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FIELDS AND QUADRATIC FORM SCHEMES WITH THE INDEX OF RADICAL NOT EXCEEDING 16

Abstract. Let g be an elementary 2-group with $-1 \in g$ and let d be a mapping of g into the family of all subgroups of g. The triple $S = \langle g, -1, d \rangle$ is called a *quadratic form scheme* if $C_1 - C_3$ are fulfilled. The main result is: if $|g| \leqslant 16$ or $[g : R] \leqslant 16$ then all these schemes can be obtained as the product of schemes or schemes of form S^t from the schemes of fields C, R, F_3, F_5, Q_2, $Q_2 \sqrt{(-1)}$, $Q_2 (\sqrt{-2})$ and radical schemes S_i^β. We give a complete list of schemes for $|g| \leqslant 16$, $[g : R] \leqslant 16$ with all invariants.

Throughout the paper, k denotes a field of characteristic $\neq 2$, $g = g(k) = k^*/k^{*2}$ is the group of square classes and $q = q(k) = |g(k)|$ its cardinality. We denote by $R(k)$ the Kaplansky's radical of the field k.

In the papers [1], [7] and [11] the classification of quadratic forms with respect to their behaviour is given for all fields with $q \leqslant 8$. It is shown that there are exactly 27 equivalence classes of fields with $q \leqslant 8$ (17 non-real and 10 formally real fields). Moreover, in [2] Cordes determined all possible sets of parameters q, t, m, s for any non-real field with $q < \infty$ and $[g : R] \leqslant 8$.

In this paper we give the complete classification of all fields with $q = 16$ and of all fields with $[g : R] \leqslant 16$. More precisely, we classify all schemes with $q \leqslant 16$ and $[g : R] \leqslant 16$ and show that these schemes are realized by fields. Complete list of schemes with $[g : R] \leqslant 16$ (and their invariants) is contained in 4 tables at the end of this paper. We see that there are 51 non-equivalent schemes with $q = 16$ (27 non-real and 24 formally real schemes). Moreover, all these schemes can be obtained as a product of schemes or schemes of form S^t from the schemes $S(C)$, $S(R)$, $S(F_3)$, $S(F_5)$, $S(Q)$, $S(Q_2 \sqrt{-1}))$, $S(Q_2(\sqrt{-2}))$ and S_i^β, $i = 1, 2$

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(where S_i^β denotes the radical scheme of cardinality β in the sense that $|R(S_i^\beta)| = q(S_i^\beta) = \beta$ and $s(S_i^\beta) = i$).

Quadratic form schemes were introduced by C. Cordes in [2], but the original definition of Cordes admits schemes, which are not realized by fields (cf. [9]). Here we use the following definition of quadratic form scheme ([9], Def. 1.1). Let g be an elementary 2-group with a distinguished element $-1 \in g$, $-a$ denotes the product $-1 \cdot a$ for $a \in g$. Let d be any mapping from g into the set of all subgroups of g. The triplet $\langle g, -1, d \rangle$ will be denoted by S. An n-tuple $\varphi = (a_1, \ldots, a_n)$, $a_i \in g$ is said to be *a form* (*over* S). For forms (a), (a, b) and $(a_1, \ldots, a_n)$ we denote $D_S(a) = \{a\}$, $D_S(a, b) = a \cdot d(ab)$, $D_S(a_1, \ldots, a_n) = \bigcup_{x \in D_S(a_2, \ldots, a_n)} D_S(a_1, x)$ $(n \geqslant 3)$. The set $D_S(\varphi)$ is said to be the *set of elements represented by the form* φ (*over* S). If $a \in D_S(\varphi)$, we also write $\varphi \approx a$. $S = \langle g, -1, d \rangle$ is said to be a *quadratic form scheme* if it satisfies the following conditions:

$C_1 : a \in D(1, a)$ for any $a \in g$.

$C_2 : a \in D(1, b) \Leftrightarrow -b \in D(1, -a)$ for any $a, b \in g$.

$C_3 : D(a, b, c) = D(b, a, c)$ for any $a, b, c \in g$.

It is evident that, for any field k, $S(k) = \langle g(k), -k^{*2}, d_k \rangle$ is quadratic form scheme ($d_k(a)$ denotes the subgroup of $g(k)$ consisting of elements represented by the form $(1, a)$. This scheme is said to be the *scheme of the field* k.

Two schemes $S_1 = \langle g_1, -1_1, d_1 \rangle$ and $S_2 = \langle g_2, -1_2, d_2 \rangle$ are said to be *equivalent* (write $S_1 \cong S_2$) if there exists a group isomorphism $f : g_1 \to g_2$ such that $f(-1_1) = -1_2$ and $f(d_1(a)) = d_2 f(a))$ for any $a \in g_1$. The phrase *fields* k_1 *and* k_2 *are equivalent with respect to quadratic forms* means $S(k_1) \cong S(k_2)$. If for a scheme S there is a field k such that $S \cong S(k)$ then we say that S *is realized by the field* k. Fundamental properties of quadratic for schemes (in the sense of our definition) are given in [9].

In this paper we adopt notation and terminology as introduced in [9]. In particular $q = q(S)$ is the cardinality of the group g, $q_2 = q_2(S) = |D_S(1, 1)|$, $R = R(S) = \{a \in g : D_S(1, -a) = g\}$ denotes the radical of the scheme S. The minimal number n such that $-1 \in D(n \times (1))$ is denoted by $s = s(S)$ and called the *stufe of* S. If $s < \infty$, then s is a power of two ([9, Th. 3.4]) and S is said to be a *non-real scheme*. Otherwise S is *formally real*. For any subgroup $P \subset g$ of index 2 we say that P is an *ordering of* S if $D(a, b) \subset P$ for $a, b \in P$. We write $r = r(S)$ for the cardinality of the set of orderings of the scheme S. In [9] we defined also *equivalent forms*, *Pfister forms* and *torsion forms*. We denote by $m = m(S)$ the number of equivalence classes of 2-fold Pfister forms and $u = u(S)$ is the maximal dimension of anisotropic and torsion forms over S.

For any schemes $S_1 = \langle g_1, -1_1, d_1 \rangle$ and $S_2 = \langle g_2, -1_2, d_2 \rangle$ we write $S_1 \sqcap S_2$ to denote the product of these schemes, i.e. $S_1 \sqcap S_2 = \langle g_1 \times g_2, (-1_1, -1_2), d \rangle$, where $d(a, b) = d_1(a) \times d_2(b)$. Lastly we define the scheme S^t in the following way: if $S = \langle g, -1, d\rangle$ and $\{1, t\}$ is a 2-element group, then we put $g^t = g \times \{1, t\}$ and, for $a \in g$, $d^t(a) = d(a)$ if $a \neq -1$, $d^t(-1) = g^t$ and $g^t(at) = \{1, at\}$. Then $\langle g^t, -1, d^t\rangle$ is a quadratic form scheme and we denote it by S^t. If the scheme S' is equivalent to S^t for some scheme S, then S' is said to be a *power scheme*. Fundamental properties of the product of schemes and of power schemes have been proved by M. Kula in [6].

In the tables 1 and 3 we give all the schemes with $q \leqslant 16$ which can be constructed as the power schemes or the products of the schemes $S(C)$, $S(R)$, $S(F_3)$, $S(F_5)$ and $S(Q_2)$. From [5] we conclude that all these schemes are realized by fields. Further, by [9, Th.3.9 and Th.3.11], we calculate the invariants q_2, R, m, s, r and u of these schemes. Using the results of [9] and the methods of [1], [7] and [11] we can prove.

THEOREM 1. *If* $S = \langle g, -1, d\rangle$ *is a quadratic form scheme and* $g \leqslant 8$, *then* S *is equivalent to one of the schemes in the table* 1. *All these schemes are realized by fields.*

Now we use Remark 4.9 of [9] and give a classification of the schemes with $[g : R] \leqslant 8$. Let S_i^β be the radical schemes of cardinality β, i.e. $|R(S_i^\beta)| = q(S_i^\beta) = \beta$ and $s(S_i^\beta) = i$, $i = 1, 2$ ([9], Def.4.2). We define the sets of schemes $X_1 = \{S_0, S_{23}, S_{35}, S_{36}\}$ and $X_2 = \{S_{13}, S_{24}, S_{26}, S_{38}, S_{39}, S_{310}, S_{312}, S_{313}, S_{315}, S_{316}, S_{317}\}$ (use the numeration of the schemes as in the table 1). According to Remark 4.9 of [9] we get that the set of all schemes with $[g : R] \leqslant 8$ is $X_1' \cup X_2' \cup X_1''$, where $X_1' = \{S \sqcap S_1^\beta : S \in X_1\}$, $X_2' = \{S \sqcap S_1^\beta : S \in X_2\}$ and $X_1'' = \{S \sqcap S_2^\beta : S \in X_1\}$. We have.

THEOREM 2. *If* $[g : \mathrm{R}] \leqslant 8$ *then* S *is equivalent to one of the schemes in the table* 2. *All these schemes are realized by fields.*

The main result of this paper says that the table 3 contains all the schemes with $q = 16$. This table presents also the fields $Q_2(\sqrt{-1})$ and $Q_2(\sqrt{-2})$ whose schemes cannot be obtained from the simplest schemes by two standard operations on the schemes. The invariants of these two exceptional fields we calculate by using the theory of quadratic forms over dyadic local fields as developed in Lam [8], pp. 152—166.

First we investigate all the schemes with $q = 16$ and non-trivial radical. If $|R(S)| \neq 1$, then $[g : R] \leqslant 8$ and from Theorem 4.7 [9] we get the decomposition $S \cong S' \sqcap S_i^\beta$, $\beta = |R|$, $i = 1, 2$ and $q(S') \leqslant 8$. Thus S' is one of the schemes in the table 1. We get.

PROPOSITION 3. *The table* 3 *contains all the schemes with* $q = 16$ *and* $|R| \neq 1$. *All these schemes are realized by fields.*

Next, let S be a power scheme. Then $S = S_0^t$ for some scheme S_0 with $q(S_0) = 8$. Thus S_0 is equivalent to one of the schemes S_{31}—S_{317} in the table 1. Using 3.15 from [9] we get

PROPOSITION 4. *If* S *is a power scheme with* $q = 16$, *then* $S \cong S(k((t)))$ *for some field* k *with* $q(k) = 8$.

Thus we can assume that S is a scheme with $q(S) = 16$, $R(S) = \{1\}$ and S is not a power scheme. We shall use two following lemmas.

LEMMA 5. *If $q \geqslant 4$ then the following statements are equivalent:*

(a) *S is not a power scheme;*

(b) *for any subgroup $h \subset g$ of index 2 there exist elements $a \notin h$ and $b \in h$, $b \neq 1$ such that $(1, a) \approx b$;*

(c) *for any $a \in g$, $a \neq \pm 1$, $|D(1, a)| \geqslant 4$* or *$|D(1, -a)| \geqslant 4$.*

Moreover, if S is a non-real scheme, then each of (a), (b), (c) *is equivalent to*

(c′) *for any $a \in g$, $a \neq 1$, $|D(1, a)| \geqslant 4$.*

The proof of Lemma 5 is omitted since the result follows directly from 3.13 and 3.14 of [9].

LEMMA 6. *If $q(S) = 16$ and there exist $b, c \in g$ such that $(1, a)$ does not represent b, c, bc, then $|D(1, a)| \leqslant 4$.*

This is a direct conclusion from the group theory.

Now we investigate all the non-real schemes with $q = 16$, $|R| = 1$ which are not power schemes. We classify the possible cases depending on the values of two invariants: $s(S)$ and $q_2(S)$.

PROPOSITION 7. *If $q(S) = 16$, $s(S) = 1$, $|R(S)| = 1$ and S is not a power scheme, then S is equivalent to S_{49} or S_{410} in table* 3.

P r o o f. If $s = 1$ then $-1 = 1$ and $(1, 1)$ is the unique binary form (up to equivalence) which represents all the 16 elements of g. First we assume that all binary forms (which are not equivalent to $(1, 1)$) represent 4 elements and let $(1, a) \approx b$. Then $D(1, a) = D(1, b) = D(1, ab) = \{1, a, b, ab\}$. We consider a subgroup $h \subset g$, $[g : h] = 2$ containing a and b. Then there exist $c, d \in g$, $1 \neq c \in h$ such that $(1, d) \approx c$ (Lemma 5(b)). It is clear that $c \neq a, b, ab$. We have $D(1, c) = D(1, d) = D(1, cd) = \{1, c, d, cd\}$ and $\{a, b, c, d\}$ is an F_2-basis of g. Because of $(a, b) \approx 1$, we have $d \in D(1, c) \subset D(1, ac, bc)$ and there exists an $x \in D(1, ac)$ such that $(x, bc) \approx d$, i.e. $(1, bcd) \approx dx$. But $(1, ac) \not\approx a, b, ab, c, d, cd, abc, bc, acd, ad$ and $(1, bcd) \not\approx a, b, ab, c, d, cd, abcd, acd, bd, bc$, so $x = abcd$. We get $D(1, ac) = D(1, bd) = D(1, abcd) = \{1, ac, bd, abcd\}$ and $D(1, bcd) = D(1, abc) = D(1, ad) = \{1, bcd, abc, ad\}$. From $(1, ad) \approx abc$ it follows that $a \in D(abcd, bc) \subset D(1, ac, bc) = \bigcup_{x \in D(ac, bc)} D(1, x) = \bigcup_{x \in acD(1, ab)} D(1, x) = \bigcup_{x \in cD(1, a)} D(1, x) = D(1, c) \cup D(1, ac) \cup D(1, bc) \cup D(1, abc)$, a contradiction to $D(1, a) = \{1, a, b, ab\}$.

We conclude that at least one binary form represents 8 elements of g and let $D(1, a) = h$, $|h| = 8$. Then there exists $d \notin h$ such that $(1, d) \approx b \in h$ and $b \neq 1, a$. Let $\{a, b, c\}$ be an F_2-basis of $D(1, a)$. Then $\{a, b, d\}$ is F_2-basis of $D(1, b)$ and

(A) $(1, x) \approx a, ax$ for $x = c, ac, bc, abc$,

(B) $(1, y) \approx b, by$ for $y = d, ad, bd, abd$.

Now we have

$$(1, a) \approx c \Rightarrow (cd, acd) \approx d \Rightarrow b \in D(1, d) \subset D(1, cd, acd) \Rightarrow \text{there exists a } z \in D(1, cd) \text{ such that } (z, acd) \approx b \Rightarrow (1, abcd) \approx bz.$$

But $(1, cd) \not\approx a, b, acd, bcd$ and $(1, abcd) \not\approx a, b, bcd, acd$, hence $z \in \{c, d\} \times \times \{1, a, b, ab\}$.

First we assume that $z = d$, i.e. $(1, cd) \approx d$ and $(1, abcd) \approx bd$. We shall show that all binary forms (except $(1, 1)$) represent 8 elements. For $x = c, ac$ and $y = d, bd$ in (A) and (B) we get

(A′) $D(1, c) = \{1, a, c, ac, d, ad, cd, acd\}$, $D(1, d) = \{1, b, c, bc, d, bd, cd, bcd\}$,

(B′) $D(1, ac) = \{1, a, c, ac, bd, abd, bcd, abcd\}$, $D(1, bd) = \{1, b, d, bd, ac, abc, acd, abcd\}$.

We have $(1, bc) \approx d$ and $(1, ad) \approx c$, by (A′), hence the forms $(1, bc)$, $(1, ad)$ represent 8 elements. Similarly, by (B′), $(1, abc) \approx bd$ and $(1, abc) \approx \approx a$ implies $(1, abd) \approx abc$, thus the forms $(1, abc)$ and $(1, abd)$ represent 8 elements too. This implies $D(1, cd) = \{1, c, d, cd, ab, abc, abd, abcd\}$ and $D(1, ab) = \{1, a, b, ab, cd, acd, bcd, abcd\}$, hence $D(1, abcd) = \{1, ab, cd, abcd, bd, ad, bc, ac\}$. Analogously $(1, acd) \approx c, ab$ and $(1, bcd) \approx d, ab$, so these forms represent 8 elements. We conclude that, if $z = d$, then all the binary forms (except $(1, 1)$) represent 8 elements. Similarly, for $z = ad$, all the binary forms represent 8 elements. It is easy to prove that the group isomorphism $a \to a$, $b \to b$, $c \to ac$, $d \to ad$ is an equivalence map (cf. [9], Def.3.7). Thus we obtain equivalent schemes for $z = d$ and $z = ad$. Analogously we get equivalent schemes if $z = bd$ or abd.

Further we assume that $z = c$. Then $(1, cd) \approx c$ and $(1, abcd) \approx bc$. Putting $x = c, bc$ and $y = d, ad$ in (A) and (B) we get

$$D(1, c) = \{1, D(1, a, c, ac, d, ad, cd, acd\},$$
$$D(1, bc) = \{1, a, bc, abc, d, ad, bcd, abcd\}.$$

$$D(1\ d) = \{1, b, c, bc, d, bd, cd, bcd\},$$
$$D(1, ad) = \{1, b, ad, abd, c, bc, acd, abcd\}.$$

Suppose that $|D(1, ab)| = 8$. From $(1, ab) \not\approx c, d$ we get $(1, ab) \approx cd$, hence $(1, cd) \approx abd$. *Since* $(1, abcd) \approx ad$ we have the case, which has been discussed for $z = abd$. So the case $|D(1, ab)| = 4$, i.e. $D(1, ab) = \{1, a, b, ab\}$ remains to be considered. We want to show that all the remaining binary forms represent 4 elements. The forms $(1, cd)$, $(1, acd)$, $(1, bcd)$ and $(1, abcd)$ do not represent a, b, ab, hence they do not represent $q/2 = 8$ elements. Further, $(1, acd) \approx c$ and $(1, bcd) \approx b$, hence $|D(1, cd)| = |D(1, acd)| = |D(1, bcd)| = 4$. This implies that $(1, ac)$ and $(1, abc)$ do not represent c, d, cd and neither of $(1, bd)$ and $(1, abd)$ represents a, c, ac, so also these forms represent 4 elements. If $z = ac, bc, abc$, we obtain the equivalent schemes, as above.

We conclude that for $s = 1$ there exist at most 2 non-equivalent schemes with trivial radical which are not power schemes. Hence $S = S_{49}$ if all binary forms (except (1, 1)) represent 8 elements and $S \cong S_{410}$ otherwise.

In the proof of the next proposition we use the following.

LEMMA 8. *If $S = \langle g, -1, d \rangle$ is a non-real quadratic form scheme, $2 < q < \infty$ and $q_2 = 2$ then $s(S) = 2$ and S is realized by the power series field $F_3((t_1)) \dots ((t_n))$, where $q = 2^{n+1}$.*

The proof is the same as in [3, Prop. 1]. It is easy to see that S is realized by the field $F_3((t_1)) \dots ((t_n))$.

COROLLARY 9. *Let S be a non-real scheme and $\beta = |R(S)| \neq 1$.*

(i) *If $[D_S(1,1) : R(S)] = 1$, then $S \cong S_1^\beta$ or $S \cong S_2^\beta$.*

(ii) *If $[D_S(1,1) : R(S)] = 2$, then $S \cong S' \sqcap S_1^\beta = S' \sqcap S_2^\beta$, where S' is the scheme from* Lemma 8 *and S_i^β are the radical schemes of cardinality β.*

P r o o f. The first result follows directly from [9, Th.4.7]. As far as the second result is concerned we observe that, if S_0 is any non-real scheme with $|R(S_0)| = 1$, then $q(S_0) \neq 2$. Using $[D_S(1,1) : R(S)] = 2$ we get $-1 \notin R(S)$, hence $S = S/R \sqcap S_1^\beta = S \sqcap S_2^\beta$ by [9, Th.4.7 (i)]. Moreover $q(S/R) = [g(S) : R(S)] \geqslant 4$ and $q_2(S/R) = [D_S(1,1) : R(S)] = 2$, thus the scheme $S' = S/R$ satisfies the assumptions of Lemma 8, as required.

PROPOSITION 10. *If $q(S) = 16$, $s(S) = 2$, $|R(S)| = 1$ and S is not a power scheme, then S is equivalent to S_{418}, S_{419} or S_{423} in the table 3.*

P r o o f. If $s(S) = 2$ then $1 \neq -1$ and the form (1, 1) is not universal, i.e. $|D(1,1)| = 8$ or 4, by Lemma 8. Moreover, if $x \in D(1,1)$ then $(1,x) \approx \pm 1, \pm x$.

1. $|D(1,1)| = 8$. First we shall show that $D(1,x) \neq D(1,1)$ for any $x \in D(1,1)$, $x \neq 1$. Otherwise, if $D(1,x') = D(1,1)$ for some $x' \in D(1,1)$, $x' \neq 1$, then $(1,x') \approx y$ for any $y \in D(1,1)$, so $(1,-y) \approx -1, y, -x'$ and $D(1,-y) = D(1,1)$ for any $y \in D(1,1)$, $y \neq 1, x'$. Moreover, $D(1,1) = D(1,1) \cap D(1,x') \subset D(1,-x')$, hence $D(1,-x') = D(1,1)$. Thus we obtain $D(1,x) = D(1,1)$ for any $x \in D(1,1)$, $x \neq -1$. But $[g : D(1,1)] = 2$ implies that there exist $z \notin D(1,1)$ and $y \in D(1,1)$, $y \neq 1$ such that $(1,z) \approx y$ (Lemma 5), hence $(1,-y) \approx -z$, $-y \in D(1,1)$, $-z \notin D(1,1)$, a contradiction. Thus we have $D(1,x) \neq D(1,1)$ for any $x \in D(1,1)$, $x \neq 1$.

Now suppose that $|D(1,a)| = |D(1,-a)| = 8$ for some $a \in D(1,1)$. Let $\{-1, a, b\}$ be an F_2-basis for $D(1,1)$ and $(1,a) \approx c \notin D(1,1)$. Then $g = D(1,1) \times \{1, c\}$. Since $D(1,1) \neq D(1,a) \neq D(1,-a)$ (if $D(1,a) = D(1,-a)$ then $D(1,a) \subset D(1,a) \cap D(1,-a) \subset D(1,1)$, a contradiction), we have $(1,-a) \not\approx b, c$ and $(1,-a) \approx bc$ (by $D(1,-a) = q/2$), so $D(1,-a) = \{\pm 1, \pm a, \pm bc, \pm abc\}$. If $D(1,x) = \{\pm 1, \pm x\}$ for any $x \in \{\pm b, \pm ab\}$, then $(1,c) \not\approx -1, \pm b$, hence $D(1,c) = \{1, c, -a, -ac\}$. From $(1,-a) \approx \pm bc$ and $(c,c) \approx bc$ we get $a \in D(1,bc) \subset D(1,c,c) = \bigcup_{w \in D(1,c)} D(w,c) = (1,c) \cup$

$\cup D(c, c) \cup D(-a, c) \cup D(-ac, c)$, hence $(1, 1) \approx ac$ or $(1, -a) \approx ac$, a contradiction. Thus at least one form $(1, x)$, $x \in \{\pm b, \pm ab\}$ represents 8 elements and we may assume that $|D(1, b)| = 8$. Because $(1, b) \not\approx \pm a, \pm ab$ we have $(1, b) \approx \pm c$ or $(1, b) \approx \pm ac$. Let $D(1, b) = \{\pm 1, \pm b, \pm c, \pm bc\}$ (in other case we obtain the equivalent scheme). Then $(1, -ab) \approx c$ and $D(1, -ab) = \{\pm 1, \pm ab, \pm c, \pm abc\}$. Now $abc \in D(1, -ab) \cap D(1, -a) \subset D(1, -b)$, hence $D(1, -b) = \{\pm 1, \pm b, \pm abc, \pm ac\}$. Similarly, $ac \in D(1, -b) \cap D(1, a) \subset D(1, ab)$ and so $|D(1, ab)| = 8$. We get that all the binary forms $(1, x)$, $x \in D(1, 1)$, $x \neq -1$ represent 8 elements. A direct calculation shows that the remaining binary forms represent 8 elements. Thus we have proved that all the schemes with $|D(1, a)| = |D(1, -a)| = 8$ for some $a \in D(1, 1)$ are equivalent.

Now we assume that $|D(1, a)| = 4$ or $|D(1, -a)| = 4$ for any $a \in D(1, 1)$, $a \neq \pm 1$. We shall prove that all these schemes are equivalent too. Choose $c \notin D(1, 1)$ such that $(1, c) \approx a \in D(1, 1)$, $a \neq 1$. Write $D(1, 1) = \{\pm 1, \pm a, \pm b, \pm ab\}$. Then $D(1, -a) = \{\pm 1, \pm a, \pm c, \pm ac\}$ and $D(1, a) = \{\pm 1, \pm a\}$. As previously, we shall show that $|D(1, x)| = 8$ for some $x \in \{\pm b, \pm ab\}$. Otherwise, $D(1, x) = \{\pm 1, \pm x\}$ and $(1, bc)$ does not represent $-1, \pm a, \pm b, \pm ab, -bc, \pm abc, \pm c, \pm ac$, so $|D(1, bc)| = 2$, a condradiction according to Lemma 5. Thus we may assume that $|D(1, b)| = 8$. Since $(1, a) \not\approx -b$ we get $(1, b) \not\approx \pm a, \pm ab$, hence $(1, b) \approx \pm c$ or $(1, b) \approx \pm ac$. Let $D(1, b) = \{\pm 1, \pm b, \pm c, \pm bc\}$ (in other case we obtain the equivalent scheme). Then $D(1, ab) = \{\pm 1, \pm ab, \pm c, \pm abc\}$ (because $c \in D(1, -a) \cap D(1, b) \subset D(1, ab)$), hence the forms $(1, -b)$ and $(1, -ab)$ represent 4 elements. Further, $(1, \pm c) \approx a, -b$ thus these forms represent 8 elements. Since $(1, \pm bc) \approx -b$, $(1, \pm abc) \approx -ab$, $(1, \pm ac) \approx a$ and $(1, \pm bc)$, $(1, \pm abc)$ do not represent $-1, \pm a$ and $(1, \pm ac) \not\approx -1, \pm b$, hence these forms represent 4 elements.

Thus we proved that there exist at most 2 non-equivalent schemes with trivial radical, $s = 2$ and $q_2 = 8$ which are not power schemes. This implies that $S \cong S_{418}$ if $|D(1, a)| = |D(1, -a)| = 8$ for some $a \in D(1, 1)$ and $S \cong S_{419}$ otherwise.

2. $|D(1, 1)| = 4$. Write $D(1, 1) = \{\pm 1, \pm a\}$. Then $(1, a) \approx \pm 1, \pm a$ and $(1, -a) \approx \pm 1, \pm a$.

First we assume that $|D(1, a)| = |D(1, -a)| = 4$. if $x \neq \pm 1, \pm a$, then the forms $(1, x)$ do not represent $-1, \pm a$, thus they represent 4 elements. Let us consider a subgroup $h \subset g$ of index 2 containing $D(1, 1)$. There exist $b \notin h$, and $c \in h$, $c \neq 1$ such that $(1, b) \approx c$. Obviously $c \neq -1, \pm a$, thus $\{-1, a, b, c\}$ is F_2-basis of g. Since $(1, -c) \approx -b$ and $(c, c) \approx -c$ we have $(1, c, c) \approx -b$. Then there exists $y \in D(1, c)$ such that $(y, c) \approx -b$, so $(1, bc) \approx -by$. Since $(1, b) \approx c$ and $(1, b) \not\approx -1$ we get $(1, b) \not\approx -c$, hence $(1, c) \not\approx -1, -c, -b, -bc, \pm a \pm ac$. Similarly $(1, -c) \not\approx b$, hence $(1, bc) \not\approx b$. Moreover $(1, b) \not\approx -c, -bc$ implies $(1, bc) \not\approx -b, -c$ and so $(1, bc) \not\approx -1$,

$-bc, b, c, -c, -b, \pm a, \pm abc$. Thus we have $y = abc$ or $y = -abc$. If $y = abc$, i.e. $(1, c) \approx abc$ and $(1, bc) \approx -ac$ then $D(1, -abc) = \{1, -abc, -c, ab\}$. From $c \in D(1, -b) \cap D(1, c) \subset D(1, -bc)$ and $(-abc, -abc) \approx -bc$ we get $c \in D(1, -bc) \subset D(1, -abc, -abc) = \bigcup_{w \in D(1, -abc)} D(w, -abc) = D(1, -abc) \cup D(-abc, -abc) \cup D(-c, -abc) \cup D(ab, -abc)$, hence $(1, 1) \approx -ab$ or $(1, -c) \approx abc$, a contradiction since $(1, -c) \approx bc$ and $(1, -c) \not\approx a$. This implies that $y = -abc$ and $(1, c) \approx -abc, -ab$ and $(1, bc) \approx ac$, hence $D(1, -ac) = \{1, -ac, -bc, ab\}$ and $D(1, ab) = \{1, ab, -c, -abc\}$. From $(1, -c) \approx -b, bc$ and $(-ac, -ac) \approx -c$ we obtain $bc \in D(1, -c) \subset D(1, -ac, -ac) = \bigcup_{w \in D(1, -ac)} D(w, -ac) = D(1, -ac) \cup D(-ac, -ac) \cup D(-bc, -ac) \cup D(ab, -ac)$ and so $(1, 1) \approx -ab$ or $(1, -ac) \approx bc$, a contradiction.

We conclude that at least one form $(1, a)$ or $(1, -a)$ represents 8 elements. Let $D(1, -a) = \{\pm 1, \pm a\}$ and $|D(1, a)| = 8$. Then there exists a $c \notin D(1, a)$ such that $(1, c) \approx b \in D(1, a)$, $b \neq \pm 1, \pm a$. We have: $\{-1, a, b\}$ is F_2-basis for $D(1, a)$, $\{-b, -a, -c\}$ is F_2-basis for $D(1, -b)$, hence $D(1, abc) = \{1, abc, b, ac\}$ by $(1, abc) \not\approx -1, \pm a$. Now $(1, -b) \approx -c$, hence $(1, -b) \not\approx c$ and so $(1, -c) \not\approx b$. Similarly $(1, -b) \approx bc$ follows that $(1, b) \not\approx bc, c$, thus $(1, -c) \not\approx -b$. This implies that $(1, -c)$ does not represent $-1, c, -abc, ab, b, -bc, -b, bc, \pm a, \pm ac$ and so $D(1, -c) = \{1, -c, -ab, abc\}$. From $(-c, -c) \approx -ac$ and $(1, -b) \approx ac$ we have $b \in D(1, -ac) \subset D(1, -c, -c) = \bigcup_{w \in D(1, -c)} D(w, -c) = D(1, -c) \cup D(-c, -c) \cup D(-ab, -c) \cup D(abc, -c)$, hence $(1, 1) \approx -bc$ or $(1, abc) \approx -a$ or $(1, -ab) \approx ac$, a contradiction in view of $(1, -b) \approx ac$ and $(1, -a) \not\approx ac$.

Thus we have proved that both the forms $(1, a)$ and $(1, -a)$ represent 8 elements. Let $D(1, a) = \{\pm 1, \pm a, \pm b, \pm ab\}$. But $(1, -a) \not\approx b$, by $(1, 1) \not\approx b$, and it follows that there exists $c \notin D(1, a)$ such that $(1, -a) \approx c$. Hence $\{-1, a, b, c\}$ is F_2-basis of g and $D(1, -a) = \{\pm 1, \pm a, \pm c, \pm ac\}$. Now, for $x = \pm b, \pm ab$ we have $\{1, x, -a, -ax\} \subset D(1, x)$. If the equality is satisfied for any x, then, according to $(1, b) \approx -a$ and $(1, -a) \approx c$, we have $(1, b) \approx -ab$ and $(b, -ab) \approx bc$, hence $bc \in D(b, -ab) \subset D(1, b, b) = \bigcup_{z \in D(b, b)} D(1, z) = D(1, b) \cup D(1, -b) \cup D(1, ab) \cup D(1, -ab)$, a contradiction. Thus we may assume that $(1, b)$ represents 8 elements (for remaining x we obtain the equivalent schemes). Then $(1, b) \approx c$ or $(1, b) \approx -c$, by $(1, b) \not\approx -1$. Let $(1, b) \approx c$ (in other case we get the equivalent scheme). We have $D(1, b) = \{1, b, -a, -ab, c, bc, -ac, -abc\}$ and, by $(1, -a) \approx c$, $D(1, ab) = \{1, ab, -a, -b, c, abc, -ac, -bc\}$ hence $D(1, ac) = \{1, ac, a, c, -ab, -bc, -b, -abc\}$ and $D(1, -c) = \{1, -c, a, -ac, -ab, abc, -b, bc\}$. We shall show that the remaining forms represent 4 elements. The forms $(1, \pm bc)$ and $(1, \pm abc)$ do not represent $-1, \pm a$, hence they represent 4 elements. In particular,

$D(1, abc) = \{1, abc, -b, -ac\}$ and $D(1, -abc) = \{1, -abc, c, -ab\}$. This implies that the forms $(1, c)$, $(1, -b)$, $(1, -ab)$ and $(1, -ac)$ do not represent -1, $\pm abc$ and so they represent 4 elements too. We conclude that there exists at most on (up to equivalence) scheme with trivial radical, $s = 2$, $q_2 = 4$ which is not a power scheme. Hence $S = S_{423}$.

PROPOSITION 11. *If* $q(S) = 16$ *and* $s(S) = 4$ *then* $|R(S)| \neq 1$ *or* S *is a power scheme.*

Proof. If $4 \times (1) \approx -1$ then there exist $a, b \in D(1, 1)$ such that $(a, b) \approx -1$ (cf. [9], Th..1.6). Hence $(1, 1) \approx 1, a, b, ab$, $(1, a) \approx -b, -ab$, $(1, b) \approx -a, -ab$, $(1, ab) \approx -a, -b$ and $(1, x) \approx \pm 1, \pm x$ for any $x \in D(1, 1)$.

We suppose that the proposition is not true so that S is not a power scheme and $|R(S)| = 1$. We consider two cases.

1. $|D(1, 1)| = 8$. Write $H = \{1, a, b, ab\}$. First observe that $|D(1, x)| = 4 \Leftrightarrow |D(1, -x)| = 4$ for any $x \in H$. For example, for $x = a$ we have $(1, a) \not\approx -1, -a, b, ab$ (by $(1, 1) \not\approx -a$) and $(1, -a) \not\approx \pm b, \pm ab$ (otherwise $b \in D(1, -a) \cap D(1, 1) \subset D(1, a)$). If $|D(1, a)| = 4$ and $(1, -a) \approx z$, $z \neq \pm 1, \pm a, \pm b, \pm ab$, then $(1, -a) \approx \pm z$. But $(1, 1) \not\approx -1$, hence $(1, 1) \approx z$ or $-z$ and so $z \in D(1, -a) \cap D(1, 1) \subset D(1, a)$ or $-z \in D(1, a)$, a contradiction. Similarly, if $|D(1, -a)| = 4$ and $(1, a) \approx w$, $w \neq \pm 1, \pm a, \pm b, \pm ab$, then $(1, 1) \approx w$ or $-w$. If $(1, 1) \approx w$ then $w \in D(1, -a)$ and, if $(1, 1) \approx -w$, then $-bw \in D(1, 1) \cap D(1, a) \subset D(1, -a)$. In both cases we get a contradiction. Thus we obtain $|D(1, a)| = 4 \Leftrightarrow |D(1, -a)| = 4$. For $x = b, ab$ the proof is the same.

Now we suppose that all the forms $(1, a)$, $(1, b)$ and $(1, ab)$ represent 4 elements. Then also $(1, -a)$, $(1, -b)$ and $(1, -ab)$ represent 4 elements and for $d \in (1, 1)\backslash H$ we have $(1, d) \not\approx -1, \pm b, \pm a, \pm ab, \pm ad, \pm bd, \pm abd$, hence $|D(1, d)| = 2$, a contradiction.

Thus we can assume that $|D(1, a)| = 8$ and choose $d \in D(1, a)$, $d \neq 1, a, -b, -ab$ such that $d \in D(1, 1)$ (if $d \notin D(1, 1)$, then $-d \in D(1, 1)$, $-bd \in D(1, 1)$ and $-bd \in D(1, a)$). We have $D(1, a) = \{1, a, -b, -ab, d, ad, -bd, -abd\}$, $D(1, -a) = \{\pm 1, \pm a, \pm d, \pm ad\}$ (by $d \in D(1, 1) \cap D(1, a) \subset D(1, -a)$) $D(1, -d) = \{\pm 1, \pm d, \pm a, \pm ad\}$ and $\{-1, a, b, d\}$ is F_2-basis of g.

If $|D(1, b)| = 4$ then $|D(1, -b)| = 4$ and so $(1, d) \approx \pm ab$ (otherwise $(1, d) \approx a$ implies $(1, d) \approx \pm b$, hence $(1, b) \approx -d$ or $(1, -b) \approx -d$, a contradiction). We obtain $(1, d) \not\approx -1, -d, \pm b, \pm bd, \pm ab, \pm abd$, hence $|D(1, d)| = 4$, i.e. $D(1, d) = \{1, d, a, ad\}$. We have $-a \in D(1, b) \subset \bigcup_{x \in D(1,1)} D(1, x) = \bigcup_{x \in D(d,d)} D(1, x) = D(1, d, d) = \bigcup_{x \in D(1,d)} D(x, d) = D(1, d) \cup D(a, d) \cup D(d, d) \cup D(ad, d) = D(1, d) \cup D(a, d) \cup D(1, 1) \cup D(1, a)$, a contradiction. Similarly, if $|D(1, ab)| = 4$, we obtain a contradiction too.

Thus we have proved that the forms $(1, a)$, $(1, b)$ and $(1, ab)$ represent 8 elements, so the forms $(1, -a)$, $(1, -b)$ and $(1, -ab)$ represent 8 elements too. In particular, $(1, b) \approx d$ or $(1, b) \approx -d$ (by $(1, 1) \not\approx -1$). In the first case we have $(1, -d) \approx -1, -b, a, -d$ and $|D(1, -d)| = 16$, a contradiction. In the second case $(1, b) \approx -d, ad$, hence $(1, -ad) \approx -1, -b, -ad, a$ and so $|D(1, -ad)| = 16$, a contradiction too.

2. $|D(1, 1)| = 4$. Since $D(1, 1) = \{1, a, b, ab\}$, $(a, b) \approx -1$ and $(a, ab) \approx -1$ we see that for any $x \in D(1, 1)$, $x \neq 1$, there exists an $y \in D(1, 1)$ such that $(x, y) \approx 1$. Write $H = \{1, -1\} \times D(1, 1)$. Let $c \notin H$ and suppose that $(1, c) \approx a$ for some $a \in D(1, 1)$. Choose $b \in D(1, 1)$ such that $(a, b \approx -1$. We have $D(1, -a) = \{\pm 1, \pm a, \pm c, \pm ac\}$ and $D(1,a) = \{1, a, -b, -ab\}$ from $(1, a) \not\approx -1, \pm c$ (if $(1, a) \approx \pm c$ then $(1, 1) \approx \pm c$). Similarly the forms $(1, bc)$ and $(1, -abc)$ do not represent $-1, \pm a$ and so they represent 4 elements.

From $c \in D(1, -a) \cap D(1, c) \subset D(1, ac)$ we have $-bc \in cD(1, a) = D(c, ac) \subset D(1, ac, ac) = \bigcup_{x \in D(a, a)} D(1, cx) = \bigcup_{x \in D(1, 1)} D(1, cx) = D(1, c) \cup D(1, ac) \cup D(1, bc) \cup D(1, abc)$ and so $(1, bc) \approx -c$ or $-ac$ or $-abc$, by $(1, bc) \not\approx -bc$. We shall show that none of these cases can occur.

If $(1, bc) \approx -c$ then $D(1, bc) = \{1, bc, -c, -b\}$. It follows that $-b \in D(1, a) \cap D(1, bc) \subset D(1, -abc)$ and $D(1, -abc) = \{1, -abc, -b, ac\}$ As previously we have $a \in D(1, c) \subset D(1, c, c) = D(1, bc, bc) = \bigcup_{x \in D(1, bc)} D(x, bc) = D(1, bc) \cup D(bc, bc) \cup D(-c, bc) \cup D(-b, bc)$. From $(1, -a) \not\approx -bc$ and $(1, 1) \not\approx abc$ we get $(-c, bc) \approx a$ or $(-b, bc) \approx a$, hence $(-a, bc) \approx c$ or b and so $(1, -abc) \approx -ac$ or $-ab$, a contradiction.

If $(1, bc) \approx -ac$ we obtain $D(1, bc) = \{1, bc, -ac, -ab\}$ and $D(1, -abc) = \{1, -abc, -ab, c\}$ (by $-ab \in D(1, bc) \cap D(1, a)$) so $a \in D(1, bc, bc) = \bigcup_{x \in D(1, bc)} D(x, bc) = D(1, bc) \cup D(bc, bc) \cup D(-ac, bc) \cup D(-ab, bc)$. This implies $(-ab, bc) \approx a$ or $(-ac, bc) \approx a$, hence $(-a, bc) \approx ab$ or ac and so $(1, -abc) \approx -b$ or $-c$, a contradiction.

Lastly, if $(1, bc) \approx -abc$ then $D(1, bc) = \{1, bc, -abc, -a\}$ and we have $a \in D(1, bc, bc) = D(1, bc) \cup D(bc, bc) \cup D(-abc, bc) \cup D(-a, bc)$. Thus $(bc, -abc) \approx a$ or $(-a, bc) \approx a$, hence $(1, -a) \approx abc$ or $(1, 1) \approx abc$, a contradiction.

We conclude that $D(1, z) \cap D(1, 1) = \{1\}$ for any $z \in H$. But $|H| = q/2$ and so there exists a $c \notin H$ such that $(1, c) \approx w \in H$, $w \neq 1$. Thus we can choose $a \in D(1, 1)$ such that $(1, c) \approx -a$ and next choose $b \in D(1, 1)$ such that $(a, b) \approx -1$. We have $-a \in D(1, c) \cap D1, b) \cap D(1, -a)$, hence $-a \in D(1, -abc) \subset D(1, -abc, -abc) = D(1, -c, -c)$. Thus there exists an $x \in D(1, -c)$ such that $(x, -c) \approx -a$, hence $(1, -c) \approx x$ and $(1, -ac) \approx -ax$. From $-c, -ac \notin H$ we obtain $(1, -c) \not\approx -1, c, a, b, ab, -ac, -bc, -abc$ and $(1, -ac) \not\approx -1, ac, a, b, c, -c, -abc, -bc$ and so $x = 1$ or $-a$. In both cases we get $(1, -c) \approx -a$, hence $(1, a) \approx c$. But $(1, a) \approx -c$ and $(1, 1) \not\approx -1$, a condradiction.

Thus we have proved that there does not exist a scheme with $s = 4$ and trivial radical which is not a power scheme and the proof of Proposition 11 is finished. Thus we get

THEOREM 12. *If S is any non-real quadratic form scheme with $q = 16$ then S is equivalent to one of the schemes $S_{41}-S_{427}$ in the table 3. All these schemes are realized by fields.*

Now we consider the formally real case.

LEMMA 13. *Let S be a formally real scheme, $q(S) = 2^n$, $q_2(S) = 2^{n-1}$ and $|D_S(1, a)| = 2$ for any $a \in D_S(1, 1)$, $a \neq 1$. Then $S \cong S(R) \sqcap S(k)$, where k is the power series field $F_5((t_1)) \dots ((t_{n-2}))$.*

Proof. Write $S(R) \sqcap S(k) = S'$. It is clear that $q(S') = 2^n$. Since $s(k) = 1$ we get $q_2(S') = q_2(R)\, q_2(k) = 2^{n-1}$. If $a \in D_{S'}(1, 1)$, $a = (x, y)$, where $x \in D_{S(R)}(1_R, 1_R) = \{1_R\}$ and $y \in D_{S(k)}(1_k, 1_k) = g(k)$. For $a \neq 1$ we have $y \neq 1_k$ and $D_{S'}(1, a) = D_{S(R)}(1_R, x) \times D_{S(k)}(1_k, y) = \{1_R\} \times \{1_k, y\} = \{1, a\}$ and $D_{S'}(1, -a) = D_{S(R)}(1_R, -x) \times D_{S(k)}(1_k, -y) = g(R) \times \{1_k, -y\} = \{\pm 1, \pm a\}$. We shall show that $D_S(1, -a) = \{\pm 1, \pm a\}$ for any $a \in D_S(1, 1)$, $a \neq 1$. Suppose $(1, -a) \approx b \neq \pm 1$. Then $(1, -a) \approx -b$, hence $(1, b) \approx a$ and $(1, -b) \approx a$. From $|D_S(1, 1)| = q/2$ we have $(1, 1) \approx b$ or $(1, 1) \approx -b$. If $(1, 1) \approx b$ then $D(1, b) = \{1, b\}$ and so $a = b$. If $(1, 1) \approx -b$ then $D(1, -b) = \{1, -b\}$ and $a = -b$. Thus $D(1, -a) = \{\pm 1, \pm a\}$. This implies that any group isomorphism $f: g(S) \to g(S')$ such that $f(-1) = -1'$ and $f(D_S(1, 1)) = D_{S'}(1', 1')$ is an equivalence map and so $S \cong S'$.

PROPOSITION 14. *If S is a formally real scheme with $q(S) = 16$, $q_2(S) = 8$ and $R(S) = \{1\}$ then $S \cong S_{430}$ or $S \cong S_{431}$.*

Proof. If $|D(1, a)| = 2$ for any $a \in D(1, 1)$, $a \neq 1$, then $S \cong S_{431}$, by Lemma 13. Thus we can assume that there exists $a \in D(1, 1)$, $a \neq 1$ such that $(1, a) \approx b$, $b \neq 1, a$. We observe that, if $|D(1, 1)| = q/2$, then there exists exactly one ordering $P = D(1, 1)$ on S and $D(1, x) \subset P$ for any $a \in P$. Thus $b \in D(1, 1)$ and write $H = \{1, a, b, ab\}$ and $D(1, 1) = H \times \{1, c\}$. We have $D(1, -b) = D(1, -a) = D(1, -ab) = \{\pm 1, \pm a, \pm b, \pm ab\}$ and $(1, x) \not\approx c$ for any $x \in H$, $x \neq 1$ (if $(1, x) \approx c$ then $c \in D(1, x) \cap D(1, 1) \subset D(1, -x)$, a contradiction). From $D(1, x) \subset P = D(1, 1)$ we get $D(1, x) = H$ for any $x \in H$, $x \neq 1$. Moreover, $(1, -cx) \not\approx a, b, ab$, hence $D(1, -cx) = \{\pm 1, \pm cx\}$. Further, $(1, cx) \not\approx a, b, ab, acx, bcx, abcx$ and $D(1, cx) \subset P$ thus $D(1, cx) = \{1, cx\}$.

We conclude that there exists at most one scheme with $q_2 = 8$, $|R| = 1$ and $|D(1, a)| = 4$ for some $a \in D(1, 1)$, $a \neq 1$. Hence $S \cong S_{430}$.

PROPOSITION 15. *If S is a formally real non-power scheme with $q(S) = 16$, $|R(S)| = 1$ and $q_2(S) = 4$ then S is equivalent to S_{435}, S_{436} or S_{437}.*

Proof. Write $H = \{1, -1\} \times D(1, 1)$. Suppose that $D(1, -x) = \{\pm 1, \pm x\}$ for any $x \in D(1, 1)$, $x \neq 1$. We choose $c \notin H$ such that $(1, c) \approx y$ for some $y \in H$, $y \neq 1$. Hence because $(1, -y) \approx -c$ we have $y \notin D(1,$

1), thus $y \in -D(1,1)$. Let $y = -a$ and $D(1,1) = \{1, a, b, ab\}$. We denote be $\sigma(S)$ the intersection of all orderings of S. From $a \in D(1,1)$ and $(1,a) \approx -c$ we get $-c \in \sigma(S)$ and $\sigma(S) = D(1,1) \times \{1, -c\}$. We shall show that $D(1,b) = \{1,b\}$. Since $b \in D(1,1)$ we have $D(1,b) \subset \sigma(S)$. *If* $(1,b) \approx -c$ or $-ac$ then, by $(1,a) \approx -c, -ac$, we get $(1,-ab) \approx -c$ or $-ac$, a contradiction (by $ab \in D(1,1)$). Moreover, $(1,-a) \not\approx -b$, hence $(1,b) \not\approx a, ab$. We have $(1,b) \not\approx a, b, ab, -c, -bc, -ac, -abc$ and so $D(1,b) = \{1,b\}$. Now $-c \in D(1,a) \subset D(1,b,b) = \bigcup_{y \in D(1,b)} D(y,b) = D(1,b) \cup D(b,b) = D(1,1)$, a contradiction.

Thus we have proved that $|D(1,-a)| = 8$ for some $a \in D(1,1)$, $a \neq 1$. Write $D(1,1) = \{1, a, b, ab\}$.

First we assume that $D(1,-a) = H = \{1,-1\} \times D(1,1)$. Then $(1,-a) \approx \pm b, \pm ab$, hence $D(1,-a) = D(1,-b) = D(1,-ab) = H$ and so $\{a, b, ab\} \subset D(1,a) \cap D(1,b) \cap D(1,ab)$. Now there exists a $c \notin H$ such that $(1,c) \approx y \in H$, $y \neq 1$, hence $(1,-y) \approx -c$. But $D(1,-y) = H$ for any $y \in D(1,1)$ and $-c \notin H$ hence we may assume that $y = -a \in -D(1,1)$ and $(1,c) \approx -a$. We obtain $b \in D(1,-a) \cap D(1,1) \subset D(1,a)$, hence $D(1,a) = \{1, a, b, -c, ab, -ac, -bc, -abc\} \subset \sigma(S)$, by $a \in D(1,1)$, hence $\sigma(S) = D(1,a)$. From $(1,-ab) \not\approx -c$ and $(1,-b) \not\approx -c$ we get $(1,b) \not\approx -c$ and $(1,ab) \not\approx -c$. Thus $D(1,b) = D(1,ab) = \{1, a, b, ab\}$ according to $D(1,b) \cup D(1,ab) \subset \sigma(S)$. Also $D(1,-cx) \subset \sigma(S)$ for any $x \in D(1,1)$ and $(1,-cx) \not\approx a, b, ab, -acx, -bcx, -abcx$ and we have $D(1,-cx) = \{1, -cx\}$. Similarly $(1,cx) \not\approx a, b, ab$, hence $|D(1,cx)| < 8$ and so $D(1,cx) = \{1, cx, -a, -acx\}$, $x \in D(1,1)$. Thus we conclude that there exists at most one scheme with $q_2 = 4$ such that $D(1,-a) = H$.

Now we assume that $|D(1,-a)| = 8$ and $(1,-a) \approx d \notin H$. We have $D(1,-a) = \{\pm 1, \pm a, \pm d, \pm ad\}$.

Suppose that $|D(1,-b)| = |D(1,-ab)| = 4$. Then

(A) $(1,bd) \not\approx -1, -bd, a, b, ab, abd, d, ad$ and $(1,-abd) \not\approx -1, abd, a, b, ab, -bd, -ab, -d$.

Since $(1,1) \approx ab$ and $(1,ad) \approx a$ we get $a \in D(1,bd,bd)$. Thus there exists an $x \in D(1,bd)$ and $(x,bd) \approx a$, so $(1,-abd) \approx ax$. (A) implies that $x = -d$ or $-ab$. If $(1,bd) \approx -ab$ then $(1,ab) \approx -bd$. Using $b, ab \in \sigma(S)$ we get $-d \in \sigma(S)$ and $-abd \in \sigma(S)$. But $(1,-abd) \approx ax$, hence $x = -ab \in \sigma(S)$, a contradiction. Thus $x = -d$, i.e. $(1,-abd) \approx -ad, b$. This implies $(1,-b) \approx abd$, thus $D(1,-b) = \{\pm 1, \pm b, \pm abd, \pm ad\}$, a contradiction because $|D(1,-b)| = 4$.

Thus we conclude that either $D(1,-b)$ or $D(1,-ab)$ represents 8 elements and we can assume that $|D(1,-b)| = 8$. If $(1,-b) \approx d$ then $(1,-ab) \approx d$ (by $(1,-a) \approx d$) and we have $d \in D(1,-a) \cap D(1,-b) \subset D(1,-ab)$. If $(1,-b) \not\approx d$ then $(1,-b) \not\approx \pm d, \pm bd$. Since $(1,-b) \not\approx a$ and $|D(1,-b)| = 8$ we get $(1,-b) \approx \pm ad$. Now $(1,-a) \approx ad$ implies $(1,-ab) \approx ad$, hence $ad \in D(1, -b) \cap D(1, -a) \subset D(1, -ab)$. Using the group isomorphism $-1 \to -1, a \to a, b \to b, d \to ad$ we get an equivalent scheme. This implies that we can assume that $(1, -b) \approx d$ and we have $(1, -ab) \approx d$, $D(1, -x) = \{\pm 1, \pm x, \pm d, \pm dx\}$ for any $x \in \{a, b, ab\}$, $D(1, \pm d) = \{1, a, b, ab\} \times \{1, \pm d\}$ and $(1, dx) \approx d$ and $(1, -dx) \approx -d$, $x \in D(1, 1)$.

Now we consider two cases: either all the forms $(1, x)$, $x \in D(1, 1)$, $x \neq 1$ represent 2 elements or $|D(1, x)| \geqslant 4$ for some $x \in D(1, 1)$, $x \neq 1$. In the first case we have $(1, \pm ad) \not\approx -1, -b, b$, $(1, \pm bd) \not\approx -1, a, -a$ and $(1, \pm abd) \not\approx -1, a, -a$ hence these forms do not represent 8 elements and so, for $x \in D(1, 1)$, $x \neq 1$, we obtain $D(1, dx) = \{1, dx, d, x\}$ and $D(1, -dx) = \{1, -dx, -d, x\}$.

In the second case we can assume that the form $(1, a)$ represents at least 4 elements. Since $(1, a) \not\approx -1, -a, b, ab, \pm d, \pm ad, -b, -ab$, by $a, b, ab \in \sigma(S)$, we get $D(1, a) = \{1, a, bd, abd\}$ or $D(1, a) = \{1, a, -bd, -abd\}$. It suffices to consider the first case, because using the group isomorfism $-1 \to -1$, $a \to a$, $b \to b$, $d \to -d$ we get an equivalent scheme. If $(1, a) \approx bd$ then $d \in \sigma(S)$. Moreover the forms $(1, -bd)$ and $(1, -abd)$ represent $-a, -d$, hence they represent 8 elements. In particular $ad \in D(1, -bd) \cap D(1, -abd)$ and so $(1, -ad)$ represents bd, abd. Further, $abd \in D(1, -ad) \cap D(1, abd) \subset D(1, b)$, $bd \in D(1, -ad) \cap D(1, bd) \subset D(1, ab)$, the forms $(1, b)$, $(1, ab)$ do not represent d and $D(1, b) \cup D(1, ab) \subset \sigma(S)$, hence $|D(1, b)| = |D(1, ab)| = 4$. Similarly, $D(1, ad) \cup D(1, bd) \cup D(1, abd) \subset \sigma(S)$ and $(1, ad) \not\approx b$, $(1, bd) \not\approx a$ and $(1, abd) \not\approx a$, thus these forms represent 4 elements.

We have proved that there exist at most 3 non-equivalent formally real schemes with $q_2 = 4$, $R = \{1\}$ which are not power schemes. Hence $S \cong S_{435}$, S_{436} or S_{437}.

PROPOSITION 16. *If S is a formally real non-power scheme with $q(S) = 16$, $|R| = 1$ and $q_2 = 2$ then S is equivalent to S_{442}, S_{443} or S_{444}.*

P r o o f. Write $D(1, a) = \{1, a\}$. Then we observe that

(A) $(1, -a) \approx \pm 1, \pm a$, $D(1, a) \cap D(1, -a) = \{1, -1\}$ and, for any $x \in g$, $D(1, x) \cap D(1, -x) \subset \{1, a\}$.

We consider some cases depending on the values of $D(1, -a)$ and $D(1, a)$.

First we assume that $D(1, -a) = \{\pm 1, \pm a\}$ and we shall show that $|D(1, a)| = 8$.

Suppose, that

(B) $D(1, a) = \{1, a\}$. Then, for $x \neq \pm 1, \pm a$, $(1, x) \not\approx -1, \pm a$, hence $|D(1, x)| \leqslant 4$.

Consider a subroup H of g such that $D(1, -a) \subset H$. Let $b \notin H$ and $(1, b) \approx c \in H$, $c \neq 1$. Clearly, $c \notin D(1, -a)$ so $H = \{\pm 1, \pm a, \pm c, \pm ac\}$ and $g = H \times \{1, b\}$. Now $c \in D(1, b)$ and $b \in D(ab, ab)$, hence $c \in (1, ab, ab)$ and so there exists $x \in D(1, ab)$ such that $(x, ab) \approx c$, hence $(1, ab) \approx x$ and $(1, -abc) \approx cx$. But $D(1, b) = \{1, b, c, bc\}$, $D(1, -c) = \{1, -c, -b,$

$bc\}$ and $D(1, -bc) = \{1, -bc, -b, c\}$ (by (B)), thus $(1, -abc) \not\approx -1$, $abc, \pm a, \pm bc, -b, ac, c, -ab$ and $(1, ab) \not\approx -1, -ab, \pm a, \pm b, c, abc, bc, ac$. a contradiction. Thus $|D(1, a)| \geqslant 4$ and suppose that $|D(1, a)| = 4$. Write $H = \{1, -1\} \times D(1, a)$. Then there exists a $b \notin H$ such that $(1, b) \approx c$, $c \neq \pm 1$ and $c \in D(1, a)$ or $-c \in D(1, a)$. We shall show that this is impossible. We have $-b \in D(1, -b) \cap D(1, -c) \subset D(1, -bc)$. Moreover $(-abc, -abc) \approx -bc$, hence $-b \in D(1, -abc, -abc)$ and so there exists $x \in D(1, -abc)$ such that $(x, -abc) \approx -b$, hence $(1, -ac) \approx -bx$. If $c \in D(1, a)$, then $D(1, a) = \{1, a, c, ac\}$, $D(1, b) = \{1, b, c, bc\}$, hence $D(1, -ab) = \{1, -ab, c, -abc\}$ and $D(1, abc) = \{1, abc, ab, c\}$ (by $(1, b)$, $(1, -ab)$, $(1, abc)$ do not represent $-1, \pm a$) and so $D(1, -c) = \{1, -c, -a, ac, -b, bc, ab, -abc\}$. This implies that $(1, -abc) \not\approx -1, abc, \pm a, \pm bc, ab, -c, c, -ab, -b, ac$, hence $D(1, -abc) \subset \{1, -abc, b, -ac\}$. But $(1, -abc) \approx x$ so $-bx \in \{-b, ac, -1, abc\}$. Using $(1, -ac) \approx -bx$ we get $(1, b) \approx ac$ or $(1, 1) \approx ac$, a contradiction. If $-c \in D(1, a)$ then $D(1, a) = \{1, a, -c, -ac\}$, $D(1, b) = \{1, b, c, bc\}$, $D(1, -c) = \{1, -c, -b, bc\}$ and $D(1, -bc) = \{1, -bc, -b, c\}$ (since these forms do not represent $-1, \pm a$). We have $(1, -abc) \not\approx -1, abc, \pm a, \pm bc, -b, ac, c, -ab$ and $(1, -abc) \approx x$ implies that $x \in \{1, -abc, b, -ac, -c, ab\}$. Moreover $(1, -ac) \not\approx -1, ac, -b, abc, bc, -a$ and $(1, -ac) \approx -bx$, a contradiction.

We have proved that $|D(1, -a)| = 4$ implies $|D(1, a)| = 8$, hence $D(1, a) = \sigma(S)$, by $a \in D(1, 1) \subset \sigma(S)$. We shall show that, for any $x \in D(1, a)$, $x \neq 1, a$, $|D(1, x)| = 2$. If not, let $b \in D(1, a)$, $b \neq 1, a$ and $|D(1, b)| \geqslant 4$. From $D(1, b) \subset \sigma(S)$ and $(1, b) \approx a$ we have $(1, b) \approx c$, $c \in \sigma(S)$ and $D(1, a) = \sigma(S) = \{1, a, b, ab, c, ac, bc, abc\}$. This implies $c \in D(1, a) \cap D(1, b) \subset D(1, -ab)$ and $c \in D(1, -bc)$, hence $(1, -ab) \not\approx -c$, $(1, -bc) \not\approx -c$ (since $(1, 1) \not\approx ab, bc$) and so $(1, c) \not\approx ab, bc, abc, b$. From $D(1, -a) = \{\pm 1, \pm a\}$ and $D(1, c) \subset \sigma(S)$ we get $D(1, c) = \{1, c\}$. Now $(1, a) \approx ab$ and $(1, -ab) \approx c$ imply $(1, ac) \approx ab$ and so $ab \in D(1, ac) \subset D(1, c, c) = \bigcup_{x \in D(1, c)} D(x, c) = D(1, c) \cup D(c, c) = \{1, c, ac\}$, a contradiction. Thus we have $D(1, x) = \{1, x\}$ for any $x \in D(1, a)$, $x \neq 1, a$. It follows that $D(1, -x) = \{1, -x, -a, ax\}$ because these forms do not represent 8 elements ($(1, -b) \not\approx b, -c, -bc$ and, if $x \neq b$, $(1, -x) \not\approx -1, \pm b$). We conclude that there exists at most one scheme with $|D(1, -a)| = 4$.

Next we assume that $|D(1, -a)| = 8$ and $D(1, a) = \{1, a\}$. We choose $c \notin D(1, -a)$ such that $(1, c) \approx b$, $b \neq 1$ and $b \in D(1, -a)$. From $(1, c) \not\approx -1, \pm a$ we get $D(1, -a) = \{\pm 1, \pm a, \pm b, \pm ab\}$ and $g = D(1, -a) \times \{1, c\}$. Since the forms $(1, c)$ and $(1, -bc)$ do not represent $-1, \pm a$, we have $D(1, c) = \{1, c, b, bc\}$ and $D(1, -bc) = \{1, -bc, -c, b\}$, hence $D(1, -b) = \{1, -b, a, -ab, -c, bc, -ac, abc\}$. Similarly, the forms $(1, ac)$ and $(1, -abc)$ do not represent $-1, \pm a$, hence $D(1, -abc) = \{1, -abc, b, -ac\}$ and $D(1, ac) = \{1, ac, b, abc\}$. From $(1, -a) \approx \pm b, \pm ab$ we have $(1, b) \approx a$ and $(1, ab) \approx a, b$ and so $(1, b)$, $(1, ab) \not\approx -ab, -c, abc$ implies $D(1, b) =$

$= D(1, ab) = \{1, a, b, ab\}$. Further, $(1, -ab) \approx a$ and $(1, -ab) \not\approx b, -c$. We shall show that $(1, -ab) \not\approx -bc$. If not, then $(1, bc) \approx ab$, hence $(b, c) \approx a$. From $(1, c) \approx b$ we get $a \in D(1, c, c) = D(1, c) \cup D(1, ac)$ and so $(1, -a) \approx \approx -c$ or $-ac$, a contradiction. Thus $(1, -ab) \not\approx b, -c, -bc$, hence $D(1, -ab) = \{1, -ab, a, -b\}$. Now, for $x = bc, abc, -c, -ac$ we have $(1, x) \not\approx \not\approx -1, -x, \pm a, \pm b, \pm ax, \pm bx, \pm abx$, hence $D(1, x) = \{1, x\}$. We conclude that there exists at most one scheme with $|D(1, -a)| = 8$ and $|D(1, a)| = 2$.

Lastly we consider $D(1, -a) = \{\pm 1, \pm a, \pm b, \pm ab\}$ and $|D(1, a)| \geqslant 4$. By (A) we get $(1, a) \not\approx -1, \pm b$ and so $D(1, a) = \{1, a, c, ac\}$, $c \notin D(1, -a)$. From $(1, b) \approx a, ba$ and $(ab, b) \approx bc$ we have $bc \in D(1, b, b) = D(1, b) \cup \cup D(1, ab)$ hence $(1, b) \approx bc$ or $(1, ab) \approx bc$. It is sufficient to consider the first case, because using the group isomorphism $a \to a$, $b \to ab$, $c \to c$, $-1 \to -1$ we get the equivalent scheme. Then we obtain $D(1, b) = \{1, b, a, ab, c, bc, ac, abc\}$, $c \in D(1, b) \cap D(1, a) \subset D(1, -ab)$, thus $D(1, -ab) = = \{1, -ab, a, -b, c, -abc, ac, -bc\}$, $D(1, -c) = \{1, -c, -b, bc, -a, ac, ab, -abc\}$ and $D(1, -ac) = \{1, -ac, -b, abc, -a, c, ab, -bc\}$. Further, $ab \in \in D(1, bc) \cap D(1, abc)$, $-b \in D(1, -bc) \cap D(1, -abc)$ and the forms $(1, \pm bc)$ and $(1, \pm abc)$ do not represent $-1, \pm a$ and so these forms represent 4 elements. Similarly, $a \in D(1, -b) \cap D(1, ab)$ and $(1, -b)$, $(1, ab)$ do not represent $-1, \pm bc$, hence they represent 4 elements. This implies that $|D(1, c)| = |D(1, ac)| = 2$.

We have proved that there exist at most 3 non-equivalent schemes with $q_2 = 2$, $|R| = 1$ which are not power schemes. Hence $S \cong S_{442}$ or S_{443} or S_{444}.

LEMMA 17. *If $S = \langle g, -1, d\rangle$ is a formally real scheme, $q_2(S) = 1$ and $[g : D_S(1, a)] = 2$ for some $a \in g$, $a \neq 1$, then $|D_S(1, -a)| = 2$ and there exists a scheme S' such that $S \cong S' \sqcap S(R)$.*

Proof. Write $g = \{1, -1\} \times D(1, a)$. If $x \in D(1, a) \cap D(1, -a) \subset \subset D(1, 1)$, then $x = 1$. If $x \in -D(1, a) \cap D(1, -a)$ then $-x \in D(1, a) \cap \cap D(1, -a)$ and $x = -1$. Thus we have $D(1, -a) = \{1, -a\}$. Let $b \in D(1, a)$. Then $D(1, b) \subset D(1, 1, a) = \bigcup_{x \in D(1,1)} D(x, a) = D(1, a)$. We shall show that

(A) $D(1, -ab) = D(1, b) \times \{1, -a\}$ for any $b \in D(1, a)$.

From $(1, a) \approx b, ab$ and $D(1, b) \subset D(1, a)$ we have $D(1, b) \subset D(1, b) \cap \cap D(1, a) \subset D(1, -ab)$. Moreover, $(1, -ab) \approx -a$, hence $-a \cdot D(1, b) \subset \subset D(1, -ab)$ and so $\{1, -a\} \times D(1, b) \subset D(1, -ab)$. For the other inclusion suppose that $x \in D(1, -ab)$. If $x \in D(1, a)$ then $x \in D(1, a) \cap D(1, -ab) \subset \subset D(1, b)$. If $-x \in D(1, a)$ then $-abx \in D(1, a) \cap D(1, -ab) \subset D(1, b)$, hence $-ax \in D(1,b)$ so $x \in -a\, D(1, b)$ and (A) is proved.

Now denote $g' = D(1, a)$, $-1' = a$ and $d'(b) = D_S(1, b)$ for any $b \in g'$. If $b, c, d \in D_S(1, a)$, then $D_S(b, c, d) \subset D_S(1, a, 1, a, 1, a) \subset D(1, a)$. Moreover, by (A) we have $(1, b) \approx c \Rightarrow (1, -ab) \approx -ac \Rightarrow (1, ac) \approx ab \Rightarrow (1,$

$-1'c) \approx -1'b$. This implies that $S' = \langle g' - 1'\, d' \rangle$ is a quadratic form scheme. We shall show that $S \cong S' \sqcap S(R)$. Obviously, $g \cong g' \times \{1_R, -1_R\}$. We observe that the mapping $f: g' \times \{1_R, -1_R\} \to g$, $f(b, 1_R) = b$ and $f(b, -1_R) = -ab$, $b \in g'$ is a group isomorphism and $f(-1', -1_R) = -1$. Moreover, for $b \in g'$, we have $f(D_{S' \sqcap S(R)}((1, 1_R), (b, 1_R))) = f(D_{S'}, (1, b) \times D_{S(R)}(1_R, 1_R)) = f(D_{S'}(1, b) \times \{1_R\}) = D_S(1, b)$ and $f(D_{S' \sqcap S(R)}((1, 1_R), (b, -1_R))) = f(D_{S'}(1, b) \times D_{S(R)}(1_R, -1_R)) = f(D_{S'}(1, b) \times \{1_R, -1_R\}) = D_S(1, b) \times \{1, -a\} = D_S(1, -ab)$. This proves that f is an equivalence map and so $S \cong S' \sqcap S(R)$.

PROPOSITION 18. *If S is a non-power scheme with $q(S) = 16$, $q_2(S) = 1$ then $S \cong S_{448}$* or S_{449}.

Proof. We choose $a, b \in g$ such that $(1, a) \approx b$, $a \neq \pm 1$, $b \neq -1$, $-a$. Let $H = \{\pm 1, \pm a, \pm b, \pm ab\}$. There exist $c \notin H$, $x \in H$, $x \neq 1$ such that $(1, x) \approx c$. If $x \in \{a, -b, -ab\}$, then $|D(1, x)| = 8$, by $-a \in D(1, -b) \cap D(1, -ab)$. Let $x = -a$ (similarly for $x = b, ab$). Then $(1, -a) \approx c, -ac$, hence $(1, ac) \approx a, c$. We have $(-c, bc) \approx ac \Rightarrow (1, bc, -c) \approx c \Rightarrow$ there exists $y \in D(1, bc)$ such that $(y, -c) \approx c \Rightarrow y \in D(1, bc) \cap D(c, c) = D(1, bc) \times \{c\} \Rightarrow (1, bc) \approx c \Rightarrow (1, c) \approx -bc, b \Rightarrow \{-c, a, b\}$ is F_2-basis of $D(1, -c) \Rightarrow |D(1, -c)| = 8$.

We have proved that there exists $x \in g$ such that $|D(1, x)| = 8$ and using Lemma 17 we conclude that $S \cong S' \sqcap S(R)$ for some S', hence $S = S_{448}$ or $S = S_{449}$. Thus we get

THEOREM 19. *If S is any formally real quadratic form scheme with $q = 16$ then S is equivalent to one of schemes $S_{428} - S_{451}$ in the table 3. All these schemes are realized by fields.*

Now we shall give the classification of all schemes with $[g : R] = 16$, $|R| \neq 1$. As previously we denote by X_1 and X_2 the sets of all schemes with $q = 16$, $|R| = 1$ ans $s = 1$ and $s = 2$, respectively. Then the set of all schemes with $[g : R] = 16$ and $|R| = \beta \neq 1$ is $X_1' \cup X_1'' \cup X_2'$, where $X_1' = \{S \sqcap S_1^\beta : S \in X_1\}$, $X_1'' = \{S \sqcap S_2^\beta : S \in X_1\}$, $X_2' = \{S \sqcap S_1^\beta : S \in X_2\}$ and S_i^β, $i = 1, 2$, denote the radical schemes of cardinality β. We have

THEOREM 20. *The table 4 contains all the schemes with $[g : R] = 16$ and $|R| = \beta \neq 1$. All these schemes are realized by fields.*

Lastly we describe the Grothendieck and Witt groups for any field k with $[g : R] \leqslant 16$, by giving a decomposition of $G(k)$ and $W(k)$ into a direct sum of cyclic groups. For Q_2, $Q_2(\sqrt{-1})$ and $Q_2(\sqrt{-2})$ this can be found in [8] and we use Theorem 21 and 22 for the remaing fields

THEOREM 21. *If k is any field of characteristic $\neq 2$ and $K = k((t))$ is the power series field, then*

$$W(K) = W(k) \oplus W(k) \text{ and } G(K) = G(k) \oplus W(k).$$

For the proof of Theorem 21 see [10].

THEOREM 22. *If k, K, L are any fields of characteristic $\neq 2$, $s(L) \leqslant s(K)$ and $S(k) \cong S(K) \sqcap S(L)$, then*

$$G(k) = Z \oplus G_0(K) \oplus G_0(L) \text{ and } W(k) = W(K) \oplus G_0(L)$$

where $G_0(K)$ and $G_0(L)$ denote the subgroups of 0-dimensional elements of $G(K)$ and $G(L)$.

Proof. Let $S(K) = \langle g(K), -1_K, d_K \rangle$ and $S(L) = \langle g(L), -1_L, d_L \rangle$ be the schemes of the fields K and L and $S = S(K) \sqcap S(L)$ be represented by the field k (i.e. $S \cong S(k) = \langle g(k), -1_k, d_k \rangle$). Let $f : g(K) \times g(L) \to g(k)$ be an equivalence map and we denote $a^K b^L = f(a, b)$, $a \in g(K)$, $b \in g(L)$. In particular $1_k = f(1_K, 1_L) = 1^K 1^L$ and $-1_k = f(-1_K, -1_L) = (-1^K)(-1^L)$.

We define two subgroups of the group $G_0(k)$ in the following way:

$$G_1 = \{\langle a^K_1 \cdot 1^L, \ldots, a^K_n \cdot 1^L \rangle - \langle b^K_1 \cdot 1^L, \ldots, b^K_n \cdot 1^L \rangle \in G_0(k) : a_i, b_i \in g(K),\ n \in \mathbf{N}\},$$

$$G_2 = \{\langle 1^K \cdot c^L_1, \ldots, 1^K \cdot c^L_n \rangle - \langle 1^K \cdot d^L_1, \ldots, 1^K \cdot d^L_n \rangle \in G_0(k) : c_j, d_j \in g(L),\ n \in \mathbf{N}\}.$$

From Theorem 3.9 [9] we conclude that the mappings $f_1 : G_0(K) \to G_1$,

$$f_1(\langle a_1, \ldots, a_n \rangle - \langle b_1, \ldots, b_n \rangle) = \langle a^K_1 \cdot 1^L, \ldots, a^K_n \cdot 1^L \rangle - \langle b^K_1 \cdot 1^L, \ldots, b^K_n \cdot 1^L \rangle$$

and $f_2 : G_0(L) \to G_2$,

$$f_2(\langle c_1, \ldots, c_n \rangle - \langle d_1, \ldots, d_n \rangle) = \langle 1^K \cdot c^L_1, \ldots, 1^K \cdot c^L_n \rangle - \langle 1^K \cdot d^L_1, \ldots, 1^K \cdot d^L_n \rangle$$

are group isomorphisms. Clearly $G_1 \cap G_2 = 0$. We shall show that $G_1 \oplus G_2 = G_0(k)$. From Theorem 3.9 we have also $1^K 1^L \in f(D_K(a, 1) \times D_L(1_L, b)) = D_k(a^K 1^L, 1^K b^L)$ for any $a \in g(K)$, $b \in g(L)$, hence $(1^K 1^L, a^K 1^L, 1^K b^L) \cong (1^K 1^L, 1^K 1^L, a^K b^L)$ and so $\langle 1^K 1^L \rangle - \langle a^K b^L \rangle = (\langle 1^K 1^L \rangle - \langle a^K 1^L \rangle) + (\langle 1^K 1^L \rangle - \langle 1^K b^L \rangle) \in G_1 \oplus G_2$. Since the group $G_0(k)$ is generated by elements $\langle 1^K 1^L \rangle - \langle a^K b^L \rangle$, $a \in g(K)$, $b \in g(L)$, we have $G_0(k) \subset G_1 \oplus G_2$, so $G_0(k) = G_1 \oplus G_2 = f_1(G_0(K)) \oplus f_2(G_0(L)) \cong G_0(K) \oplus G_0(L)$ and the first part of the theorem is proved.

In the sequel we write $C(X)$ for the cyclic subgroup generated by an element X. From Theorem 1.1 [10] we have that there exist subgroups G_{OK} and G_{OL} of the groups $G_0(K)$ and $G_0(L)$ such that $G_0(K) = C(\langle 1_K \rangle - \langle -1_K \rangle) \oplus G_{OK}$ and $G_0(L) = C(\langle 1_L \rangle - \langle -1_L \rangle) \oplus G_{OL}$ and we get

$$G_0(k) = C(\langle 1^K 1^L \rangle - \langle -1^K 1^L \rangle) \oplus f_1(G_{OK}) \oplus C(\langle 1^K 1^L \rangle - \langle 1^K(-1^L) \rangle) \oplus \oplus f_2(G_{OL}).$$

We shall prove that

$$C(\langle 1^K 1^L \rangle - \langle -1^K 1^L \rangle) \oplus C(\langle 1^K 1^L \rangle - \langle 1^K(-1^L) \rangle) = = C(\langle 1^K 1^L \rangle - \langle (-1^K)(-1^L) \rangle) \oplus C(\langle 1^K 1^L \rangle - \langle 1^K(-1^L) \rangle).$$

Since $\langle 1^K 1^L \rangle - \langle (-1^K)(-1^L) \rangle = \langle 1^K 1^L \rangle - \langle 1^K(-1^L) \rangle + \langle 1^K 1^L \rangle - \langle (-1^K) 1^L \rangle$ it is sufficient to show that $C(\langle 1^K 1^L \rangle - \langle (-1^K)(-1^L) \rangle) \cap C(\langle 1^K 1^L \rangle - \langle 1^K(-1^L) \rangle) = 0$. Suppose that $\varphi - \psi = m \times (\langle 1^K 1^L \rangle - \langle (-1^K)(-1^L) \rangle) =$

$= n \times (\langle 1^K 1^L \rangle - \langle 1^K(-1^L)\rangle)$ for some forms φ and ψ over k. Using also Theorem 3.9 [9] we get $m \times (1_K) \cong m \times (-1_K)$. If $s(K) = \infty$ then $m = 0$ and $\varphi - \psi = 0$. If $s(K) < \infty$ then $s(K)$ divides m. But $s(k) = s(K)$ by Theorem 3.9 [9] hence $s(k)$ divides m and $\varphi - \psi = 0$ too. Thus we have

$$G_0(k) =$$
$$= C(\langle 1^K 1^L \rangle - \langle (-1^K)(-1^L)\rangle) \oplus C(\langle 1^K 1^L \rangle - \langle 1^K(-1^L)\rangle) \oplus f_1(G_{OK}) \oplus f_2(G_{OL}) =$$
$$= C(\langle 1^K 1^L \rangle - \langle (-1^K)(-1^L)\rangle) \oplus f_1(G_{OK}) \oplus f_2(G_0(L)).$$

Using Theorem 4.1 [10] we conclude that

$$W(k) \cong Z \oplus f_1(G_{OK}) \oplus f_2(G_0(L)) \cong Z \oplus G_{OK} \oplus G_0(L) \cong W(K) \oplus G_0(L)$$

if $s(k) = s(K) = \infty$ and

$$W(k) \cong Z/2\, s(k)\, Z \oplus f_1(G_{OK}) \oplus f_2(G_0(L)) \cong Z/2\, s(K)\, Z \oplus G_{OK} \oplus G_0(L) \cong$$
$$\cong W(K) \oplus G_0(L)$$

if $s(k) = s(K) < \infty$.

REMARK. In [1, Th.7.1] Cordes proved that, if $|W| = 32$, then q and the Witt group determine only one non-real field up to equivalence with respect to quadratic forms. For fields with $|W| = 64$ the invariants W and q do not suffice for characterizing the scheme. For example, if $S(K) \cong S_{415}$ and $L = Q_2(\sqrt{-2})$, then $q(K) = q(L)$ and $W(K) = W(L)$ and the fields K and L are not equivalent.

APPENDICES

We enclose four tables containing all non-equivalent schemes with $[g : R] \leqslant 16$. The following notation is used in the tables:

$q = |g|$ — the cardinality of the group g,

$q_2 = |D(1, 1)|$,

$|R|$ — the cardinality of the radical of the scheme S,

m — the number of the equivalence classes of 2-fold Pfister forms,

s — the stufe of the scheme S,

r — the number of orderings of the scheme S,

u — the maximal dimension of anisotropic torsion form over S.

For any schemes S, we denote by W and G the Witt group and the Grothendieck group of a corresponding field. Moreover, in the tables 2 and 4 we use

$q_2' = [D(1, 1) : R]$,

S_i^β, $i = 1, 2$ — the radical schemes of cardinality β and stufe i,

h — an elementary 2-group of cardinality β.

TABLE 1

Schemes with $q \leqslant 8$

Schemes	q	q_2	$\lvert R\rvert$	m	s	r	u	G	W
$S_0 = S(C)$	1	1	1	1	1	0	1	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
$S_{11} = S(F_5)$	2	2	2	1	1	0	2	$\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z})^2$
$S_{12} = S(F_3)$	2	2	2	1	2	0	2	$\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$
$S_{13} = S(R)$	2	1	1	2	∞	1	0	$\mathbf{Z}+\mathbf{Z}$	$\mathbf{Z}$
$S_{21} = S_{11} \sqcap S_{11}$	4	4	4	1	1	0	2	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^2$	$(\mathbf{Z}/2\mathbf{Z})^3$
$S_{22} = S_{12} \sqcap S_{12}$	4	4	4	1	2	0	2	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^2$	$\mathbf{Z}/4\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$
$S_{23} = S_{11}^t$	4	4	1	2	1	0	4	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^3$	$(\mathbf{Z}/2\mathbf{Z})^4$
$S_{24} = S_{12}^t$	4	2	1	2	2	0	4	$\mathbf{Z}+\mathbf{Z}/4\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$	$(\mathbf{Z}/4\mathbf{Z})^2$
$S_{25} = S_{11} \sqcap S_{13}$	4	2	2	2	∞	1	2	$\mathbf{Z}^2+\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$
$S_{26} = S_{13} \sqcap S_{13}$	4	1	1	4	∞	2	0	$\mathbf{Z}^3$	$\mathbf{Z}^2$
$S_{31} = S_{11} \sqcap S_{21}$	8	8	8	1	1	0	2	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^3$	$(\mathbf{Z}/2\mathbf{Z})^4$
$S_{32} = S_{12} \sqcap S_{22}$	8	8	8	1	2	0	2	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^3$	$\mathbf{Z}/4\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^2$
$S_{33} = S_{11} \sqcap S_{23}$	8	8	2	2	1	0	4	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^4$	$(\mathbf{Z}/2\mathbf{Z})^5$
$S_{34} = S_{12} \sqcap S_{23}$	8	8	2	2	2	0	4	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^4$	$\mathbf{Z}/4\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^3$
$S_{35} = S_{21}^t$	8	8	1	4	1	0	4	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^5$	$(\mathbf{Z}/2\mathbf{Z})^6$
$S_{36} = S_{23}^t$	8	8	1	8	1	0	8	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^7$	$(\mathbf{Z}/2\mathbf{Z})^8$
$S_{37} = S_{11} \sqcap S_{24}$	8	4	2	2	2	0	4	$\mathbf{Z}+\mathbf{Z}/4\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^2$	$(\mathbf{Z}/4\mathbf{Z})^2+\mathbf{Z}/2\mathbf{Z}$
$S_{38} = S(Q_2)$	8	4	1	2	4	0	4	$\mathbf{Z}+\mathbf{Z}/4\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^2$	$\mathbf{Z}/8\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^2$
$S_{39} = S_{22}^t$	8	4	1	4	2	0	4	$\mathbf{Z}+\mathbf{Z}/4\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^3$	$(\mathbf{Z}/4\mathbf{Z})^2+(\mathbf{Z}/2\mathbf{Z})^1$
$S_{310} = S_{24}^t$	8	2	1	8	2	0	8	$\mathbf{Z}+(\mathbf{Z}/4\mathbf{Z})^3+\mathbf{Z}/2\mathbf{Z}$	$(\mathbf{Z}/4\mathbf{Z})^4$
$S_{311} = S_{13} \sqcap S_{21}$	8	4	4	2	∞	1	2	$\mathbf{Z}^2+(\mathbf{Z}/2\mathbf{Z})^2$	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^2$
$S_{312} = S_{13} \sqcap S_{23}$	8	4	1	4	∞	1	4	$\mathbf{Z}^2+(\mathbf{Z}/2\mathbf{Z})^3$	$\mathbf{Z}+(\mathbf{Z}/2\mathbf{Z})^3$
$S_{313} = S_{13} \sqcap S_{24}$	8	2	1	4	∞	1	4	$\mathbf{Z}^2+\mathbf{Z}/4\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}+\mathbf{Z}/4\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$
$S_{314} = S_{11} \sqcap S_{26}$	8	2	2	4	∞	2	2	$\mathbf{Z}^3+\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}^2+\mathbf{Z}/2\mathbf{Z}$
$S_{315} = S_{25}^t$	8	2	1	7	∞	2	4	$\mathbf{Z}^3+(\mathbf{Z}/2\mathbf{Z})^2$	$\mathbf{Z}^2+(\mathbf{Z}/2\mathbf{Z})^2$
$S_{316} = S_{13} \sqcap S_{26}$	8	1	1	8	∞	3	0	$\mathbf{Z}^4$	$\mathbf{Z}^3$
$S_{317} = S_{26}^t$	8	1	1	12	∞	4	0	$\mathbf{Z}^5$	$\mathbf{Z}^4$

TABLE 2

Schemes with radical of cardinality $\beta \neq 1$ and $[g:R] \leqslant 8$

Schemes	$[g:R]$	q_2'	m	s	r	u	G	W
$S_0' = S_0 \sqcap S_1^\beta \cong S_1^\beta$	1	1	1	1	0	2	$Z+h$	$Z/2Z+h$
$S_0'' = S_0 \sqcap S_2^\beta \cong S_2^\beta$	1	1	1	2	0	2	$Z+h$	$Z/4Z+h'$ where $[h:h'] = 2$
$S_{13}' = S_{13} \sqcap S_1^\beta$	2	1	2	∞	1	2	Z^2+h	$Z+h$
$S_{23}' = S_{23} \sqcap S_1^\beta$	4	4	2	1	0	1	$Z+(Z/2Z)^3+h$	$(Z/2Z)^4+h$
$S_{23}'' = S_{23} \sqcap S_2^\beta$	4	4	2	2	0	4	$Z+(Z/2Z)^3+h$	$Z/4Z+(Z/2Z)^2+h$
$S_{24}' = S_{24} \sqcap S_1^\beta$	4	2	2	2	0	4	$Z+Z/4Z+Z/2Z+h$	$(Z/4Z)^2+h$
$S_{26}' = S_{26} \sqcap S_1^\beta$	4	1	4	∞	2	2	Z^3+h	Z^2+h
$S_{35}' = S_{35} \sqcap S_1^\beta$	8	8	4	1	0	4	$Z+(Z/2Z)^5+h$	$(Z/2Z)^6+h$
$S_{35}'' = S_{35} \sqcap S_2^\beta$	8	8	4	2	0	4	$Z+(Z/2Z)^5+h$	$Z/4Z+(Z/2Z)^4+h$
$S_{36}' = S_{36} \sqcap S_1^\beta$	8	8	8	1	0	8	$Z+(Z/2Z)^7+h$	$(Z/2Z)^8+h$
$S_{36}'' = S_{36} \sqcap S_2^\beta$	8	8	8	2	0	8	$Z+(Z/2Z)^7+h$	$Z/4Z+(Z/2Z)^6+h$
$S_{38}' = S_{38} \sqcap S_1^\beta$	8	4	2	4	0	4	$Z+Z/4Z+(Z/2Z)^2+h$	$Z/8Z+(Z/2Z)^2+h$
$S_{39}' = S_{39} \sqcap S_1^\beta$	8	4	4	2	0	4	$Z+Z/4Z+(Z/2Z)^3+h$	$(Z/4Z)^2+(Z/2Z)^2+h$
$S_{310}' = S_{310} \sqcap S_1^\beta$	8	2	8	2	0	8	$Z+(Z/4Z)^3+Z/2Z+h$	$(Z/4Z)^4+h$
$S_{312}' = S_{312} \sqcap S_1^\beta$	8	4	4	∞	1	4	$Z^2+(Z/2Z)^3+h$	$Z+(Z/2Z)^3+h$
$S_{313}' = S_{313} \sqcap S_1^\beta$	8	2	4	∞	1	4	$Z^2+Z/4Z+Z/2Z+h$	$Z+Z/4Z+Z/2Z+h$
$S_{315}' = S_{315} \sqcap S_1^\beta$	8	2	7	∞	2	4	$Z^3+(Z/2Z)^2+h$	$Z^2+(Z/2Z)^2+h$
$S_{316}' = S_{316} \sqcap S_1^\beta$	8	1	8	∞	3	2	Z^4+h	Z^3+h
$S_{317}' = S_{317} \sqcap S_1^\beta$	8	1	12	∞	4	2	Z^5+h	Z^4+h

TABLE 3

Schemes with $q = 16$

Schemes	q_2	$\lvert R\rvert$	m	s	r	u	G	W
$S_{41} = S_{11} \sqcap S_{31}$	16	16	1	1	0	2	$Z+(Z/2Z)^4$	$(Z/2Z)^5$
$S_{42} = S_{12} \sqcap S_{32}$	16	16	1	2	0	2	$Z+(Z/2Z)^4$	$Z/4Z+(Z/2Z)^3$
$S_{43} = S_{21} \sqcap S_{23}$	16	4	2	1	0	4	$Z+(Z/2Z)^5$	$(Z/2Z)^6$
$S_{44} = S_{22} \sqcap S_{23}$	16	4	2	2	0	4	$Z+(Z/2Z)^5$	$Z/4Z+(Z/2Z)^4$
$S_{45} = S_{11} \sqcap S_{35}$	16	2	4	1	0	4	$Z+(Z/2Z)^6$	$(Z/2Z)^7$
$S_{46} = S_{12} \sqcap S_{35}$	16	2	4	2	0	4	$Z+(Z/2Z)^6$	$Z/4Z+(Z/2Z)^5$
$S_{47} = S_{11} \sqcap S_{36}$	16	2	8	1	0	8	$Z+(Z/2Z)^8$	$(Z/2Z)^9$
$S_{48} = S_{12} \sqcap S_{36}$	16	2	8	2	0	8	$Z+(Z/2Z)^8$	$Z/4Z+(Z/2Z)^7$
$S_{49} = S(Q_2(\sqrt{-1}))$	16	1	2	1	0	4	$Z+(Z/2Z)^5$	$(Z/2Z)^6$
$S_{410} = S_{23} \sqcap S_{23}$	16	1	4	1	0	4	$Z+(Z/2Z)^6$	$(Z/2Z)^7$
$S_{411} = S^t_{31}$	16	1	8	1	0	4	$Z+(Z/2Z)^7$	$(Z/2Z)^8$
$S_{412} = S^t_{33}$	16	1	15	1	0	8	$Z+(Z/2Z)^9$	$(Z/2Z)^{10}$
$S_{413} = S^t_{35}$	16	1	26	1	0	8	$Z+(Z/2Z)^{11}$	$(Z/2Z)^{12}$
$S_{414} = S^t_{36}$	16	1	36	1	0	16	$Z+(Z/2Z)^{15}$	$(Z/2Z)^{16}$
$S_{415} = S_{21} \sqcap S_{24}$	8	4	2	2	0	4	$Z+Z/4Z+(Z/2Z)^3$	$(Z/4Z)^2+(Z/2Z)^2$
$S_{416} = S_{11} \sqcap S_{38}$	8	2	2	4	0	4	$Z+Z/4Z+(Z/2Z)^3$	$Z/8Z+(Z/2Z)^3$
$S_{417} = S_{11} \sqcap S_{39}$	8	2	4	2	0	4	$Z+Z/4Z+(Z/2Z)^4$	$(Z/4Z)^2+(Z/2Z)^3$
$S_{418} = S(Q_2((\sqrt{-2}))$	8	1	2	2	0	4	$Z+Z/4Z+(Z/2Z)^3$	$(Z/4Z)^2+(Z/2Z)^2$
$S_{419} = S_{2}$[illegible] $\sqcap S_{24}$	8	1	4	2	0	4	$Z+Z/4Z+(Z/2Z)^4$	$(Z/4Z)^2+(Z/2Z)^3$
$S_{420} = S^t_{32}$	8	1	8	2	0	4	$Z+Z/4Z+(Z/2Z)^5$	$(Z/4Z)^2+(Z/2Z)^4$
$S_{421} = S^t_{34}$	8	1	15	2	0	8	$Z+Z/4Z+(Z/2Z)^7$	$(Z/4Z)^2+(Z/2Z)^6$
$S_{422} = S_{11} \sqcap S_{310}$	4	2	8	2	0	8	$Z+(Z/4Z)^3+(Z/2Z)^2$	$(Z/4Z)^4+Z/2Z$
$S_{423} = S_{24} \sqcap S_{24}$	4	1	4	2	0	4	$Z+(Z/4Z)^2+(Z/2Z)^2$	$(Z/4Z)^3+Z/2Z$
$S_{424} = S^t_{37}$	4	1	15	2	0	8	$Z+(Z/4Z)^3+(Z/2Z)^3$	$(Z/4Z)^4+(Z/2Z)^2$
$S_{425} = S^t_{38}$	4	1	16	4	0	8	$Z+Z/8Z+Z/4Z+(Z/2Z)^4$	$(Z/8Z)^2+(Z/2Z)^4$
$S_{426} = S^t_{39}$	4	1	26	2	0	8	$Z+(Z/4Z)^3+(Z/2Z)^5$	$(Z/4Z)^4+(Z/2Z)^4$
$S_{427} = S^t_{310}$	2	1	36	2	0	16	$Z+(Z/4Z)^7+Z/2Z$	$(Z/4Z)^8$
$S_{428} = S_{13} \sqcap S_{31}$	8	8	2	∞	1	2	$Z^2+(Z/2Z)^3$	$Z+(Z/2Z)^3$
$S_{429} = S_{11} \sqcap S_{312}$	8	2	4	∞	1	4	$Z^2+(Z/2Z)^4$	$Z+(Z/2Z)^4$
$S_{430} = S_{13} \sqcap S_{35}$	8	1	8	∞	1	4	$Z^2+(Z/2Z)^5$	$Z+(Z/2Z)^5$
$S_{431} = S_{13} \sqcap S_{36}$	8	1	16	∞	1	8	$Z^2+(Z/2Z)^7$	$Z+(Z/2Z)^7$
$S_{432} = S_{21} \sqcap S_{26}$	4	4	4	∞	2	2	$Z^3+(Z/2Z)^2$	$Z^2+(Z/2Z)^2$
$S_{433} = S_{11} \sqcap S_{313}$	4	2	4	∞	1	4	$Z^2+Z/4Z+(Z/2Z)^2$	$Z+Z/4Z+(Z/2Z)^2$
$S_{434} = S_{11} \sqcap S_{315}$	4	2	7	∞	2	4	$Z^3+(Z/2Z)^3$	$Z^2+(Z/2Z)^3$

TABLE 3 (continued)

Schemes	q_2	$\lvert R\rvert$	m	s	r	u	G	W
$S_{435} = S_{13} \sqcap S_{38}$	4	1	4	∞	1	4	$Z^2+Z/4Z+(Z/2Z)^2$	$Z+Z/4Z+(Z/2Z)^2$
$S_{436} = S_{13} \sqcap S_{39}$	4	1	8	∞	1	4	$Z^2+Z/4Z+(Z/2Z)^3$	$Z+Z/4Z+(Z/2Z)^3$
$S_{437} = S_{13} \sqcap S_{312}$	4	1	8	∞	2	4	$Z^3+(Z/2Z)^3$	$Z^2+(Z/2Z)^3$
$S_{438} = S^t_{311}$	4	1	13	∞	2	4	$Z^3+(Z/2Z)^4$	$Z^2+(Z/2Z)^4$
$S_{439} = S^t_{312}$	4	1	24	∞	2	8	$Z^3+(Z/2Z)^6$	$Z^2+(Z/2Z)^6$
$S_{440} = S_{11} \sqcap S_{316}$	2	2	8	∞	3	2	$Z^4+Z/2Z$	$Z^3+Z/2Z$
$S_{441} = S_{11} \sqcap S_{317}$	2	2	12	∞	4	2	$Z^5+Z/2Z$	$Z^4+Z/2Z$
$S_{442} = S_{13} \sqcap S_{313}$	2	1	8	∞	2	4	$Z^3+Z/4Z+Z/2Z$	$Z^2+Z/4Z+Z/2Z$
$S_{443} = S_{13} \sqcap S_{315}$	2	1	14	∞	3	4	$Z^4+(Z/2Z)^2$	$Z^3+(Z/2Z)^2$
$S_{444} = S_{13} \sqcap S_{310}$	2	1	16	∞	1	8	$Z^2+(Z/4Z)^3+Z/2Z$	$Z+(Z/4Z)^3+Z/2Z$
$S_{445} = S^t_{314}$	2	1	21	∞	4	4	$Z^5+(Z/2Z)^2$	$Z^4+(Z/2Z)^2$
$S_{446} = S^t_{313}$	2	1	24	∞	2	8	$Z^3+(Z/4Z)^2+(Z/2Z)^2$	$Z^2+Z/4Z^2+(Z/2Z)^2$
$S_{447} = S^t_{315}$	2	1	33	∞	4	8	$Z^5+(Z/2Z)^4$	$Z^4+(Z/2Z)^4$
$S_{448} = S_{13} \sqcap S_{316}$	1	1	16	∞	4	0	Z^5	Z^4
$S_{449} = S_{13} \sqcap S_{317}$	1	1	24	∞	5	0	Z^6	Z^5
$S_{450} = S^t_{316}$	1	1	34	∞	6	0	Z^7	Z^6
$S_{451} = S^t_{317}$	1	1	44	∞	8	0	Z^9	Z^8

TABLE 4

Schemes with radical of cardinality $\beta \neq 1$ and $[g:R] = 16$

Schemes	q_2'	m	s	r	u	G	W
$S'_{49} = S_{49} \sqcap S_1^\beta$	16	2	1	0	4	$Z+(Z/2Z)^5+h$	$(Z/2Z)^6+h$
$S''_{49} = S_{49} \sqcap S_2^\beta$	16	2	2	0	4	$Z+(Z/2Z)^5+h$	$Z/4Z+(Z/2Z)^4+h$
$S'_{410} = S_{410} \sqcap S_1^\beta$	16	4	1	0	4	$Z+(Z/2Z)^6+h$	$(Z/2Z)^7+h$
$S''_{410} = S_{410} \sqcap S_2^\beta$	16	4	2	0	4	$Z+(Z/2Z)^6+h$	$Z/4Z+(Z/2Z)^5+h$
$S'_{411} = S_{411} \sqcap S_1^\beta$	16	8	1	0	4	$Z+(Z/2Z)^7+h$	$(Z/2Z)^8+h$
$S''_{411} = S_{411} \sqcap S_2^\beta$	16	8	2	0	4	$Z+(Z/2Z)^7+h$	$Z/4Z+(Z/2Z)^6+h$
$S'_{412} = S_{412} \sqcap S_1^\beta$	16	15	1	0	8	$Z+(Z/2Z)^9+h$	$(Z/2Z)^{10}+h$
$S''_{412} = S_{412} \sqcap S_2^\beta$	16	15	2	0	8	$Z+(Z/2Z)^9+h$	$Z/4Z+(Z/2Z)^8+h$
$S'_{413} = S_{413} \sqcap S_1^\beta$	16	26	1	0	8	$Z+(Z/2Z)^{11}+h$	$(Z/2Z)^{12}+h$
$S''_{413} = S_{413} \sqcap S_2^\beta$	16	26	2	0	8	$Z+(Z/2Z)^{11}+h$	$Z/4Z+(Z/2Z)^{10}+h$
$S'_{414} = S_{414} \sqcap S_1^\beta$	16	36	1	0	16	$Z+(Z/2Z)^{15}+h$	$(Z/2Z)^{16}+h$
$S''_{414} = S_{414} \sqcap S_2^\beta$	16	36	2	0	16	$Z+(Z/2Z)^{15}+h$	$Z/4Z+(Z/2Z)^{14}+h$
$S'_{418} = S_{418} \sqcap S_1^\beta$	8	2	2	0	4	$Z+Z/4Z+(Z/2Z)^4+h$	$(Z/4Z)^2+(Z/2Z)^2+h$
$S'_{419} = S_{419} \sqcap S_1^\beta$	8	4	2	0	4	$Z+Z/4Z+(Z/2Z)^3+h$	$(Z/4Z)^2+(Z/2Z)^3+h$
$S'_{420} = S_{420} \sqcap S_1^\beta$	8	8	2	0	4	$Z+Z/4Z+(Z/2Z)^5+h$	$(Z/4Z)^2+(Z/2Z)^4+h$
$S'_{421} = S_{421} \sqcap S_1^\beta$	8	15	2	0	8	$Z+Z/4Z+(Z/2Z)^7+h$	$(Z/4Z)^2+(Z/2Z)^6+h$
$S'_{423} = S_{423} \sqcap S_1^\beta$	4	4	2	0	4	$Z+(Z/4Z)^2+(Z/2Z)^2+h$	$(Z/4Z)^3+Z/2Z+h$
$S'_{424} = S_{424} \sqcap S_1^\beta$	4	15	2	0	8	$Z+(Z/4Z)^3+(Z/2Z)^3+h$	$(Z/4Z)^4+(Z/2Z)^2+h$
$S'_{425} = S_{425} \sqcap S_1^\beta$	4	16	4	0	8	$Z+Z/8Z+Z/4Z+$ $+(Z/2Z)^4+h$	$(Z/8Z)^2+(Z/2Z)^4+h$
$S'_{426} = S_{426} \sqcap S_1^\beta$	4	26	2	0	8	$Z+(Z/4Z)^3+(Z/2Z)^5+h$	$(Z/4Z)^4+(Z/2Z)^4+h$
$S'_{427} = S_{427} \sqcap S_1^\beta$	2	36	2	0	16	$Z+(Z/4Z)^7+Z/2Z+h$	$(Z/4Z)^8+h$
$S'_{430} = S_{430} \sqcap S_1^\beta$	8	8	∞	1	4	$Z^2+(Z/2Z)^5+h$	$Z+(Z/2Z)^5+h$
$S'_{431} = S_{431} \sqcap S_1^\beta$	8	16	∞	1	8	$Z^2+(Z/2Z)^7+h$	$Z+(Z/2Z)^7+h$
$S'_{435} = S_{435} \sqcap S_1^\beta$	4	4	∞	1	4	$Z^2+Z/4Z+(Z/2Z)^2+h$	$Z+Z/4Z+(Z/2Z)^2+h$
$S'_{436} = S_{436} \sqcap S_1^\beta$	4	8	∞	1	4	$Z^2+Z/4Z+(Z/2Z)^3+h$	$Z+Z/4Z+(Z/2Z)^3+h$
$S'_{437} = S_{437} \sqcap S_1^\beta$	4	8	∞	2	4	$Z^3+(Z/2Z)^3+h$	$Z^2+(Z/2Z)^3+h$
$S'_{438} = S_{438} \sqcap S_1^\beta$	4	13	∞	2	4	$Z^3+(Z/2Z)^4+h$	$Z^2+(Z/2Z)^4+h$
$S'_{439} = S_{439} \sqcap S_1^\beta$	4	24	∞	2	8	$Z^3+(Z/2Z)^6+h$	$Z^2+(Z/2Z)^6+h$
$S'_{442} = S_{442} \sqcap S_1^\beta$	2	8	∞	2	4	$Z^3+Z/4Z+Z/2Z+h$	$Z^2+Z/4Z+Z/2Z+h$
$S'_{443} = S_{443} \sqcap S_1^\beta$	2	14	∞	3	4	$Z^4+(Z/2Z)^2+h$	$Z^3+(Z/2Z)^2+h$
$S'_{444} = S_{444} \sqcap S_1^\beta$	2	16	∞	1	8	$Z^2+(Z/4Z)^3+Z/2Z+h$	$Z+(Z/4Z)^3+Z/2Z+h$
$S'_{445} = S_{445} \sqcap S_1^\beta$	2	21	∞	4	4	$Z^5+(Z/2Z)^2+h$	$Z^4+(Z/2Z)^2+h$
$S'_{446} = S_{446} \sqcap S_1^\beta$	2	24	∞	2	8	$Z^3+(Z/4Z)^2+(Z/2Z)^2+h$	$Z^2+(Z/4Z)^2+(Z/2Z)^2+h$
$S'_{447} = S_{447} \sqcap S_1^\beta$	2	33	∞	4	8	$Z^5+(Z/2Z)^4$	$Z^4+(Z/2Z)^4+h$

TABLE 4 (continued)

Schemes	q_2	m	s	r	u	G	W
$S'_{448} = S_{448} \sqcap S_1$	1	16	∞	4	2	Z^5+h	Z^4+h
$S'_{449} = S_{449} \sqcap S_1^\beta$	1	24	∞	5	2	Z^6+h	Z^5+h
$S'_{450} = S_{450} \sqcap S_1^\beta$	1	34	∞	6	2	Z^7+h	Z^6+h
$S'_{451} = S_{451} \sqcap S_1^\beta$	1	44	∞	8	2	Z^9+h	Z^8+h

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