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ON THE RANDOM INTEGRAL EQUATION WITH ADVANCING ARGUMENT

Abstract. In this paper the existence and uniqueness of stochastic solutions of equation (1) are presented. Our theorem is an application of Banach fixed point theorem to the integral equation with advancing argument.

Introduction. Let $(\Omega, \mathcal{B}, P)$ be a probability space. Denote by $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$. We are mainly concerned with the existence and uniqueness of a random solution (defined below) of the random integral equation with advancing argument

$$x(\omega, t) = -y(\omega, t) + \int_0^{t+\delta(t)} K(\omega, u, x(\omega, u))\, du, \tag{1}$$

where the kernel K is defined on $\Omega \times R^+ \times R$.

A mapping $x : \Omega \times R^+ \to R$ is called a *random solution of equation* (1) if $x(\omega, t)$ is a random variable for every t and if x satisfies equation (1) a.e. (P). The mapping $\bar{x}$ is the unique random solution of equation (1) if for every solution x of equation (1) and for every $t \in R^+$, the condition

$$\bar{x}(\omega, t) = x(\omega, t) \text{ a.e.}$$

holds.

Determined functional differential equations with advancing argument have been investigated by other authors (see [2], [3], [4]). The aim of this paper is to obtain the existence and uniqueness of the random solution of a random integral equation with advancing argument.

The main result of the paper is the following theorem.

THEOREM. *Suppose that the following assumptions are satisfied:*

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1° *For each* $(\omega, u) \in \Omega \times R^+$, $K(\omega, u, \cdot)$ *is a continuous function on* R, *and for each fixed* $v \in R$, $K(\cdot, \cdot, v)$ *is product-measurable on* $\Omega \times R^+$ *and*

$$\int_0^\infty \sup_v K(\omega, u, v)\, du < \infty.$$

2° $\delta(t)$ *is a nonnegative continuous function,* $\delta(0) = 0$.

3° *There exists a function* $a : \Omega \times R^+ \to R^+$ *such that for each* $\omega \in \Omega$ $a(\omega, \cdot)$ *is an integrable function and for each fixed* $u \in R^+$ $a(\cdot, u)$ *is measurable on* Ω; *in addition the inequality*

$$|K(\omega, u, v_1) - K(\omega, u, v_2)| \leqslant a(\omega, u)|v_1 - v_2|, \quad \omega \in \Omega, \ u \in R^+, \ v_1, v_2 \in R,$$

holds.

4° *There exists* $0 < \Lambda < \frac{1}{e}$ *such that,*

$$|b(\omega, t + \delta(t)) - b(\omega, t)| < \Lambda, \quad \omega \in \Omega, \ t \in R^+$$

where

$$b(\omega, t) = \int_0^t a(\omega, u)\, du.$$

5° *For every* $\omega \in \Omega$, $u \in R^+$ *the inequality*

$$|K(\omega, u, 0)| \leqslant k a(\omega, u) \ e^{eb(\omega, u)}$$

holds.

6° *For each* $\omega \in \Omega$ *the function* $y(\omega, \cdot)$ *is continuous and for each* $t \in R^+$ $y(\cdot, t)$ *is a random function; in addition, for fixed* p *the inequality*

$$\sup_{t \geqslant 0} |y(\omega, t)\, e^{-b(\omega, t)}| \leqslant \frac{p}{2}$$

holds, where

$$p > \frac{2ke^{e\Lambda}}{e-2}.$$

Then equation (1) *has the unique random solution.*

P r o o f. Denote by Φ the class of all mappings $x : \Omega \times R^+ \to R$ such that for each $\omega \in \Omega$ $x(\omega, \cdot) \in C(R^+, R)$. For $x \in \Phi$ and for fixed $\omega \in \Omega$, let

$$\|x\|_\omega = \sup_{t \geqslant 0} |x(\omega, t)|\, e^{-eb(\omega, t)}.$$

Let, for fixed $\omega \in \Omega$, Φ_ω, V_ω denote subsets of $C(R^+, R)$ such that $\|x\|_\omega < \infty$ and $\|x\|_\omega < p$, respectively.

With the norm $\|\cdot\|_\omega$, Φ_ω is a Banach space. It is easy to verify that V_ω is a nonempty closed bounded subset of Φ_ω. Let E be the set of all mappings $x \in \Phi$ such that, for each $\omega \in \Omega$, $x(\omega, \cdot) \in V_\omega$ and for each $t \in R^+$ $x(\cdot, t)$ is a random variable. Then $x \in E$ is product measurable on $\Omega \times R^+$. By 1°, the function

$$(\omega, t) \to \int_0^{t+\delta(t)} K(\omega, u, x(\omega, u))\, du$$

is a random variable for each $t \in R^+$ and for every ω it is continuous in t. Define the operator $T : \Phi \to \Phi$

$$(Tx)(\omega, t) = -y(\omega, t) + \int_0^{t+\delta(t)} K(\omega, u, x(\omega, u))\, du.$$

The operator T has the following properties:

(i) $T(E) \subset E$.

(ii) For every $x, z \in E$

$$\|Tx - Tz\|_\omega \leqslant L\, \|x - z\|_\omega,$$

where $0 < L < 1$.

Now, we shall prove (i). Let $x \in E$. Applying condition 3° we obtain

$$|K(\omega, u, x(\omega, u)) - K(\omega, u, 0)| \leqslant a(\omega, u)\, e^{eb(\omega, u)}\, \|x\|_\omega .$$

In virtue of 4° and 5° we have

$$|(Tx)(\omega, t)| < |y(\omega, t)| + \int_0^{t+\delta(t)} |K(\omega, u, x(\omega, u))|\, du \leqslant$$

$$\leqslant |y(\omega, t)| + \int_0^{t+\delta(t)} e\, |a(\omega, u)|\, e^{eb(\omega, u)}\, du \frac{\|x\|_\omega}{e} + \frac{k}{e} \int_0^{t+\delta(t)} e\, |a(\omega, u)|\, e^{eb(\omega, u)}\, du \leqslant$$

$$\leqslant |y(\omega, t)| + \left(\frac{\|x\|_\omega}{e} + \frac{k}{e} \right) e^{eb(\omega, t+\delta(t))}.$$

By conditions 4° and 5° it follows that

$$\|Tx\|_\omega \leqslant \|y\|_\omega + \left(\frac{\|x\|_\omega}{e} + \frac{k}{e} \right) e^{eA} \leqslant \frac{p}{2} + \left(\frac{p}{e} + \frac{k}{e} \right) e^{eA} \leqslant p.$$

Now, we shall prove (ii). Let $x, z \in E$. Applying the Lipschitz conditions 2° and 3° we have:

$$|(Tx)(\omega, t) - (Tz)(\omega, t)| \leqslant \int_0^{t+\delta(t)} |a(\omega, u)|\, e^{eb(\omega, u)}\, du\, \|x - z\|_\omega \leqslant$$

$$\leqslant \frac{\|x - z\|_\omega}{e}\, e^{eb(\omega, t+\delta(t))}.$$

This inequality implies that

$$\|Tx - Tz\|_\omega \leqslant \frac{e^{eA}}{e}\, \|x - z\|_\omega .$$

Now, define

$$L = \frac{e^{eA}}{e} < 1.$$

Then defining recursively the sequence:

$$x_0(\omega, t) = -y(\omega, t),$$
$$\dots\dots\dots\dots\dots\dots,$$
$$x_{n+1}(\omega, t) = -y(\omega, t) + \int_0^{t+\delta(t)} K(\omega, u, x_n(\omega, u))\, du,$$

we see that all mappings x_n are in E and, for each $\omega, x_n(\omega, \cdot)$ is a Cauchy sequence in the Banach space $(\Phi_\omega, \|\cdot\|_\omega)$. Hence, there exists an x in E such that for each $\omega \in \Omega$

$$x(\omega, \cdot) = \lim_{n \to \infty} x_n(\omega, \cdot) \text{ in } (\Phi_\omega, \|\cdot\|_\omega).$$

Finally, we shall show that x is a fixed point of T. Indeed, for n large enough, we have

$$\|x_n - Tx\|_\omega \leqslant \|Tx_{n-1} - Tx\|_\omega \leqslant L\, \|x_{n-1} - x\|_\omega$$

and

$$\|Tx - x\|_\omega \leqslant \|Tx - x_n\|_\omega + \|x_n - x\|_\omega,$$

hence $\|Tx - x\|_\omega = 0$ i.e.

$$\sup_{t \geqslant 0} e^{-eb(\omega, t)} |(Tx)(\omega, t) - x(\omega, t)| = 0 \text{ for every } \omega \in \Omega.$$

Consequently,

$$(Tx)(\omega, t) - x(\omega, t) = 0.$$

Thus x is the unique solution of (1). This completes the proof.

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