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BAIRE IRRESOLVABLE SPACES AND IDEAL THEORY

Abstract. In the New Scottish Book M. Katětov asked whether there exists a Hausdorff space X without isolated points such that every real-valued function on X is continuous at some point? In the paper it is shown that the existence of such a space is equiconsistent to the existence of measurable cardinal.

The "structure theory of ideals" is concerned with the combinatorial properties of ideals on cardinals. It has many connections with large cardinal theory and with infinitary combinatorics. It has recently come into prominence as a separate branch of set theory with the appearance of [1]. Its first non-trivial applications to topology can be found in [18].

Irresolvable spaces (spaces in which any two dense sets meet) were studied extensively first by Hewitt [4]. In the 60's and 70's, El'kin and Malyhin published a number of papers on this subject and its connections with various topological problems. One of the problems considered by Malyhin [14] concerns the existence of irresolvable spaces satisfying the Baire Category Theorem. He noted that there is such a space if and only if there is one on which every real-valued function is continuous at some point. The question about the existence of a Hausdorff space on which every real-valued function is continuous at some point was posed by M. Katětov in [9], and then repeated in 1958 in the New Scottish Book, Problem 109. Here we consider this question from the point of view of ideals theory. It will be convenient to assume (as we have tacitly done above) that all spaces are without isolated points. It will be convenient to assume also, that open sets in the sense of a topology on a cardinal $\varkappa$ will have size $\varkappa$.

1. Basic topological facts. A space is *irresolvable* if it does not admit disjoint dense sets. A space is *strongly irresolvable* if every open subspace is irresolvable.

Let $\varkappa$ be a cardinal. A space is *$\varkappa$-Baire* if the intersection of fewer than $\varkappa$ dense open sets is dense. Thus Baire spaces are the ω_1-Baire spaces. A $(\varkappa-)$ SIB is a strongly irresolvable $(\varkappa-)$Baire space; $\omega-$SIB's coincide with strongly irresolvable spaces, which exist in ZFC [4].

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PROPOSITION 1.1 ([4]). *Every irresolvable space includes a non-empty open strongly irresolvable subspace. In an irresolvable space, every dense set has non-empty interior.*

Since ($\varkappa-$) Baireness is inherited by open sets, existence questions may as well be confined to ($\varkappa-$) SIB's.

PROPOSITION 1.2. *For any space X and a cardinal $\varkappa$ the following conditions are equivalent*

(1) *X is $\varkappa$-SIB,*

(2) *the intersection of $<\varkappa$ dense sets in X is dense,*

(3) *for any space Y of weight $<\varkappa$ and for any function $F: X \to Y$ the set of points of continuity of F contains a dense and open set.*

The easy proof is left to the reader; the equivalence (1)$\Leftrightarrow$(3) is, in fact, contained in [14].

In a strongly irresolvable space any set is the union of an open set and a nowhere dense set. In particular, any subset of a strongly irresolvable space has the Baire property. On the other hand we have

PROPOSITION 1.3. *If X is $\varkappa$-Baire and each subset has the Baire property, then there is an extension τ of the topology on X such that (X, τ) is $\varkappa$-SIB.*

Proof. Let us denote the topology on X by Θ. The topology τ in question is generated by sets of the form $U-E$, where $U \in \Theta$ and E is of first category in Θ. Indeed, the space (X, τ) has no isolated points; any set of first category in Θ becomes nowhere dense in τ.

CLAIM. *If B is a boundary set in τ, then B is of first category in Θ.*

Indeed, since each subset of X has the Baire property, $B = (U-E) \cup F$, where $U \in \Theta$ and E, F are of first category in Θ. Since B is boundary in τ, $U-E = \emptyset$. Hence $B = F$ and the claim is proved.

Now, to prove that (X, τ) is $\varkappa$-SIB, take boundary sets B_α, where $\alpha < \lambda$ and $\lambda < \varkappa$, in τ. By the claim, $\bigcup \{B_\alpha : \alpha < \lambda\}$ is the union of $\leqslant \omega \cdot \lambda = \lambda$ nowhere dense sets in Θ. Since X is $\varkappa$-Baire, $\bigcup \{B_\alpha : \alpha < \lambda\}$ is boundary in Θ. It is also boundary in τ. So, again by the claim, $\bigcup \{B_\alpha : \alpha < \lambda\}$ is of first category in Θ and therefore is nowhere dense in τ.

NOTE. The proof of Proposition 1.3 for $\varkappa = \omega_1$ was performed using only the countable axiom of choice.

2. Basic facts about ideals. Let $\varkappa$ be an uncountable cardinal. A set $I \subset \mathcal{P}(\varkappa)$ is an *ideal* over $\varkappa$ if

(1) $\alpha \in I$ for all $\alpha < \varkappa$ and $\varkappa \notin I$,

(2) if $X \in I$ and $Y \subset X$, then $Y \in I$,

(3) if $X \in I$ and $Y \in I$, then $X \cup Y \in I$.

Let I be a given ideal over a cardinal $\varkappa$. Then $I^+ = \{X \subset \varkappa : X \notin I\}$ and $I^* = \{X \subset \varkappa : \varkappa - X \in I\}$. A family $R \subset I^+$ is *I-almost disjoint* if for every distinct $S, S' \in R$, $S \cap S' \in I$; R is *I-dense* if for every $X \in I^+$ there is an $S \in R$ such that

$S-X\in I$; R is *I-proper* if for any finite subfamily $R'\subset R$ either $\bigcap R'=\emptyset$ or $\bigcap R'\in I^+$.

I is *λ-saturated* if every I-almost disjoint collection has size $<\lambda$. I is *λ-complete* if it is closed under unions of size $<\lambda$. Let $A\in I^+$. An *I-partition* of A is a maximal I-almost disjoint family of subsets of A. An I-partition P_2 of A is a *refinement* of an I-partition P_1 of A, $P_1\leqslant P_2$, if every $X\in P_2$ is a subset of some $Y\in P_1$.

The ideal I is *weakly-precipitous* iff it is ω_1-complete and whenever A is in I^+ and $\{P_n: n\in\omega\}$ are I-partitions of A such that $P_0\leqslant P_1\leqslant\ldots\leqslant P_n\leqslant\ldots$, then there exists a sequence of sets $W_0\supset W_1\supset\ldots\supset W_n\supset\ldots$, such that $W_n\in P_n$ for each n, and $\bigcap\{W_n: n\in\omega\}\neq\emptyset$. It should be noted that our definition of weakly precipitous ideals differs from the notion of precipitous ideal used by other authors ([8], [20]) only by assumption of ω_1-completeness instead of $\varkappa$-completeness. For weakly precipitous ideals, the associated Boolean ultrapower is well-founded.

THEOREM 2.0 (T. Jech, M. Magidor, W. Mitchell and K. Prikry [7]). *If κ is a regular cardinal that carries a weakly precipitous ideal, then there is a measurable cardinal in some transitive model of* ZFC.

Originally, the theorem was formulated for precipitous ideals but the proof works for weakly precipitous ideals, as well.

The ideal I has a *lifting* if there is a homomorphism $h:\mathscr{P}(\varkappa)/I\to\mathscr{P}(\varkappa)$ of the factor Boolean algebra $\mathscr{P}(\varkappa)/I$ into the power algebra $\mathscr{P}(\varkappa)$ such that $h(\xi)\in\xi$ for each $\xi\in\mathscr{P}(\varkappa)/I$.

THEOREM 2.1. *For any cardinal $\varkappa$, the following conditions are equivalent*

(1) *there is a topology on $\varkappa$ which is λ-SIB,*

(2) *there is a λ-complete ideal I over $\varkappa$ for which there is a family R which is both I-dense and I-proper,*

(3) *there is a λ-complete ideal over $\varkappa$ which has a lifting.*

Proof. We shall show that (1)$\Leftrightarrow$(2), (1)$\Rightarrow$(3) and (3)$\Rightarrow$(2).

Suppose τ is a topology on $\varkappa$ which is λ-SIB. Then the ideal I consisting of nowhere dense subsets of $\varkappa$ with respect to τ is λ-complete while the family τ is both I-dense and I-proper.

To prove the converse, having a λ-complete ideal I over $\varkappa$ and a family R which is both I-dense and I-proper, define the topology τ on $\varkappa$ by declaring as open base sets of the form $\bigcap P-X$, where $P\in[R]^{<\omega}$ and $X\in I$. To prove that τ is λ-SIB it suffices only to check that dense sets in τ are in I^*. Thus (1)$\Leftrightarrow$(2) is proved.

To prove that (1)$\Rightarrow$(3), suppose again that τ is a topology on $\varkappa$ which is λ-SIB and I is the ideal consisting of nowhere dense sets in τ. Then I is λ-complete. Note that if τ' is any extension of the topology τ such that τ' has no isolated points, then there are no "new" boundary (= nowhere dense) sets in τ'. By the Kuratowski-Zorn Lemma, there is a maximal dense-in-itself topology $\tilde{\tau}$ on $\varkappa$ extending the topology τ. By Proposition 1.2, this topology is also λ-SIB. Since $\tilde{\tau}$ is maximal, it is, in particular, extremally disconnected. Let us put $h(\xi)$ to be the closed-open set with respect to $\tilde{\tau}$ which is in $\xi\in\mathscr{P}(\varkappa)/I$. Then h is the required homomorphism for I to have a lifting.

To prove (3)⇒(2), observe that if $h:\mathscr{P}(\varkappa)/I \to \mathscr{P}(\varkappa)$ is a lifting for I, then $\{h(\xi): \xi \in \mathscr{P}(\varkappa)/I\}$ is a family which is both I-dense and I-proper.

If X is a space, then $F \subset X$ is a $P(\lambda)$-*set* iff it is closed and for any family R of neighborhoods of F, $|R| < \lambda$, there is $F \subset \operatorname{int} \bigcap R$; $P(\omega_1)$-sets are called *P-sets.*

LEMMA 2.2. *Let $\varkappa$ be regular and $F \subset \mathscr{U}(\varkappa)$ a $P(\lambda)$-set which is a retract of $\beta\varkappa$. Then $I_F = \{A \subset \varkappa : \operatorname{cl}_{\beta\varkappa} A \cap F = \emptyset\}$ is a λ-complete ideal over $\varkappa$ for which there is a family R which is both I_F-dense and I_F-proper.*

Proof. If r: $\beta\varkappa \to F$ is a retraction, then the family R in question consists of sets of the form $\varkappa \cap r^{-1}(V)$, where V is a non-empty closed-open subset of F. Indeed, $\varkappa \cap r^{-1}(V)$ is a dense subset of $r^{-1}(V)$ and $\emptyset \neq V \subset r^{-1}(V)$; thus $\varkappa \cap \cap r^{-1}(V) \in I_F^+$. If $V_1, \ldots, V_n \in R$, say $V_i = \varkappa \cap r^{-1}(U_i)$, then $V_1 \cap \ldots \cap V_n = \varkappa \cap \cap r^{-1}(U_1 \cap \ldots \cap U_n)$ is either empty (if $U_1 \cap \ldots \cap U_n = \emptyset$) or belongs to I_F^+(if $U_1 \cap \ldots \cap U_n \neq \emptyset$). The above two properties of R show that R is I_F-proper. To prove that R is also I_F-dense, let $A \in I_F^+$. Then $U = \operatorname{cl}_{\beta\varkappa} A \cap F \neq \emptyset$ is closed-open in F. We will show that $B = \varkappa \cap r^{-1}(U) - A \in I_F$. Assume otherwise, then $V = \operatorname{cl}_{\beta\varkappa} B \cap F \neq \emptyset$. Since $A \cap B = \emptyset$, $U \cap V = \emptyset$. Since r is a retraction, $r^{-1}(U)$ is closed-open in $\beta\varkappa$ and therefore it contains V. Since $r^{-1}(U) \cap F = U$, $\emptyset \neq V \cap F \subset U$. But then $U \cap V \neq \emptyset$; a contradiction.

NOTE 2.3. It should be remarked that if the retraction r, as above, is 1-1 on $\varkappa$, then the topology τ on $\varkappa$ induced by constructed family R for I_F is Hausdorff.

LEMMA 2.4. *Suppose that τ is a λ-SIB topology on $\varkappa$ such that any subset of $\varkappa$ of size $< \varkappa$ is nowhere dense in τ. Then there exists a $P(\lambda)$-set in $\mathscr{U}(\varkappa)$ which is a retract of $\beta\varkappa$.*

Proof. We may assume that $(\varkappa, \tau)$ is an extremally disconnected space. For $\alpha < \varkappa$ let $\xi_\alpha = \{x \subset \varkappa : \alpha \in \operatorname{cl}_\tau \operatorname{int}_\tau x\}$. Then ξ_α is a uniform ultrafilter on $\varkappa$. Let $Y = \{\xi_\alpha : \alpha < \varkappa\} \subset \mathscr{U}(\varkappa)$.

CLAIM. *For any $x \subset \varkappa$, $Y \cap \operatorname{cl}_{\beta\varkappa} x = \emptyset$ iff x is nowhere dense in τ.*

Indeed, $Y \cap \operatorname{cl}_{\beta\varkappa} x = \emptyset$ iff $x \notin \xi_\alpha$ for each $\alpha < \varkappa$ iff $\alpha \notin \operatorname{cl}_\tau \operatorname{int}_\tau x$ for each $\alpha < \varkappa$ iff $\operatorname{cl}_\tau \operatorname{int}_\tau x = \emptyset$ iff $\operatorname{int}_\tau x = \emptyset$ iff x is boundary in τ iff x is nowhere dense in τ.

Define $r: \varkappa \cup Y \to Y$ by setting $r(y) = \xi_y$ if $y \in \varkappa$ and $r(y) = y$ if $y \in Y$. The function r is continuous. To see this, it suffices to show that if U is a closed-open subset of Y, then $Y \cap \operatorname{cl}_{\beta\varkappa}(\varkappa \cap r^{-1}(U)) = U$. For this purpose take a set $x \subset \varkappa$ such that $U = Y \cap \operatorname{cl}_{\beta\varkappa} x$. Then $U = \{\xi_\alpha : \alpha \in \operatorname{cl}_\tau \operatorname{int}_\tau x\}$ and hence $\varkappa \cap r^{-1}(U) = \{\alpha : \xi_\alpha \in U\} = \{\alpha : \alpha \in \operatorname{cl}_\tau \operatorname{int}_\tau x\}$. The set $A = (x - \operatorname{cl}_\tau \operatorname{int}_\tau x) \cup (\operatorname{cl}_\tau \operatorname{int}_\tau x - x)$ is nowhere dense in τ and, by Claim, $Y \cap \operatorname{cl}_{\beta\varkappa}(\varkappa \cap r^{-1}(U)) = Y \cap \operatorname{cl}_{\beta\varkappa} x = U$; the continuity of r is proved.

There exists a continuous extension $\beta r: \beta\varkappa \to \operatorname{cl}_{\beta\varkappa} Y$ of r. Since $r|Y = \mathrm{id}$, $\beta r | \operatorname{cl}_{\beta\varkappa} Y = \mathrm{id}$. Thus βr is a retraction. Since the topology τ is λ-SIB we get, by the claim, that $\operatorname{cl}_{\beta\varkappa} Y$ is a $P(\lambda)$-set in $\mathscr{U}(\varkappa)$.

NOTE 2.5. It should be remarked that if the topology τ, above, is Hausdorff, then the constructed retraction is $1-1$ on $\varkappa$.

Finally, by Lemmas 2.2 and 2.4 and Notes 2.3 and 2.5 we have

THEOREM 2.6. *There exists a (Hausdorff) λ-SIB iff there exist a cardinal $\varkappa$, a $P(\lambda)$-set $F \subset \mathscr{U}(\varkappa)$ and a retraction $r: \beta\varkappa \to F$ (which is $1-1$ on $\varkappa$).*

It is a well-known fact (proved probably first by Gleason [3]) that any compact extremally disconnected space of the density $\varkappa$ can be embedded in $\beta\varkappa$ as a retract of $\beta\varkappa$. Moreover, a retraction can be chosen to be $1-1$ on $\varkappa$.

NOTE 2.7. In the case when an ideal I has a family which is both I-dense and I-proper, the resulting (λ-)SIB topology τ is such that I coincides with the family of all nowhere dense sets in τ.

PROPOSITION 2.8. *If I is a $\varkappa$-complete ideal over $\varkappa$ which has an I-dense family of size $\leqslant\varkappa$, then there is a family $R\subset I^+$ which is both I-dense and I-proper.*

Proof. If $S=\{a_\alpha:\alpha<\varkappa\}$ is I-dense, then $R=\{b_\alpha:\alpha<\varkappa\}$, where $b_\alpha=a_\alpha-\bigcup\{a_\alpha\cap\bigcap\{a_\xi:\xi\in s\}:s\in[\alpha]^{<\omega}$ and $a_\alpha\cap\bigcap\{a_\xi:\xi\in s\}\in I\}$ is both I-dense and I-proper.

PROPOSITION 2.9. *Let I be an ideal over $\varkappa$ for which there is a family R which is both I-dense and I-proper. Then I is $\varkappa^+$-saturated. If, in addition, I is ω_1-complete, then I is weakly precipitous.*

Proof. Both parts of the proposition follow easily from the following observation:

For any I-partition P_1 of $A\in I^+$ there exists an I-partition P_2 of A which consists of disjoint sets and is a refinement of P_1.

Before we state our next result we introduce the following (standard) notation: $\mathbf{sat}(I)=\inf\{\lambda:I$ is λ-saturated$\}$; if $X, Y, Z_t, t\in T$, are sets, then ${}^XY=\{f:f$ is a function and $\mathrm{dom}(f)=X$ and $\mathrm{rng}(f)\subset Y\}$, $\Pi\{Z_t:t\in T\}=\{f:f$ is a function and $\mathrm{dom}(f)=T$ and $f(t)\in Z_t\}$.

THEOREM 2.10. *Let I be a $\varkappa$-complete ideal over $\varkappa$ such that the Boolean algebra $\mathscr{P}(\varkappa)/I$ is atomless. If either* $\mathrm{sat}(I)<\varkappa$ *or $\varkappa$ is weakly compact and* $\mathrm{sat}(I)\leqslant\varkappa$, *then there are I-partitions P_α, $\alpha<\lambda$, of $\varkappa$ such that*:

(i) *if $\alpha<\beta<\lambda$, then $P_\alpha\leqslant P_\beta$,*

(ii) *if $\varphi\in\Pi\{P_\alpha:\alpha<\lambda\}$, then $|\bigcap\{\varphi(\alpha):\alpha<\lambda\}|\leqslant 1$.*

Proof. In both cases $\varkappa$ cannot be inaccessible. So there exists the smallest λ that $2^\lambda\geqslant\varkappa$. Let $f:\varkappa\to{}^\lambda 2$ be 1-1. For any $\alpha<\varkappa$ and $\varphi\in{}^\lambda 2$, $A_{\varphi,\alpha}=\{\psi\in{}^\varkappa 2:\varphi\subset\psi\}$. We set $P_\alpha=\{f^{-1}(A_{\varphi,\alpha}):\varphi\in{}^\alpha 2$ and $f^{-1}(A_{\varphi,\alpha})\in I^+\}$. Clearly, conditions (i) and (ii) are then satisfied. It remains to prove that each P_α, $\alpha<\lambda$, is an I-partition of $\varkappa$. To see this, note that for any $\alpha<\lambda$, $2^\alpha<\varkappa$. Thus $|P_\alpha|<\varkappa$ for every $\alpha<\lambda$. Since $\bigcup\{A_{\varphi,\alpha}:\varphi\in{}^\alpha 2\}={}^\lambda 2$ and I is $\varkappa$-complete, P_α is an I-partition of $\varkappa$, for every $\alpha<\varkappa$

3. Existence and consistency results. We shall present some methods for obtaining λ-complete ideals over regular cardinals having both dense and proper sets. Then we exhibit some consistent examples.

One of the methods is the one using maximal θ-independent families, with $\theta\geqslant\omega_1$, introduced and considered by K. Kunen in [12].

Let $\varkappa$ and θ be infinite cardinals with θ regular. A family $S\subset\mathscr{P}(\varkappa)$ is called *θ-independent* iff whenever $S_0, S_1\subset S$ with $S_0\cap S_1=\emptyset$ and $|S_0\cup S_1|<\theta$, we have $\bigcap\{A\in S_0\}\cap\bigcap\{\varkappa-A:A\in S_1\}\neq\emptyset$. A family $S\subset\mathscr{P}(\varkappa)$ is called *maximal θ-independent* iff it is θ-independent but no proper superset is.

It will be convenient to introduce the following notation. $\mathrm{Fn}(X, Y, \theta)=\{p:p$ is a function and $\mathrm{dom}(p)\subset X$ and $\mathrm{rg}(p)\subset Y$ and $0<|p|<\theta\}$. For $S\subset\mathscr{P}(\varkappa)$ and

$p \in \mathrm{Fn}(S, 2, \theta)$ we set $\varphi(p) = \bigcap \{A : A \in \mathrm{dom}(p) \text{ and } p(A) = 1\} \cap \bigcap \{\varkappa - A : A \in \mathrm{dom}(p) \text{ and } p(A) = 0\}$. Then $S \subset \mathscr{P}(\varkappa)$ is θ-independent iff $\varphi(p) \neq \emptyset$ for all $p \in \mathrm{Fn}(S, 2, \theta)$. The following is proved in [12].

THEOREM 3.1. *Suppose $S \subset \mathscr{P}(\varkappa)$ is a maximal θ-independent family with $|S| \geqslant \theta \geqslant \omega_1$. Then there is a $\lambda \leqslant \varkappa$ and $S' \subset \mathscr{P}(\lambda)$ such that:*

(1) *S' is maximal θ-independent and $|S'| \geqslant \theta$,*

(2) *the family $J = \{X \subset \lambda : \neg(\exists\, p \in \mathrm{Fn}(S', 2, \theta)\, (\varphi(p) \subset X)\}$ is a λ-complete ideal over λ.*

From the definition of the ideal J it follows that the family $R = \{\varphi(p) : p \in \mathrm{Fn}(S', 2, \theta)\}$ is J-dense. Moreover, if $p, q \in \mathrm{Fn}(S', 2, \theta)$ are such that $\varphi(p) \cap \varphi(q) \neq \emptyset$, then there is an $r \in \mathrm{Fn}(S', 2, \theta)$ which extends p and q and $\varphi(r) = \varphi(p) \cap \varphi(q)$. Thus R is J-proper. According to Theorem 2.1, there is a topology τ in λ which is λ-SIB. Let us look at the topology ϱ whose base is formed by the sets $\varphi(p)$, $p \in \mathrm{Fn}(S', 2, \theta)$. This topology is slightly weaker than the topology τ but is still λ-SIB: B is boundary in ϱ iff $\neg(\exists\, p \in \mathrm{Fn}(S', 2, \theta)\, (\varphi(p) \subset B))$ iff $B \in J$. In the sense of the topology ϱ the sets $\varphi(p)$, $p \in \mathrm{Fn}(S', 2, \theta)$ are closed-open. We shall show that for some $p_0 \in \mathrm{Fn}(S', 2, \theta)$, the topology ϱ restricted to $\varphi(p_0)$ is T_1, thus 0-dimensional and completely regular. For this purpose, for any $\alpha < \lambda$ we put $s_\alpha = \bigcap \{\varphi(p) : p \in \mathrm{Fn}(S', 2, \theta) \text{ and } \alpha \in \varphi(p)\}$. Then $\{s_\alpha : \alpha < \lambda\}$ is a decomposition of λ into sets belonging to J. Observe also that $A = \{\alpha : s_\alpha = \{\alpha\}\} \in J^+$. Indeed, let X be a set intersecting each s_α in precisely one point. By maximality of S', there is a $p_0 \in \mathrm{Fn}(S', 2, \theta)$ such that $\varphi(p_0) \subset X$ or $\varphi(p_0) \subset \lambda - X$. Clearly, there must be $\varphi(p_0) \subset A$. Therefore we have shown

THEOREM 3.2. *If a maximal θ-independent family S exists with $|S| \geqslant \theta \geqslant \omega_1$, then there is a cardinal $\lambda > \omega$ and a topology τ on λ which is completely regular* 0*-dimensional and* λ-SIB.

In [12] it is also shown that if ZFC plus the existence of measurable cardinal is consistent, so is ZFC plus the existence of a maximal ω_1-independent family $S \subset \mathscr{P}(2^{\omega_1})$. Hence

THEOREM 3.3. *If* ZFC *is consistent with the existence of a measurable cardinal, so is* ZFC *with the existence of a completely regular* 0*-dimensional* SIB.

In [15] it is shown that the consistency of ZFC + an inaccessible implies the consistency of ZF + "each subset of reals has the Baire property". By Note and Proposition 1.3 we have

THEOREM 3.4. *Con*(ZFC + $\exists$ *inaccessible*) $\Rightarrow$ *Con*(ZF + $\exists a$ SIB *extension of the topology on reals*).

Let us make a remark on spaces being regular SIB's. If X is an irresolvable space such that any non-empty open set has size $\geqslant \varkappa$, then X has π-weight $> \varkappa$. Thus if X is also regular, the regular open algebra of X cannot have a dense set of size $\leqslant \varkappa$. On particular, the SIB topology obtained from Proposition 2.8 is not regular.

Now, we shall present a method which yields a T_2 λ-SIB.

THEOREM 3.5. *Let I be a $\varkappa$-complete ideal over $\varkappa$ such that the Boolean algebra*

$\mathscr{P}(\kappa)/I$ is atomless and either $\mathrm{sat}(I) < \kappa$ *or* $\mathrm{sat}(I) \leqslant \kappa$ *and* κ *is weakly compact. If I has a lifting, then there is a* κ-SIB *topology on some* $Z \in I^*$ *which is Hausdorff.*

Proof. Let I-partitions P_α, $\alpha < \lambda$, of κ fulfil the conditions (i) and (ii) from Theorem 2.10. By Theorem 2.1 and Note 2.7, there is a κ-SIB topology τ on κ such that the ideal I coincides with the family of all nowhere densessets in τ. Let $\mathrm{int}P_\alpha = \{\mathrm{int}_\tau X : X \in P_\alpha\}$. Thus $\kappa - \bigcup \mathrm{int}P_\alpha \in I$ for all $\alpha < \lambda$. Since $\lambda < \kappa$, the set $Z = \bigcap \{\bigcup \mathrm{int}P_\alpha : \alpha < \lambda\}$ is in I^*. The subspace Z of the space (κ, τ) is Hausdorff, since the condition (ii) holds.

COROLLARY 3.6. *Con*(ZFC + $\exists$ *real-valued measurable cardinal*) $\Rightarrow$ *Con*(ZFC + $+ \exists\, \omega < \kappa \leqslant 2^\omega$ *topology* τ *on* κ ((κ, τ) *is Hausdorff* κ-SIB)).

Proof. If it is consistent with ZFC that a real-valued measurable cardinal exists, then it is consistent with ZFC that there is a $\kappa \leqslant 2^\omega$ and a measure μ on $\mathscr{P}(\kappa)$ which is κ-complete, finite and atomless. Thus the ideal I of null sets with respect to μ is κ-complete and the factor algebra $\mathscr{P}(\kappa)/I$ is atomless. Clearly, $\mathrm{sat}(I) = \omega_1 < \kappa$. Moreover, by Maharam's result [13] (see also [19]), I has a lifting. Applying the preceding theorem to I, we get the required result.

COROLLARY 3.6. *The following conditions are pairwise equiconsistent*:

(1) $\exists$ *measurable cardinal*,

(2) $\exists$ *real-valued measurable cardinal*,

(3) $\exists$ SIB *topology on a regular* κ.

Proof. The equiconsistency of (1) and (2) is due to R. Solovay [16]. That the consistency of (2) implies the consistency of (3) is in Corollary 2.5. If we have a SIB topology on a regular cardinal κ, then, by Theorem 2.1 and Proposition 2.9, κ admits a weakly precipitous ideal. But the existence of a regular cardinal κ with such an ideal implies the consistency of a measurable cardinal (see Theorem 2.0).

We have constructed Hausdorff SIB topologies in two cases, both on cardinals which are not inaccessible. The Hausdorff property does not indicate greater strenght in the next proposition.

PROPOSITION 3.7. *If there is a* κ-SIB *topology* τ *on* κ *and* κ *is not inaccessible then there is a dense open in* τ *set* $Z \subset \kappa$ *such that the subspace Z is Hausdorff.*

Proof. Let $\lambda < \kappa$ be such that $2^\lambda \geqslant \kappa$. Take an arbitrary 1-1 function $f : \kappa \to {}^\lambda 2$. For $\alpha < \lambda$ and $i \in \{0, 1\}$ let $A(\alpha, i) = \{p \in {}^\lambda 2 : p(\alpha) = i\}$. Since τ is strongly irresolvable, $D_\alpha = \mathrm{int}(f^{-1}(A(\alpha, 0))) \cup \mathrm{inf}(f^{-1}(A(\alpha, 1)))$ is dense. Since $\lambda < \kappa$ and τ is κ-SIB, $Z = \mathrm{int}(\bigcap \{D_\alpha : \alpha < \kappa\})$ is dense. Since f is 1-1, the subspace Z is Hausdorff.

Observe that if we are given a κ-complete non-principal ultrafilter ξ on κ, where $\kappa > \omega$, then $\xi \cup \{\emptyset\}$ is a κ-SIB topology on κ, since dense sets must be in the ultrafilter. This topology is only T_1. One can ask whether the existence of a measurable cardinal implies the existence of a Hausdorff SIB topology on some cardinal κ. The answer is no under $V = L[U]$ (see [10] or [6] for the definition).

THEOREM 3.8. *If* $V = L[U]$, *then there is no Hausdorff* SIB *on a regular cardinal.*

Sketch of the proof. If U is the unique normal ultrafilter on κ in $L[U]$ then it can be shown by standard methods, that the only regular cardinal in $L[U]$ which bears a weakly precipitous ideal is the measurable cardinal κ. Moreover, S. Wagon proved [20] that in $L[U]$ any (weakly) precipitous ideal I over κ is atomic,

i.e., there is an I-partition P of $\varkappa$ such that $I|A = \{B \subset \varkappa : B \cap A \in I\}$ is prime for all $A \in P$. But the existence of a Hausdorff SIB on a regular cardinal yields a weakly precipitous ideal on that cardinal which is nowhere atomic.

PROPOSITION 3.9. *If $V = L$, then any space with $|X|$ a regular cardinal is the union of countably many boundary subsets.*

Proof. Immediate, by Propositions 1.2 and 2.9 and Theorem 2.0.

4. SIB's on ω_1. We shall now turn our attention to the SIB topologies on sets of size ω_1. We do this again from the point of view of the theory of ideals. To explain our results we need some new definitions and facts.

A $\varkappa$-complete ideal I over $\varkappa$ will be called a *P-ideal* (*selective*) provided that every $f \in {}^{\varkappa}\varkappa$ is either constant on a set from I^+ or there is an $X \in I^*$ such that $|f^{-1}(\alpha) \cap X| < \varkappa$ $(|f^{-1}(\alpha) \cap X| \leqslant 1)$ for every $\alpha < \varkappa$.

A function $f \in {}^{\varkappa}\varkappa$ is called *incompressible* for the ideal I provided that $f^{-1}(\alpha) \in I$ for every $\alpha < \varkappa$ and for every $g \in {}^{\varkappa}\varkappa$, if $\{\alpha < \varkappa : g(\alpha) < f(\alpha)\} \in I^+$, then g is constant on a set from I^+.

A $\varkappa$-complete ideal I on $\varkappa$ is *normal* provided that the identity on $\varkappa$ is incompressible for I.

It is well-known that the ideal $\mathrm{NS}_{\varkappa}$ of non-stationary subsets of a regular cardinal $\varkappa$ is normal and it is contained in any normal ideal over $\varkappa$ [5].

THEOREM (Taylor, [17]). *If I is a $\varkappa$-complete $\varkappa^+$-saturated ideal over a successor cardinal $\varkappa$, then I is a P-ideal.*

THEOREM (Solovay, [16]). *If I is a $\varkappa$-complete $\varkappa^+$-saturated ideal over $\varkappa$, then there is a function $f \in {}^{\varkappa}\varkappa$ that is incompressible for I and such that $f_*(I) = \{X \subset \varkappa : f^{-1}(X) \in I\}$ is a normal $\varkappa^+$-saturated ideal over $\varkappa$.*

THEOREM (Węglorz [21]). *If I is a normal ideal over $\varkappa$, then I is a selective ideal.*

THEOREM 4.1. *Suppose there is a topology τ on ω_1 which is* SIB. *Then there is a stationary set $S \subset \omega_1$ and a topology θ on S which is* SIB *and such that $J = \{X \subset \omega_1 : X \cap S$ is nowhere dense in $\theta\}$ is a normal ideal over $\varkappa$.*

Proof. The ideal I of nowhere dense in τ subsets of ω_1 is an ω_1-complete ω_2-saturated ideal over ω_1, By Solovay's theorem, there is an $f \in {}^{\omega_1}\omega_1$ which is incompressible for I. By Taylor's theorem, there is an $X \in I^*$ such that $|f^{-1}(\alpha) \cap X| \leqslant \omega$ for every $\alpha < \omega_1$. Hence X is the union of countably many sets on which f is 1-1. Thus one of them, say A, belongs to I^+. Clearly, $U = \mathrm{int}_{\tau} A \neq \emptyset$, so $U \in I^+$. Since $f_*(I)$ is normal, $S = f(U)$ is stationary. The topology θ on S is defined in such a way that $f|U$ becomes a homeomorphism, i.e., $V \subset S$ is in θ iff $f^{-1}(V)$ is in τ. Since U, as an open subspace of a SIB, is SIB, θ is a SIB on S. Since the ideal $f_*(I)$ is normal, the ideal $J = f_*(I)|S$ is normal, as well.

Taking a function $h \in {}^{\omega_1}S$ which is 1-1 and onto and defining a topology Σ on ω_1 in such a way that h becomes a homeomorphism from (ω_1, Σ) onto (S, θ), we get that Σ is a SIB and the ideal of nowhere dense sets in Σ is selective. As is easy to observe, such a topology must be rigid.

Now we shall consider the question of the existence of a SIB on ω_1. Note that the existence of any SIB on ω_1 induces a Hausdorff one, according to Proposition 3.7.

A special case of ω_2-saturated ideals over ω_1 is those which have a dense set of size ω_1 (see [22] for a relative consistency of the existence of such ideals). According to Proposition 2.8, if I is an ω_1-complete ideal over ω_1 which has an I-dense set of size ω_1, then there is a family which is both I-dense and I-proper. In such a case, by Theorem 2.1 there is a SIB topology on ω_1. However we can obtain a SIB topology on ω_1 from such a dense set in another fashion. First we need the following, due to A. Taylor [17, Theorems 7.4 and 7.7]

LEMMA 4.3. *If there is an ω_1-complete ideal over ω_1 which has a dense set of size ω_1, then there is one which is selective and which has a dense set $\{D_\alpha : \alpha < \omega_1\}$ such that either $D_\alpha \cap D_\beta = \emptyset$ or $D_\alpha \subset D_\beta$ or $D_\beta \subset D_\alpha$ for all $\alpha, \beta < \omega_1$.*

THEOREM 4.4. *Suppose that there is an ω_1-complete ideal I over ω_1 which has a dense set of size ω_1. Then there is a topology τ on ω_1 which is Hausdorff and* SIB.

Proof. Without loss of generality we may assume that I is selective and that $\{D_\alpha : \alpha < \omega_1\}$ is I-dense and for all $\alpha, \beta < \omega_1$, either $D_\alpha \cap D_\beta = \emptyset$ or $D_\alpha \subset D_\beta$ or $D_\beta \subset D_\alpha$. Let us define a relation r on ω_1 setting $\alpha r \beta$ iff $(\alpha \in D_\gamma \Leftrightarrow \beta \in D_\gamma$ for all $\gamma < \omega_1)$. Then r is an equivalence relation over ω_1 and each equivalence class of r is in I. By selectivity of I, there is an $S \in I^*$ which has precisely one point in common with each equivalence class of the relation r. Since $\{D_\alpha : \alpha < \omega_1\}$ may be chosen so that $D_\alpha \in I^+ - I^*$ for all $\alpha < \omega_1$, the topology $\tilde{\tau}$ on S generated by sets D_α and their complements has no isolated points. Moreover, by the choice of S it is Hausdorff. Now, the topology τ in question is generated by sets of the form $U - A$, where U is open in τ and $A \in I$. Clearly, τ being an extension of a T_2 topology is again T_2. If $A \in I$, then $A \cap S$ is nowhere dense in τ. To prove that τ is a SIB it suffices to show, by virtue of ω_1-completeness of I, that if B is boundary in τ, then $B \in I$. To prove this, take $B \in I^+$. Then there is an $\alpha < \omega_1$ such that $A = D_\alpha - B \in I$. Hence $D_\alpha - A \subset B$ and $D_\alpha - A$ is non-empty and open in τ.

PROBLEM. A family $\{U_\alpha : \alpha < \lambda\}$, $\lambda \leqslant \varkappa$, of ultrafilters on $\varkappa$ is a *complete family of ultrafilters* iff $\bigcap \{U_\alpha : \alpha < \lambda\}$ is a $\varkappa$-complete filter. A problem of Taylor is: what implications are reversible among: $\exists$ ω_1-complete ideal on ω_1 which has a dense set of size $\omega_1 \Rightarrow \exists$ complete family of ultrafilters on $\omega_1 \Rightarrow \exists$ ω_2-saturated ω_1-complete ideal on ω_1?

The statement: $\exists$ SIB *on* ω_1, *falls between the first and second.* Indeed, one of the implications is done in Theorem 4.4. Let us prove the second. Let τ be a topology on ω_1 which is SIB. As in Proposition 1.2, the dense sets form an ω_1-complete filter F. Let N_α be the neighborhood filter at $\alpha \in \omega_1$. For each α, $N_\alpha \cup F$ generates a filter. Extend it to an ultrafilter U_α. Claim $F = \bigcap \{U_\alpha : \alpha < \omega_1\}$. Only one direction requires proof. If D is in each U_α, then D meets every neighbourhood of every α, so D is dense, i.e., $D \in F$. Thus ω_1 admits a complete family of ultrafilters.

Again we do not know which new arrows reverse.

REFERENCES

[1] J. BAUMGARTNER, A. TAYLOR, S. WAGON, *Structural properties of ideals*, (to appear in Dissertationes Math.).

[2] A. G. ELKIN, *Ultrafilters and irresolvable spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 24 (1969), 51—56.

[3] A. M. GLEASON, *Projective topological spaces*, Illinois J. Math. 2 (1958), 482—489.

[4] E. HEWITT, *A problem of set-theoretic topology*, Duke Math. J. 10 (1943), 309—333.

[5] T. JECH, *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic 5 (1973), 165—198.

[6] T. JECH, *Set Theory*, Academic Press 1978.

[7] T. JECH, M. MAGIDOR, W. MITCHELL, K. PRIKRY, *Precipitous ideals*, (to appear).

[8] T. JECH, K. PRIKRY, *Ideals over uncountable sets: Applications of almost disjoint functions and generic ultrapowers*, Mem. Amer. Math. Soc. 214.

[9] M. KATĚTOV, *On topological spaces containing no disjoint dense sets*, Mat. Sb. 21 (1947), 3—12.

[10] K. KUNEN, *Some applications of iterated ultrapowers in set theory*, Ann. Math. Logic 1 (1970), 179—227.

[11] K. KUNEN, *Saturated ideals*, J. Symbolic Logic, 43 (1978), 65—76.

[12] K. KUNEN, *Maximal σ-Independent Families*, Technical Report #15, Univer. of Texas at Austin, Dec. 1979.

[13] D. MAHARAM, *On a theorem of von Neumann*, Proc. Amer. Math. Soc. 9 (1958), 987—994.

[14] V. I. MALYHIN, *On the resolvability of the product of two spaces and a problem of Katětov*, Dokl. Akad. Nauk SSSR 222 (1975), 725—729.

[15] R. SOLOVAY, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. 92 (1970), 1—56.

[16] R. SOLOVAY, *Real-valued measurable cardinals*, *Axiomatic Set Theory*, Proc. Sympos. Pure Math. 13 (1971), 397—428.

[17] A. TAYLOR, *Regularity properties of ideals and ultrafilters*, Ann. Math. Logic 16 (1979), 33—55.

[18] A. TAYLOR, *Diamond principles, ideals and the normal Moore space problem*, Canad. J. Math., (to appear).

[19] A. IONESCU TULCEA, C. IONESCU TULCEA, *Topics in the theory of liftings*, Ergebnisse der Math. 48, Springer-Verlag, Berlin 1969.

[20] S. WAGON, *The structure of precipitous ideals*, Fund. Math. 106 (1980), 47—52.

[21] B. WĘGLORZ, *Some properties of filters*, (preprint).

[22] H. WOODIN, *An $\aleph_1$-dense $\aleph_1$-complete ideal on $\aleph_1$*, (preprint).