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A NOTE ON REMAINDERS OF COMPACT EXTENSIONS

Abstract. The paper contains a construction of a Tychonoff space X such that for every compact extension bX the subset $bX-X$ contains a non-empty $\mathscr{G}_\delta$-set G such that Int $G=\emptyset$.

In 1960 Fine and Gillman [4] proved that if X is a locally compact real compact space, then the remainder of the Čech-Stone compactification of X (abbreviated $\beta X-X$) has the following property: every non-empty $\mathscr{G}_\delta$-set contains a non-empty open set. Hausdorff spaces satisfying this property are called *P'-spaces,* whereas *P-spaces* are the spaces in which all $\mathscr{G}_\delta$-sets are open; see e.g. Gillman and Jerison [5], Plank [6] or Veksler [7]. Although every compact (Hausdorff) P-space is finite, there exist non-trivial compact P'-spaces. As an example of a non-trivial compact P'-space one can state $\beta N-N$, the remainder of the Čech-Stone compactification of the integers.

Recently Aniskovič [1] has shown that the result of Fine and Gillman can be improved by replacing the Čech-Stone compactification by a wide class of compactifications. He has also pointed out that by an additional set-theoretical assumption one can construct a Tychonoff space no compactification of which has the remainder being a P'-space. The aim of this note is to construct such spaces without any additional set-theoretical assumptions.

LEMMA 1. *Every countable P'-space is discrete.*

Proof. Indeed, in countable Hausdorff spaces every point is a $\mathscr{G}_\delta$-set.

LEMMA 2. *Every uncountable P'-space is non-separable.*

Proof. Let D be a countable subset of an uncountable P'-space X. Choose a point x of $X-D$. There exists a $\mathscr{G}_\delta$-set G such that $x\in G\subset X-D$. Since Int$G\neq\emptyset$, D is not dense in X.

A topological space E is *extremally disconnected* (abbreviated e.d.) if the closure of every open subset of E is open. Clearly, dense subspaces of e.d. spaces are e.d. For every topological space X there exists so called *absolute* (or *Gleason space*) of X, that is an e.d. space $G(X)$ which can be mapped onto X by a continuous

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irreducible perfect mapping; see e.g. Comfort and Negrepontis [2] for the compact case.

THEOREM 1. *There exists an e.d. locally compact space X such that for every compactification bX of the space X, the remainder $bX-X$ is either finite or is not a P'-space.*

Proof. Let E be an e.d. compact space without isolated points (e.g. the absolute of the Cantor set). Choose a countable discrete subset N of E. Clearly, $\mathrm{cl}N$ is a nowhere dense subset of E. Thus, E is a compactification of the e.d. locally compact space $X=E-\mathrm{cl}N$. By a theorem of Taĭmanov (see e.g. Engelking [3, p. 182]), every continuous mapping of X into a compact space has a continuous extension over E. Thus, E is equivalent to βX. Now, let bX be an arbitrary compactification of X. Then, there exists a continuous mapping f from E onto bX such that $f(\mathrm{cl}N)= =bX-X$. Hence $bX-X$ is a separable compact space. Assume that it is a P'-space. Then, by Lemma 2 and Lemma 1, it must be finite.

In particular, Theorem 1 says that in the Theorem of Fine and Gillman mentioned above the assumption of realcompactness cannot be removed.

A Tychonoff space is called *nowhere locally compact* whenever every compact subset of this space is nowhere dense.

THEOREM 2. *There exists an e.d. nowhere locally compact space no compactification of which has the remainder being a P'-space.*

Proof. Let E be the absolute of the Cantor set and let D be a countable dense subset of E. We set $X=E-D$. By the Taĭmanov's Theorem (see the proof of Theorem 1), E is a compactification of X equivalent to βX. Since D is countable, the remainder of any compactification of X is countable. Furthermore, X is nowhere locally compact because D is dense in E. Suppose bX is a compactification of X with $bX-X$ being a P'-space. By Lemma 1, $bX-X$ is discrete. Thus, X is locally compact in some points; we get a contradiction.

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