X) := \min\{|\mathscr{B}| : \text{every open and dense subset of } X \text{ containes an element from } \mathscr{B}\}.$$

From these definitions $\pi_d(X) \leqslant \pi_0(X)$ for every space X. In the paper we construct a space X for which $\pi_d(X) = \omega_1$ and $\pi_0(X) = 2^{\aleph_0}$.

We shall define two cardinal functions π_d and π_0. We shall give conditions which ensure that $\pi_d(X) = \pi_0(X)$ and give a consistent example of a space X such that $\pi_d(X) < \pi_0(X)$.

Recall that for a topological space X, $\pi(X)$ denotes that least cardinal of a π-base for X, i.e.:

$\min\{|\mathscr{B}|$: for each non-empty open $U \subseteq X$, there is $B \in \mathscr{B}$ such that $B \subseteq U\}$.

For a space X we denote by $\pi_0(X)$ the cardinal:

$$\min\{\pi(U): U \text{ is a non-empty open subset of } X\}.$$

And we denote by $\pi_d(X)$ the cardinal

$\min\{|\mathscr{B}|$: for each dense open $U \subseteq X$, there is $B \in \mathscr{B}$ such that $B \subseteq U\}$,

where such families $\mathscr{B}$ are called π_d-bases for X. All other terminology used in this article can be found in one of the standard textbooks [1], [3] or [5]. Furthermore we shall assume that all topological spaces under consideration are regular.

Motivation for the two new definitions comes from the following question.

QUESTION. *Given a collection $\mathscr{U}$ of non-empty open subsets of a space X, can I pick a point $x(U) \in U$ for each $U \in \mathscr{U}$ such that $\{x(U) : U \in \mathscr{U}\}$ is nowhere dense?*

Note that $x : \mathscr{U} \to X$ can be considered as a choice function. The question asks for a choice function with nowhere dense image, hence the first half of our title. Now suppose that $\mathscr{U}$ is a π-base for an open subset $G \subseteq X$. Clearly, the image

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any choice function on $\mathscr{U}$ is dense in G, giving a negative answer to the question and the second half of our title.

A moment's thought will convince the reader that the question has a "NO" answer iff $\mathscr{U}$ is a π_d-base for X. Therefore the question has a "YES" answer for all collections $\mathscr{U}$ such that $|\mathscr{U}| \leqslant \varkappa$ iff $\pi_d(X) > \varkappa$. The argument in the previous paragraph then shows that $\pi_d(X) \leqslant \pi_0(X)$.

These considerations were first made by M. van de Vel. E. K. van Douwen observed that if $\pi_d(X) = \omega$, so does $\pi_0(X)$ and communicated the general problem of the relationship between π_d and π_0 to us.

We first draw some easy conclusions.

THEOREM 1. *For each space X we have*

(a) $\pi_d(X) \leqslant \pi_0(X)$,

(b) $\pi_0(X) \leqslant 2^{\pi_d(X)}$,

(c) $\pi_0(X) \leqslant \pi_d(X) \cdot \pi_\chi(X)$.

Proof. (a) is above. To prove (b), let $\mathscr{A}$ be a π_d-base of X of cardinality $\pi_d(X)$. For each $A \in \mathscr{A}$ pick $X(A) \in A$. Then $X(A): A \in \mathscr{A}$ must be dense in some open set U. Since X is regular we must have

$$\pi_0(X) \leqslant \pi(U) \leqslant 2^{d(U)} \leqslant 2^{\pi_d(X)}.$$

Let us continue and prove (c). For each $A \in \mathscr{A}$, pick a local π-base $\mathscr{B}(A)$ for $x(X)$ such that $|\mathscr{B}(A)| \leqslant \pi_\chi(X)$. It is now easy to check that $\bigcup\{\mathscr{B}(A): A \in \mathscr{A}\}$ forms a π-base for U and verifies (c).

We shall see that for many types of spaces we actually have π_d equal to π_0. We begin with the following useful lemma.

LEMMA 2. *Suppose $\pi_0(X) = \varkappa$, $\mathscr{A}$ is a family of $< \varkappa$ non-empty open subsets of X and $\mathscr{B}$ is a family of $\leqslant \varkappa$ non-empty open subsets of X. Then there is a family $\mathscr{C}$ of λ non-empty open sets precisely refining $\mathscr{B}$ such that no $A \in \mathscr{A}$ is covered by finitely many members of $\mathscr{C}$. Furthermore $\mathscr{C}$ can be chosen as a subfamily of any given π-base for X.*

Proof. Enumerate $\mathscr{B}$ as $\{B_\alpha: \alpha < \lambda\}$. We construct $\mathscr{C} = \{C_\alpha: \alpha < \lambda\}$ by recursively constructing C_α for $\alpha < \lambda$ with the inductive hypothesis at $\beta < \lambda$ that for each $\alpha < \beta$, C_α is a non-empty open subset of B_α such that no $A \in \mathscr{A}$ is covered by finitely many members of $\{\overline{C}_\alpha: \alpha < \beta\}$.

At stage β note that $\pi(B_\beta) \geqslant \varkappa$, and

$$\left|\{A \setminus \bigcup_{\alpha \in F} \overline{C}_\alpha : A \in \mathscr{A} \text{ and } F \in [\beta]^{<\omega}\}\right| < \varkappa.$$

Hence there is some open $G \subset B_\beta$ such that for all $A \in \mathscr{A}$ and for all $F \in [\beta]^{<\omega}$

$$(A \setminus \bigcup_{\alpha \in F} \overline{C}_\alpha) \setminus G \neq \emptyset.$$

Now pick a non-empty open C in our given π-base such that $\overline{C}_\beta \subseteq G$. This completes the induction and the proof.

The next lemma reduces the problem to considerations involving π-weight.

LEMMA 3. *If X is a space, then X has an open subspace Y such that*

$$\pi_d(Y) \leqslant \pi_d(X) \leqslant \pi_0(X) \leqslant \pi_0(Y) = \pi(Y).$$

Proof. Let $\mathscr{A}$ be a π_d-base for X. It suffices to prove that there is an open $Y \subseteq X$ such that $\pi_0(Y) = \pi(Y)$ and $\{Y \cap A : A \in \mathscr{A}\}$ is a π_d-base for Y.

Suppose not. Consider a maximal pairwise disjoint family of open sets $\mathscr{U}$ such that for all $U \in \mathscr{U}$, $\pi_0(U) = \pi(U)$. For each $U \in \mathscr{U}$, pick $G(U)$, an open dense subset of U such that for all $A \in \mathscr{A}$, if $U \cap A \neq \emptyset$ then $(U \cap A) \setminus G(U) \neq \emptyset$. Since $\bigcup \mathscr{U}$ is dense in X, so is $G = \bigcup \{G(U) : U \in \mathscr{U}\}$ and no $A \in \mathscr{A}$ is contained in G, contradicting that $\mathscr{A}$ is a π_d-base.

We can now state some theorems.

THEOREM 4. *If X is a locally compact space, then $\pi_d(X) = \pi_0(X)$.*

Proof. By Lemma 3 we can assume $\pi_0(X) = \pi(X)$ without loss of generality. Let $\varkappa = \pi_0(X) > \pi_d(X) = \lambda$ and show a contradiction.

Let $\mathscr{A}$ be a collection of open sets, of size λ, such that for each dense open $V \subseteq X$ there is some $A \in \mathscr{A}$ such that $\bar{A}$ is compact and $\bar{A} \subseteq V$. Let $\mathscr{B}$ be a π-base for X of size $\varkappa$. By Lemma 2, we can choose $\mathscr{C}$ as in the statement of the lemma. Since $\mathscr{C}$ refines $\mathscr{B}$, $\bigcup \mathscr{C}$ is dense, hence there is a compact $\bar{A} \subset \bigcup \mathscr{C}$ which contradicts the other property of $\mathscr{C}$.

THEOREM 5. *$\pi_d(X) = \pi_0(X)$ if either* (i) *X is locally connected, or* (ii) *X is a linearly ordered topological space.*

Proof. We first show that in each case (i) and (ii) there is a π-base $\mathscr{U}$ for X such that if $U \in \mathscr{U}$ and $\mathscr{V}$ is a pairwise disjoint subcollection of $\mathscr{U}$ such that $U \subseteq \bigcup \mathscr{V}$ then there is some $V \in \mathscr{V}$ such that $U \subseteq V$. For case (i) this is immediate. For case (ii), let $\mathscr{U}$ be the collection: $\{\{P\} : p$ is isolated$\} \cup \{(a, b) : a$ has no immediate successor and b has no immediate predecessor$\}$.

We now use this property of $\mathscr{U}$ to complete the proof that $\pi_d(X) = \pi_0(X)$. Let $\mathscr{A} \subseteq \mathscr{U}$ be a subcollection of size $< \pi_0(X)$. For each open V there is some $U(V) \in \mathscr{U}$ such that no element of $\mathscr{A}$ is contained in $U(V)$. Let $\mathscr{V}$ be a maximal pairwise disjoint subcollection of $\{U(V) : V$ is open in $X\}$; $\bigcup \mathscr{V}$ is dense, showing that $\mathscr{A}$ is not a π_d-base for X.

It is not true that $\pi_d(X) = \pi_0(X)$ for every X, but we only have consistent counterexamples. These use the following lemma.

LEMMA 6. *If X is a non-separable Lusin space of cardinality ω_1, then $\pi_d(X) \leqslant \omega_1$.*

Proof. Enumerate X as $\{x_\alpha : \alpha \in \omega_1\}$. Since every nowhere dense subset of X is countable, the following collection forms a π_d-base:

$$\{X \setminus \mathrm{cl}(\{x_\beta : \beta \in \alpha\}) : \alpha \in \omega_1\}.$$

In [6], there is constructed a dense Lusin subspace Y of $2^\varkappa$, under the assumption BACH plus $\omega_1 < \varkappa < 2^{\omega_1}$. For this space we have $\pi_0(Y) = \varkappa$ and $\pi_d(Y) = \omega_1$.

We shall show that the inequality $\pi_0(X) \leqslant 2^{\pi_d(X)}$ is sharp by showing that it is relatively consistent that 2^{ω_1} is "anything reasonable" and there is a space X with $\pi_d(X) = \omega_1$ and $\pi_0(X) = 2^{\omega_1}$. This is accomplished by Lemma 6 and the following theorem.

THEOREM 7. CON (ZFC *plus* $2^{\omega_1} = \varkappa$) *implies* CON (ZFC *plus* $2^{\omega_1} = \varkappa$ *plus there is a dense Lusin subspace of* $2^\varkappa$ *of cardinality* ω_1).

Proof. We can suppose that we have

$$V \models \text{"ZFC plus CH plus } 2^{\omega_1} = \varkappa\text{"}.$$

We shall construct a generic extension of V in order to prove the theorem. We first describe a partial order $\mathscr{P}$ in the model V. Using CH, let X be a dense Baire (for example, countably compact) subspace of $2^\varkappa$ of size ω_1. Enumerate X as $\{x_\alpha : \alpha \in \omega_1\}$. Let $H(\varkappa)$ be the collection of all finite partial functions from $\varkappa$ into 2. For each $\varepsilon \in H(\varkappa)$, denote by $[\varepsilon]$ the set $\{f \in 2^\varkappa : \varepsilon \subseteq f\}$ which is an elementary open subset of $2^\varkappa$. Let $\mathscr{D}$ denote the set

$$\{D \in [H(\varkappa)]^{\leq\omega} : \bigcup \{[\varepsilon] : \varepsilon \in D\} \text{ is dense in } 2^\varkappa\}.$$

Finally, let $\mathscr{P}$ be the set

$$\{\langle Y, \mathscr{V}\rangle : Y \in [X]^{\leq\omega} \text{ and } \mathscr{V} \in [\mathscr{D}]^{\leq\omega}\}$$

with the ordering $\langle Y_1, \mathscr{V}_1\rangle \leq \langle Y_2, \mathscr{V}_2\rangle$ iff $Y_2 \subseteq Y_1$, $\mathscr{V}_2 \subseteq \mathscr{V}_1$ and for each $D \in \mathscr{V}_2$,

$$Y_1 \setminus Y_2 \subset \bigcup \{[\varepsilon] : \varepsilon \in D\}.$$

Let $\mathscr{G}$ be $\mathscr{P}$-generic over V. We claim that $V[\mathscr{G}] \models$"$2^{\omega_1} = \varkappa$ and there is a dense Lusin subspace of $2^\varkappa$ of size ω_1".

Let $X^* = \bigcup \{Y : \text{for some } \mathscr{V}, \langle Y, \mathscr{V}\rangle \in \mathscr{G}\}$. Observe that P is countably closed and hence $V[\mathscr{G}]$ contains no new countable subsets of V. Since $V \models$ CH and P is 2^ω-centered, all cardinals are preserved. We know that $|X^*| = \omega_1$ by considering the following dense sets:

$$\{\langle Y, \mathscr{V}\rangle : \text{for some } \alpha > \beta, x_\alpha \in Y\}, \ \beta \in \omega_1.$$

It remains to show that X^* is a dense Lusin subspace of $2^\varkappa$. X^* is dense in $2^\varkappa$ because the following sets are dense in P:

$$\{\langle Y, \mathscr{V}\rangle : Y \cap [\varepsilon] \neq \emptyset\}, \ \varepsilon \in H(\varkappa).$$

Note that for each $D \in \mathscr{D}$, the set $\{\langle Y, \mathscr{V}\rangle : D \in \mathscr{V}\}$ is dense. We will show that this implies that every dense open subset of X^* is co-countable. Let U be a dense open subset of $2^\varkappa$. Let E be a maximal pairwise disjoint collection of elementary open subsets of U. $|E| \leq \omega$ and hence $D = \{\varepsilon : [\varepsilon] \in E\} \in \mathscr{D}$. For some $\langle Y\rangle$, $\langle Y, \{D\}\rangle \in \mathscr{G}$ and since elements of $\mathscr{G}$ are compatible we have that $X^* \setminus U \subseteq Y$ and is hence countable.

We note that this proof can be generalized to obtain the following corollary.

COROLLARY 8. CON (ZFC *plus* $2^{(\lambda^+)} = \varkappa$) *implies* CON (ZFC *plus there is a space* X *with* $\pi_d(X) = \lambda^+$ *and* $\pi_0(X) = \varkappa$).

We also note that this theorem gives a consistant example of an L-space of weight 2^{ω_1} where 2^{ω_1} is arbitrarily large. See [2], [4] and [6].

Now we will show that the existance of a dense subspace X of $2^{(2^{\omega_1})}$ such that $\pi_d(X) < \pi_0(X) = 2^{\omega_1}$ is denied by Martin's Axiom and is hence independent of ZFC.

Let X be a space and $\mathcal{U}$ be a collection of subsets of X. We denote by $\mathcal{P}(X, \mathcal{U})$ the set

$$\{\langle S, \mathcal{V}\rangle : S\in[X]^{<\omega}\setminus\{\emptyset\}, \mathcal{U}\in[\mathcal{V}]^{<\omega} \text{ and } S\cap\bigcup\mathcal{V} = \emptyset\}$$

with the partial ordering $\langle S_1, \mathcal{V}_1\rangle \leqslant \langle S_2, \mathcal{V}_2\rangle$ iff $S_2\subseteq S_1$ and $\mathcal{V}_2\subseteq\mathcal{V}_1$.

THEOREM 9. *Assume* MA. *If* $\varkappa\leqslant 2^\omega$ *and* X *is a dense subspace of* $2^\varkappa$, *then* $\pi_d(X)=\pi_0(X)=\varkappa$.

Proof. We show that if $\lambda<\varkappa$, then $\pi_d(X)>\lambda$. Suppose not and derive a contradiction by assuming that $\mathcal{A}$ is a π_d-base of size λ. Without loss of generality assume that each $A\in\mathcal{A}$ is an elementary open set. Since $\lambda<\varkappa$ we can find $Y\in[\varkappa]^\omega$ such that the support of any $A\in\mathcal{A}$ is disjoint from Y. Let $\mathcal{U}$ be the collection of all elementary open sets with support contained in Y.

Let us notice the following facts. $\mathcal{U}$ is countable. If $A\in\mathcal{A}$ and $\mathcal{V}\in[\mathcal{U}]^{<\omega}$, then either $\bigcup\mathcal{V}=2^\varkappa$ or $X\cap(A\setminus\bigcup\mathcal{V})\neq\emptyset$. If $\mathcal{U}'\subseteq\mathcal{U}$ such that for each $U\in\bigcup\mathcal{U}$, $U\cap\bigcup\mathcal{U}'=\emptyset$ then $\bigcup\mathcal{U}'$ is a dense open subset of $2^\varkappa$.

Now consider $\mathcal{P}(X, \mathcal{U})$. From the above facts, we have that $\mathcal{P}(X, \mathcal{U})$ is σ-centered and that for each $A\in\mathcal{A}$, the set

$$\{\langle S, \mathcal{V}\rangle : S\cap(A\setminus\bigcup\mathcal{V})\neq\emptyset\}$$

is dense in $\mathcal{P}(X, \mathcal{U})$. Furthermore, for each $U\in\mathcal{U}$ the set

$$\{\langle S, \mathcal{V}\rangle : U\cap\bigcup\mathcal{V}\neq\emptyset\}$$

is also dense in $\mathcal{P}(X, \mathcal{U})$.

Let $\mathcal{G}\subseteq\mathcal{P}(X, \mathcal{U})$ be a filter which meets each of the dense sets above; and let $G=\bigcup\{\bigcup\mathcal{V}: \text{for some } S, \langle S, \mathcal{V}\rangle\in\mathcal{G}\}$. Then G is a dense open set contradicting that $\mathcal{A}$ is a π_d-base for X.

Only MA for σ-centered posets was used above. In the following theorem we use only MA for a countable poset.

THEOREM 10. *Assume* MA. *If* X *is separable then* $\pi_d(X)=\pi_0(X)$.

Proof. Since $\pi(X)\leqslant 2^{d(X)}\leqslant c$, it suffices to show that if $\pi_d(X)=\lambda<c$ then $\pi_0(X)=\lambda$. We suppose $\pi_0(X)=\varkappa>\lambda$ and derive a contradiction. By Lemma 3 we can assume that $\pi(X)=\varkappa$. We can also assume, without loss of generality, that X is countable and has no isolated points.

Let $\mathcal{A}$ be a π_d-base for X of cardinality λ, and let $\mathcal{B}$ be a π-base for X of cardinality $\varkappa$. Let $\mathcal{C}$ be the family obtained from Lemma 2. Let $\mathcal{D}$ be a complete pairwise disjoint subfamily of $\mathcal{C}$ such that $\bigcup\mathcal{D}$ is dense in X. Let $\mathcal{U}=\{D\setminus F: D\in\mathcal{D}$ and $F\in[X]^{<\omega}\}$.

Now consider $\mathcal{P}(X, \mathcal{U})$. This is countable, and each set $\{\langle S, \mathcal{V}\rangle\in\mathcal{P}(X, \mathcal{U}): S\cap$ $\cap(A\setminus\bigcup\mathcal{V})\neq\emptyset\}$, where $A\in\mathcal{A}$, is dense in $\mathcal{P}(X, \mathcal{U})$. Also each set $\{\langle S, \mathcal{V}\rangle\in$ $\in\mathcal{P}(X, \mathcal{U}): D\cap\bigcup\mathcal{V}\neq\emptyset\}$, where $D\in\mathcal{D}$, is also dense. MA allows us to find a filter $\mathcal{G}\subseteq\mathcal{P}(X, \mathcal{U})$ which meets each of the above dense sets.

Let $G=\bigcup\{\bigcup\mathcal{V}: \langle S,\mathcal{V}\rangle\in\mathcal{G} \text{ for some } S\}$. Since $\mathcal{G}$ meets each of the first type of dense set, no $A\in\mathcal{A}$ is contained in G. Since $\mathcal{G}$ meets each of the second type of dense set and X has no isolated points, G is a dense open subset of X. This contradicts that $\mathcal{A}$ is a π_d-base for X.

The result of van Douwen that $\pi_d(X)=\omega$ implies $\pi_0(X)=\omega$ can be gleaned from the proof of this last theorem. If $\mathscr{A}$ is a countable π_d-base for X, let Y be a countable subset of X meeting each set in $\mathscr{A}$. Now follow the proof of Theorem 10 for the subspace $\mathrm{Int}(\overline{Y})$ of X. MA is not needed since only countable many dense sets need to be met. However, van Douwen's original proof is easier and more straightforward.

We have one more result about π_d and π_0. It uses the following lemma, which is of independent interest.

LEMMA 11. *If X has no isolated points and $c(X)=\omega$, then either there is a Suslin tree of open subsets of X or there is a countable collection of open subsets of X such that for each $F\in[X]^{<\omega}$, $\bigcup\{C\in\mathscr{C}: C\cap F=\emptyset\}$ is dense.*

Proof. We build a tree of open subsets of X, by recursion on the levels of the tree, starting with $T_0=\{X\}$. If level T_α has been defined and $t\in T_\alpha$ we define the node of t, $N(t)$, to be a maximal non-trivial collection of open subsets of t such that for all $U, V\in N(t)$ $\overline{U}\cap\overline{V}=\emptyset$. Let $T_{\alpha+1}=\bigcup\{N(t): t\in T_\alpha\}$.

If $\lim(\lambda)$ and we have T_α for all $\alpha<\lambda$, consider the tree $\bigcup\{T_\alpha:\alpha<\lambda\}$. For each branch b of this tree consider $\mathrm{Int}(\bigcap b)$. Let $T_\alpha=\{\mathrm{Int}(\bigcap b): b$ is a branch of $\bigcup\{T_\alpha:\alpha<\lambda\}\setminus\{\emptyset\}$.

Note that since $c(X)=\omega$ this recursion stops after at most ω_1 steps and that the resulting tree T has no uncountable chains or antichains.

If T is not a Suslin tree, then $|T|=\omega$. In this case, let $\mathscr{C}=T$. Since $\mathscr{C}$ is closed under finite intersections, it only remains to prove that for any $x\in X\bigcup\{C\in\mathscr{C}: x\in C\}$ is dense. To this end let $p\in X$ and show that p is in the closure of $\bigcup\{C\in\mathscr{C}: x\in C\}$. However, this result is obtained by a straightforward consideration of the ways in which p and x can "leave" the tree construction and is therefore left for the reader (i.e. it is messy to write out).

Recall that the *Novak number of a space X* is

$$n(X)=\min\{\varkappa: X \text{ can be covered by } \varkappa \text{ nowhere dense sets}\}.$$

COROLLARY 12. *If X has no isolated points, then $n(X)\leqslant 2^{c(X)}$.*

Proof. This follows from the proof of the lemma since each element of T and each branch of T determine a nowhere dense set, and their union is all of X. The tree T has at most $(c(X))^+$ elements and $2^{c(X)}$ branches.

We use Lemma 11 in the following theorem.

THEOREM 13. *Assume* MA. *If $c(X)=\omega$ and $\pi(X)<c$, then $\pi_d(X)=\pi_0(X)$.*

Proof. Suppose $\pi_d(X)<\pi_0(X)$. By Lemma 3 we can assume $\pi_0(X)=\pi(X)$. Let $\mathscr{A}$ be a π_d-base for X of cardinality $\pi_d(X)$. By Lemma 2 there is a π-base $\mathscr{C}$ such that no finite subcollection of $\mathscr{C}$ covers any element of $\mathscr{A}$. Let $\mathscr{C}_1$ be a maximal pairwise disjoint subcollection of $\mathscr{C}$, By Lemma 11 obtain a countable collection $\mathscr{C}_2$ of open subsets of X such that for each $F\in[X]^{<\omega}$, $\bigcup\{C\in\mathscr{C}_2: C\cap F=\emptyset\}$ is dense.

Let $\mathscr{U}=\{C_1\cap C_2: C_1\in\mathscr{C}_1$ and $C_2\in\mathscr{C}_2\}$. Then $\mathscr{U}$ has the following properties:

(i) no finite subcollection of $\mathscr{U}$ covers an element of $\mathscr{A}$;

(ii) for each $F\in[X]^{<\omega}$, $\bigcup\{U\in\mathscr{U}: U\cap F=\emptyset\}$ is dense.

Now, consider $\mathscr{P}(X, \mathscr{U})$. Since $\mathscr{U}$ is countable, $\mathscr{P}(X, \mathscr{U})$ is σ-centered. By property (i) for each $A \in \mathscr{A}$ the set

$$\{\langle S, \mathscr{V}\rangle : S \cap (A \setminus \bigcup \mathscr{V}) \neq \emptyset\}$$

is dense in $\mathscr{P}(X, \mathscr{U})$. Fix a π-base $\mathscr{B}$ of size $< C$. By property (ii), for each $B \in \mathscr{B}$ the set

$$\{\langle S, \mathscr{V}\rangle : \bigcup \mathscr{V} \cap B \neq \emptyset\}$$

is dense in $\mathscr{P}(X, \mathscr{U})$. Let $\mathscr{G} \subseteq \mathscr{P}(X, \mathscr{U})$ be a filter meeting each of the above dense sets. Let

$$G = \{\bigcup \mathscr{V} : \langle S, \mathscr{V}\rangle \in \mathscr{G} \text{ for some } S\}.$$

Then G is a dense open subset of X witnessing that $\mathscr{A}$ is not a π_d-base for X.

We could have eliminated the hypothesis "$\pi(X) < C$" from Theorem 13 if we could have constructed $\mathscr{U}$ in the proof such that it "self-witnessed denseness" as in the proofs of Theorems 9 and 10. We need an extension of Lemma 10, which, in conclusion, we ask as a question.

QUESTION 14. *Assume* MA. *Suppose X is a space with $c(X) = \omega$ and no isolated points. Does there exist a countable family $\mathscr{U}$ of open subsets of X with the following two properties*:

1. *for each finite $F \subseteq X$, $\bigcup\{U \in \mathscr{U} : U \cap F \neq \emptyset\}$ is dense*;
2. *if $\mathscr{V} \subseteq \mathscr{U}$ such that for each $U \in \mathscr{U}$, $(\bigcup \mathscr{V}) \cap U \neq \emptyset$, then $\bigcup \mathscr{V}$ is dense in X*?

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