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RIGID GRAPHS OF MAPS

Abstract. In this note we construct maps between metric separable connected spaces X and Y such that the graphs are connected, dense and rigid subspaces of the Cartesian product $X \times Y$. From this result it follows that there is no maximal topology among metric separable connected topologies on a given set X.

In this note we shall construct maps between metric separable connect spaces X and Y such that the graphs are connected, dense and rigid subspaces of the Cartesian product $X \times Y$. The first construction of a map $f: \boldsymbol{R} \to \boldsymbol{R}$ with the connected and dense graph in the plane and satisfying the Cauchy equation $f(x)+f(y)=f(x+y)$ was given by F.B. Jones [3] in 1942. More general construction one can find in [4]. In order to obtain the existence of rigid graphs of maps, we shall utilize, in the proof, an idea of W. Sierpiński from [5]. A similar method is also used in de Groot's paper [2].

Spaces considered here are assumed to be separable and metric, i.e. we assume that they are subspaces of the Hilbert's cube I^{ω}.

A continuous map $f: X \to Y$, $X, Y \subset I^{\omega}$, is called a *continuous displacement* [2], iff there exists a subset $V \subset X$ such that

$$|f(V)| = 2^{\omega} \text{ and } V \cap f(V) = \emptyset,$$

Let us notice that each homeomorphism $f: X \to X$ different from the identity map, and where X is a connected subspace of I^{ω}, ia a continuous displacement. Indeed, since $f \neq \mathrm{id}_X$, there exists a point $x \in X$ such that $f(x) \neq x$. Choose disjoint open sets $V, W \subset X$ such that $x \in V$ and $f(x) \in f(V) \subset W$. Since X is a connected metric space hence $|V| = 2^{\omega}$. Thus, $|f(V)| = 2^{\omega}$ and $V \cap f(V) = \emptyset$.

For more exhaustive information on continuous displacements, the reader can refer to de Groot's paper [2].

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A space X is said to be *rigid* if it admits between itself no homeomorphsim different from the identity map. An abound information on rigid spaces can be found in Charatonik's paper [1].

For each map $f: X \to Y$, let $G(f)$ denotes the graph of the map f:

$$G(f) = \{(x, y) \in X \times Y : y = f(x)\}.$$

Let $\pi: X \times Y \to X$ means the projection and let the symbols Int, Bd mean respectively interior and boundary operations.

Let us start from a

LEMMA. *If $f: X \to Y$ is a map between connected metric separable spaces such that for each non-empty open set $G \subset X \times Y$ with non-empty boundary*

$$G(f) \cap \mathrm{Bd}_{X \times Y} G \neq \emptyset,$$

then the graph is connected and dense in $X \times Y$.

Proof. It is obvious that the graph must be dense in $X \times Y$, because the sets of the form $U \times V$, U open in X and V open in Y, create a base for the topology of the space $X \times Y$.

In order to see that the graph must be connected we shall utilize two results from [4]. It was proved in ([4, Lemma 1]) that if X and Y are connected spaces and G is a non-empty subset of $X \times Y$ then one of the following conditions is satisfied:

(a) $\mathrm{Int}_X \pi(\mathrm{Bd}_{X \times Y} G) \neq \emptyset$,

(b) there exists an $x \in X$ such that $\pi^{-1}(x) \subset \mathrm{Bd}_{X \times Y} G$,

(c) G is dense in $X \times Y$.

Secondly ([4, Lemma 2]), if D is a dense subset of a connected space Z such that for each non-empty open set $G \subset Z$ with $D \not\subset G$,

$$D \cap \mathrm{Bd}_Z G \neq \emptyset$$

then D is a connected set.

Put $D = G(f)$ and $Z = X \times Y$. Let us verify that the condition $D \cap \mathrm{Bd}_Z G \neq \emptyset$ is satisfied for each non-empty open set $G \subset Z$ for that $D \not\subset G$.

(1) If $\mathrm{Int}_X \pi(\mathrm{Bd}_Z G) \neq \emptyset$ then according to the assumption $D \cap \mathrm{Bd}_Z G \neq \emptyset$.

(2) If there exists an $x \in X$ such that $\pi^{-1}(x) \subset \mathrm{Bd}_Z G$ then it is clear that $D \cap \mathrm{Bd}_Z G \neq \emptyset$.

(3) If $G \not\supset D$ is dense in Z then

$$D \cap \mathrm{Bd}_Z G = D \cap (Z \setminus G) = D \setminus G \neq \emptyset.$$

Thus, the lemma is proved.

THEOREM. *Let X and Y be metric separable and connected spaces. Then there exists a family $\mathscr{C} \subset \mathrm{Map}(X, Y)$, $|\mathscr{C}| = 2^{\mathfrak{c}}$, $\mathfrak{c} = 2^{\omega}$, such that:*

(1) *each graph $G(f), f \in \mathscr{C}$, is a connected, dense and rigid subspace of the product $X \times Y$,*

(2) *no two distinct graphs $G(f)$ and $G(g)$, $f, g \in \mathscr{C}$, are homoemorphic.*

Proof. Assume that the product $X \times Y$ is a subspace of the Hilbert cube I^ω, $X \times Y \subset I^\omega$. Consider the family

$$\{(f_\alpha : S_\alpha \to I^\omega) : \alpha < 2^\omega\}$$

of all the continuous displacement $f_\alpha : S_\alpha \to I^\omega$, where S_α is a $\mathcal{G}_\delta$ subset of I^ω, such that

$$\pi[S_\alpha \cap (X \times Y)] = 2^\omega,$$

where $\pi : X \times Y \to X$ is the projection. Let us well order the set X;

$$X = \{x_\alpha : \alpha < 2^\omega\}$$

and let us put, for each $\alpha < 2^\omega$, $Q_\alpha = \{x_\alpha\} \times Y$. Let $\{P_\alpha : \alpha < 2^\omega\}$ be a well-ordering of the family

$$\{\mathrm{Bd}_{X \times Y} G : G \text{ is open in } X \times Y \text{ and } \mathrm{Int}_X \pi(\mathrm{Bd}_{X \times Y} G) \neq \varnothing\}.$$

We shall define by induction sets

$$A_\alpha = \{p_\alpha, q_\alpha, r_\alpha, s_\alpha, t_\alpha\} \subset X \times Y, \ \alpha < 2^\omega,$$

satisfying the following conditions:

(1) $p_\alpha \in P_\alpha$, $q_\alpha \in Q_\alpha$, $r_\alpha, s_\alpha \in S_\alpha \cap (X \times Y)$, $s_\alpha \neq t_\alpha$ and $\pi(s_\alpha) = \pi(t_\alpha)$,

(2) if $x, y \in \bigcup \{A_\alpha \setminus \{t_\alpha\} : \alpha < 2^\omega\}$ and $x \neq y$ then $\pi(x) \neq \pi(y)$,

(3) for each $\alpha < 2^\omega$, $f_\alpha(r_\alpha) \notin \bigcup \{A_\beta : \beta < 2^\omega\}$.

Suppose that the sets A_β have been choosen for each $\beta < \alpha$. Put

$$Z_\alpha = \bigcup \{A_\beta : \beta < \alpha\}.$$

We have $|Z_\alpha| < 2^\omega$.

(a) Let us choose a $p_\alpha \in P_\alpha$ such that

$$p_\alpha \in P_\alpha \setminus \{f_\beta(r_\beta) : \beta < \alpha\} \text{ and } \pi(p_\alpha) \notin \pi(Z_\alpha).$$

(b) Choose a $q_\alpha \in Q_\alpha$ such that

$$q_\alpha = q_0 \text{ whenever } Q_\alpha \cap (Z_\alpha \cup \{p_\alpha\}) \neq \varnothing$$

or

$$q_\alpha \in Q_\alpha \setminus \{f_\beta(r_\beta) : \beta < \alpha\} \text{ whenever } Q_\alpha \cap (Z_\alpha \cup \{p_\alpha\}) = \varnothing.$$

(c) Let $V_\alpha \subset S_\alpha$ be a set such that

$$|f_\alpha(V_\alpha)| = 2^\omega \text{ and } V_\alpha \cap f_\alpha(V_\alpha) = \varnothing.$$

Choose points r_α, $s_\alpha \in S_\alpha \cap (X \times Y)$ such that

$$r_\alpha, s_\alpha \in f_\alpha^{-1}[f_\alpha(V_\alpha) \setminus (Z_\alpha \cup \{p_\alpha, q_\alpha\})] \setminus \{f_\beta(r_\beta) : \beta < \alpha\},$$

$$\pi(r_\alpha) \neq \pi(s_\alpha) \text{ and } \pi(r_\alpha), \pi(s_\alpha) \notin \pi(Z_\alpha \cup \{p_\alpha, q_\alpha\}).$$

(d) Finally, choose $t_\alpha \in X \times Y$ such that

$$t_\alpha \in \{\pi(s_\alpha)\} \times Y \setminus \{f_\beta(r_\beta) : \beta \leqslant \alpha\}.$$

One can verify that the conditions (a)–(d) imply the conditions (1)–(3).

Let us put $S = \{s_\alpha : \alpha < 2^\omega\}$. The set S can be represented as the union

$$S = \bigcup \{B_\gamma : \gamma < 2^{\mathfrak{c}}\}, \ \mathfrak{c} = 2^\omega,$$

such that

$$\gamma \neq \gamma' \text{ implies } B_\gamma \neq B_{\gamma'}.$$

Define for each $\gamma < 2^{\mathfrak{c}}$ the set

$$K_\gamma = \bigcup \{\{p_\alpha, q_\alpha, r_\alpha, d_\alpha^\gamma\} : \alpha < 2^\omega\},$$

where

$$d_\alpha^\gamma = \begin{cases} s_\alpha, & \text{if } s_\alpha \in B_\gamma, \\ t_\alpha, & \text{if } s_\alpha \notin B_\gamma. \end{cases}$$

Let $g_\gamma : X \to Y$ be such that $G(g) = K_\gamma$.

Since each set K contains the set $\bigcup \{p_\alpha, q_\alpha\} : \alpha < 2^\omega\}$ hence according to Lemma each of the sets, $K_\gamma < 2^{\mathfrak{c}}$, is dense and connected in the product $X \times Y$.

Now, suppose that there exists a continuous displacement $f: K_\gamma \to K_{\gamma'}$, γ, $\gamma' < 2^{\mathfrak{c}}$. Since $K_{\gamma'} \subset X \times Y \subset I^\omega$, we can consider the map f as a continuous displacement $f: K_\gamma \to I^\omega$.

By Lavrientieff's Theorem there exists a continuous extension of f, $f^* : K_\gamma^* \to I^\omega$, where $K_\gamma^* \supset K_\gamma$ is a $\mathcal{G}_\delta$ subspace of I^ω. According to the construction there exists an $\alpha < 2^\omega$ such that

$$f^* = f_\alpha \text{ and } S_\alpha = K_\gamma^*.$$

Consider the point $r_\alpha \in S_\alpha$. By the construction we get

$$r_\alpha \in S_\alpha \cap K_\gamma \text{ and } f_\alpha(r_\alpha) \notin K_{\gamma'}, \text{ for each } \gamma' < 2^{\mathfrak{c}}.$$

Hence

$$f(r_\alpha) = f^*(r_\alpha) = f_\alpha(r_\alpha) \notin K_{\gamma'},$$

hat contradicts with $f(r_\alpha) \in K'_\gamma$.

COROLLARY. *There exist $2^{\mathfrak{c}}$ non-homeomorphic, connected rigid subspaces of the Hilbert cube I^ω.*

If we put in Theorem $X = Y = \mathbf{R}$ then we get

COROLLARY. *On the set of reals, there exist $2^{\mathfrak{c}}$ non-homeomorphic metric connected separable and rigid topologies which are finer than the natural topology of the space $\mathbf{R}$ of reals.*

COROLLARY. *There is no maximal topology among metric separable connected topologies on the set X.*

Proof. Suppose that X is a maximal connected metric separable space. Let $f: X \to Y$ be a map such that the graph $G(f) \subset X \times Y$ is a rigid connected and dense subspace of the product $X \times Y$. The projection $\pi : G(f) \xrightarrow{\text{onto}} X$ induces a topology on the set X which is finer than the previous topology.

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