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CONTROLLABILITY OF INFINITE-DIMENSIONAL SYSTEMS WITH DELAYS IN CONTROL

Abstract. This paper considers the various types of controllability of linear infinite-dimensional dynamical systems defined in a Banach space, with multiple time-varying delays in control. Necessary and sufficient conditions for approximate controllability, approximate relative controllability and approximate absolute controllability of these systems are obtained. Special cases of systems defined in a Hilbert space are also considered.

1. Introduction and notation. Controllability is one of the most important notions in modern systems theory. Various types of controllability of linear abstract dynamical systems defined in a Banach or Hilbert spaces have been recently extensively explored by several authors (see e.g. [2], [3], [4], [5], [8], [10], [11], [12], [13], [14], [15]). The main purpose of this paper is to examine some fundamental questions concerning approximate controllability, approximate relative controllability and approximate absolute controllability of linear abstract dynamical systems defined in Banach or Hilbert spaces, with multiple time-varying delays in control. From the well known consequence of the Hahn-Banach theorem [13], [14], necessary and sufficient conditions for various types of controllability are derived. In particular for systems defined in Hilbert spaces, stronger conditions are obtained. This paper extends the results given in the papers [3], [4], [13], [14] to the systems with multiple time-varying delays in control.

In the sequel the following notation and terminology, which are adopted from the papers [3], [4], [13], [14], will be used.

Let X and U be two Banach spaces. The Banach space of all bounded linear operators from U to X will be denoted by $L(U, X)$, in particular, $L(X)$ will stand for $L(X, X)$. If $P \in L(U, X)$, then $D(P)$, $R(P)$, $N(P)$

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will be used respectively for the domain, range and kernel of the operator P . If X is a Banach space, then X^* will be its dual and x^* an element of X^* . The adjoint of the operator $P \in L(U, X)$ is represented by $P^* \in L(X^*, U^*)$. For Hilbert space the usual identification $X^* = X$ will be assumed. The identity operator in $L(X)$ is written I , while $\|\cdot\|_X$ denoted the norm and O_X the origin in the space X . The symbol $\langle \cdot, \cdot \rangle_X$ will stand for inner product in the Hilbert space X . The symbol $L_1([t_0, t_1], U)$ denotes the Banach space of Bochner integrable functions $u(t)$, from $[t_0, t_1]$ into U , with the usual norm. The closure of the arbitrary set $E \subset X$ will be denoted by \bar{E} , or exceptionally by $\text{Cl}(E)$. If $\{E_n\}$, $n = 0, 1, 2, \dots$ are subspaces of the space X , then $\overline{\text{sp}} \{E_n, n \geq 0\}$, or $\text{Cl}(\text{sp} \{E_n, n = 0, 1, 2, \dots\})$ will indicate the closure of their span [13], [14]. Let $X_{[t_0, t_1]}$ denote the characteristic function of a set $[t_0, t_1] \subset \mathbb{R}$. The symbols $[b_{ij}]$, or $[b_j]$ will stand for matrix or vector respectively with elements b_{ij} , or b_j . Moreover the symbol T will denote the matrix or vector transpose. Now, following [14], several hypothesis and properties concerning the operator A , which will be used extensively in the next sections, will be listed. First of all, let $A : X \supset D(A) \rightarrow R(A) \subset X$, be linear, closed, unbounded operator with dense domain i.e. $D(A) = X$. The operator A is assumed throughout to satisfy the following hypothesis.

H 1. A is the infinitesimal generator of a strongly continuous semigroup or group (of class C_0), of bounded linear operators $S(t) : X \rightarrow X$, for $t \geq 0$, or $t \in \mathbb{R}$. For operator A let us define $D_\infty(A) = \bigcap_{n=1}^{\infty} D(A^n)$. Let $\eta_A(A)$ denote the totality of analytic vectors for semigroup group $S(t)$ generated by A . The next hypothesis for A are

H 2. $R(S(t)) \subset D(A) \subset X$ for each $t > 0$.

H 3. A generates an analytic semigroup $S(t)$, $t > 0$, i.e. $\eta_A(A) = X$. The relationships between hypothesis H1, H2, H3 and the differentiability of the semigroup $S(t)$ are explained in the paper [14], and here will be omitted.

For the special case, when X is a Hilbert space, we additionally list here, for convenience, another hypothesis for operator A , to which we shall refer in the sequel.

H 4. A is normal with compact resolvent $R(\lambda, A)$, $\lambda \in \rho(A)$, resolvent set.

H 5. A is selfadjoint and satisfies the hypothesis H 1. $\sigma(A)$, $P\sigma(A)$, $C\sigma(A)$, $R\sigma(A)$, will denote respectively spectrum, point spectrum, continuous spectrum and residual spectrum of the operator A . It is well known [14] that for the operator A satisfying the assumption H 4, $\sigma(A) = P\sigma(A)$ and consists entirely of distinct isolated eigenvalues of A denoted by $\{\lambda_k\}$, $k = 1, 2, 3, \dots$ each with finite multiplicity l_k , $k = 1, 2, 3, \dots$

equal to the dimensionality of the corresponding eigenmanifold. Moreover there exists a correspondent complete orthonormal set $\{x_{kq}\}$, $k = 1, 2, 3, \dots$, $q = 1, 2, \dots, l_k$ of eigenvectors of the operator A .

Finally, in order to clear the exposition, we repeat the well known consequence of the Hahn-Banach Theorem [13], [14].

PROPOSITION 1.1. *Let E be an arbitrary linear subspace of a normed linear space X . Then $E = X$, if and only if, the zero functional is the only bounded linear functional, that vanishes on a subspace E .*

2. System description and definitions. Let us consider an abstract linear dynamical system described by the following differential equation with multiple time-varying delays in control.

$$(2.1) \quad \dot{x}(t) = A x(t) + \sum_{i=0}^{i=M} B_i u(v_i(t)), \quad 0 \leq t_0 \leq t \leq t_1$$

where $x(t) \in X$, Banach space, $\dot{x}(t)$ is a time derivative with respect to the norm in the space X , A satisfies the assumption H 1, $B_i \in L(U, X)$, for $i = 0, 1, \dots, M$. The functions $v_i : [t_0, t_1] \rightarrow R$, $i = 0, 1, \dots, M$ are absolutely continuous strictly increasing and moreover fulfill the following inequalities

$$v_M(t) < v_{M-1}(t) < \dots < v_i(t) < \dots < v_1(t) < v_0(t) = t$$

for $t \in [t_0, t_1]$.

Hence, we may introduce the so called *time lead functions* $r_i : [v_i(t_0), v_i(t_1)] \rightarrow [t_0, t_1]$, such that $r_i(v_i(t)) = t$, for $t \in [t_0, t_1]$. The admissible controls for system (2.1), $u \in L_1([v_M(t_0), t_1], U)$, U is Banach space. It is well known, (see e.g. [13], [14], [15]), that for $x(t_0) \in X$, and $u \in L_1([v_M(t_0), t_1], U)$, there exists a unique so called *mild solution* of the equation (2.1), given by the following integral formula

$$(2.2) \quad x(t, x(t_0), u) = S(t-t_0) x(t_0) + \int_{t_0}^t S(t-s) \sum_{i=0}^{i=M} B_i(u(v_i(s))) ds$$

where the integral being understood in the sense of Bochner is well defined for $u \in L_1([v_M(t_0), t_1], U)$. Now we also list here the special cases of the system (2.1), to which we shall refer in the sequel. For $u \in L_1([v_M(t_0), t_1], R^p)$, and moreover

$$(2.3) \quad B_i = [b_{i1}, b_{i2}, \dots, b_{ij}, \dots, b_{ip}], \quad i = 0, 1, \dots, M, \text{ and } b_{ij} \in X$$

the system (2.1) can be expressed in the following more convenient form

$$(2.4) \quad \dot{x}(t) = A x(t) + \sum_{i=0}^{i=M} \sum_{j=1}^{j=p} b_{ij} u_j(v_i(t)), \quad 0 \leq t_0 \leq t \leq t_1.$$

For $v_i(t) = t - h_i$, $i = 0, 1, \dots, M$, $0 = h_0 < h_1 < \dots < h_i < \dots < h_M$, the systems (2.1) and (2.4) become the autonomous systems respectively of the following form

$$(2.5) \quad \dot{x}(t) = A x(t) + \sum_{i=0}^{i=M} B_i u(t-h_i), \quad t \in [0, t_1]$$

$$(2.6) \quad \dot{x}(t) = A x(t) + \sum_{i=0}^{i=M} \sum_{j=1}^{j=p} b_{ij} u_j(t-h_i), \quad t \in [0, t_1].$$

For brevity of notation, the dynamical systems (2.1), (2.4), (2.5) and (2.6), in the sequel will be referred respectively as SV_∞^M , SV_p^M , SH_∞^M , and SH_p^M , (the upper index denotes the number of delays, the lower index denotes the dimension of the space U , the letter V denotes time-varying delays, and finally the letter H denotes time-invariant delays). For example the systems without delays are denoted by SV_∞^0 and SV_p^0 , or simply S_∞ and S_p . For systems with delays in control it is desirable to introduce the notation of the so called *complete state at time t* , denoted by z_t and defined as a pair $z_t = \{x(t), u_t\}$, where the function u_t is defined for $s \in [v_M(t), 0)$ by the following formula $u_t(s) = u(t+s)$. For systems defined in a Banach spaces and with delays in control several definitions of various types of controllability may be introduced. Now, we shall introduce the precise definitions of three types of controllability and the remaining will be only mentioned. (Definitions will be given only for system (2.1), modifications are obvious).

DEFINITION 2.1. System (2.1) is said to be *approximate controllable* on $[t_0, t_1]$, if and only if for every $x(t_0) \in X$, every $x_1 \in X$ and every number $\varepsilon > 0$, there exists an admissible control $u \in L_1([v_M(t_0), t_1], U)$, such that the corresponding trajectory $x(t, x(t_0), u)$ of the system satisfies the condition

$$(2.7) \quad \|x(t_1, x(t_0), u) - x_1\|_X \leq \varepsilon.$$

DEFINITION 2.2. System (2.1) is said to be *approximate relative controllable* on $[t_0, t_1]$, if and only if for every initial complete state $z_{t_0} = \{x(t_0), u_{t_0}\}$, every $x_1 \in X$ and every real number $\varepsilon > 0$, there exists an admissible control $u \in L_1([t_0, t_1], U)$, such that the corresponding trajectory $x(t_1, z_{t_0}, u)$ of the system satisfies the condition

$$(2.8) \quad \|x(t_1, z_{t_0}, u) - x_1\|_X \leq \varepsilon.$$

DEFINITION 2.3. System (2.1) is said to be *approximate absolute controllable* on $[t_0, t_1]$, $t_0 < v_M(t_1)$ if and only if for every initial complete state $z_{t_0} = \{x(t_0), u_{t_0}\}$, every final complete state $z_{t_1} = \{x_1, u_{t_1}\}$ and every real number $\varepsilon > 0$, there exists an admissible control $u \in L_1([t_0,$

$v_M(t_1), U$), such that the corresponding trajectory $x(t, z_{t_0}, u)$ of the system satisfies the condition

$$(2.9) \quad \|x(t_1, z_{t_0}, u) - x_1\|_X \leq \varepsilon.$$

If in the above definitions we put $\varepsilon = 0$, we obtain exact controllability, exact relative controllability and exact absolute controllability respectively.

3. Approximate controllability. For given $x(t_0) \in X$, by some easy manipulation, the equality (2.2) can be expressed as

$$(3.1) \quad x(t, x(t_0), u) = x(t, x(t_0), 0) + x(t, 0, u)$$

where

$$(3.2) \quad \begin{aligned} x(t, 0, u) &= \sum_{i=0}^{t=M} \int_{v_i(t_0)}^{v_i(t)} S(t-r_i(s)) B_i \dot{r}_i(s) u(s) ds = \\ &= \int_{v_M(t_0)}^t \sum_{i=0}^{i=M} X(s) \dot{r}_i(s) S(t-r_i(s)) B_i u(s) ds \end{aligned}$$

$$(3.3) \quad x(t, x(t_0), 0) = S(t-t_0) x(t_0).$$

For brevity of notation let us introduce the function $G: [v_M(t_0), t] \rightarrow L(U, X)$ defined as follows

$$(3.4) \quad G(s) = \sum_{i=0}^{i=M} X(s) \dot{r}_i(s) S(t-r_i(s)) B_i.$$

Since the translation of a dense subspace of X is still dense in X , then if we are interested in approximate controllability, without loss of generality we can take $x(t_0) = O_X$. Hence the set attainable at time t_1 , from $x(t_0) = O_X$, denoted by $K_{[v_M(t_0), t_1]}$ is defined as follows

$$(3.5) \quad \begin{aligned} K_{[v_M(t_0), t_1]} &= \{x(t_1, 0, u) \in X : u \in L_1([v_M(t_0), t_1], U)\} = \\ &= \left\{ \int_{v_M(t_0)}^{t_1} G(s) u(s) ds \in X : u \in L_1([v_M(t_0), t_1], U) \right\}. \end{aligned}$$

THEOREM 3.1. *The following statements are equivalent:*

- (i) *the system SV_{∞}^M is approximately controllable on $[t_0, t_1]$;*
- (ii) $\bar{K}_{[v_M(t_0), t_1]} = X$;
- (iii) $x^*(G(t)) = 0$ on $[v_M(t_0), t_1]$ for all $x^* \in X$ implies $x^* = O_{X^*}$.

Proof. The equivalence (i) \Leftrightarrow (ii) follows immediately from (3.5) and Definition 2.1. The equivalence (ii) \Leftrightarrow (iii) is obtained by using Proposition 1.1. to the attainable set $K_{[v_M(t_0), t_1]}$ and taking into consideration the formula (3.5).

THEOREM 3.2. *Let $B_i \in L(U, \eta_a(A))$, for $i = 0, 1, \dots, M$ and the functions $v_i(t)$ are analytic in $[t_0, t_1]$, for $i = 0, 1, \dots, M$. Then the system*

SV_{∞}^M is approximately controllable on $[t_0, t_1]$ if and only if the linear systems without delays in control

$$(3.6) \quad \dot{x}(t) = A x(t) + B w(t)$$

where $B = [B_0, B_1, \dots, B_i, \dots, B_M]$, $w \in L_1([v_M(t_0), t_1], W)$, $W = \bigtimes_{i=0}^{i=M} U_i = U^{(M+1)}$ is approximately controllable on $[t_0, t_1]$.

Proof. Since the function $v_i(t)$ are strictly increasing and analytic in $[t_0, t_1]$, then $\dot{r}_i(s) > 0$ and are analytic in $[v_i(t_0), v_i(t_1)]$, for $i = 0, 1, 2, \dots, M$. Hence since $B_i \in L(U, \eta_a(A))$, the function $G(s)$ is piecewise analytic in $[v_M(t_0), t_1]$, and

$$x^*(G(t)) = x^* \left(\sum_{i=0}^{i=M} x(t) \begin{matrix} \dot{r}_i(t) S(t_1 - r_i(t)) B_i \\ [v_i(t_0), v_i(t_1)] \end{matrix} \right) = 0$$

on $[v_M(t_0), t_1]$ for all $x^* \in X^*$ implies $x^* = O_{X^*}$, is equivalent to the following statement

$$(3.7) \quad x^*(S(t_1 - r_i(t)) B_i) = 0 \text{ on } [v_i(t_0), v_i(t_1)] \text{ for all } x^* \in X^*, i = 0, 1, \dots, M \text{ implies } x^* = O_{X^*}.$$

The statement (3.7) by analyticity of the functions $S(t_1 - r_i(t)) B_i$ is equivalent to

$$(3.8) \quad x^*(S(t_1 - t) B_i) = 0 \text{ on } [t_0, t_1] \text{ for all } x^* \in X^*, i = 0, 1, \dots, M \text{ implies } x^* = O_{X^*}.$$

But the statement (3.8) is the necessary and sufficient condition for approximate controllability on $[t_0, t_1]$ for the system (3.6), [3], [4], [13], [14]. Hence the theorem follows.

Using the results of the paper [14] the following corollaries can be stated easily.

COROLLARY 3.1. Let the assumptions of Theorem 3.2. be satisfied. A sufficient condition for SV_{∞}^M to be approximately controllable on $[t_0, t_1]$ is given by the formula

$$(3.9) \quad \overline{\text{sp}} \{A^n B W_{\infty}, n \geq 0\} = X$$

where $W_{\infty} = \{w \in W : Bw \in D_{\infty}(A)\}$, [14] or more generally, by

$$(3.10) \quad \overline{\text{sp}} \{A^n S(t) B W_{\infty}, n \geq 0\} = X, t \in [t_0, t_1].$$

If A satisfies also hypothesis H2, then (3.10) can be relaxed as to replace BW_{∞} by BW , with arbitrary t in $[t_0, t_1]$. Conversely, assume that BW_{∞} is dense in BW . Then a necessary condition for SV_{∞}^M to be approximately controllable on $[t_0, t_1]$ is given by

$$(3.11) \quad \overline{\text{sp}} \{A^n S(t) B W_{\infty}, n \geq 0\} = X, t > 0.$$

If in addition the hypothesis H 3 is also satisfied, then (3.11) can be relaxed as to replace BW_∞ by BW . Also, if $S(t)$ is a group, (3.11) simplifies in this case to

$$(3.12) \quad \overline{\text{sp}} \{A^n BW, n \geq 0\} = X.$$

Proof. Since $B_i \in L(U, \eta_a(A))$, $i = 0, 1, \dots, M$, then $U_a = \{u \in U : B_i u \in \eta_a(A)\} = U$, and the corollary follows immediately from the results of the paper [14, Th. 2.1].

REMARK 3.1. From Theorem 3.2 and Corollary 3.1 it follows, that the type of delays do not affect on approximate controllability of the systems with delays. Hence the statements of Theorem 3.2 and Corollary 3.1 remain valid, if we replace SV_∞^M by SH_∞^M .

COROLLARY 3.2. Let the assumptions of Theorem 3.2 be satisfied and X be separable space. A sufficient condition for SV_p^N or SH_p^M to be approximately controllable on $[t_0, t_1]$ is given by

$$(3.13) \quad \overline{\text{sp}} \{A^n b_{ij}, n \geq 0, i = 0, 1, \dots, M, j = 1, 2, \dots, p\} = X, b_{ij} \in D_\infty(A)$$

or more generally, by

$$(3.14) \quad \overline{\text{sp}} \{A^n S(t) b_{ij}, n \geq 0\} = X, b_{ij} \in D_\infty(A), t \in [t_0, t_1].$$

If A satisfies also hypothesis H 2, then in (3.14) $b_{ij} \in X$, with t in $[t_0, t_1]$. Conversely, a necessary condition for SV_p^M or SH_p^M to be approximately controllable on $[t_0, t_1]$ is given by the following formula

$$(3.15) \quad \overline{\text{sp}} \{A^n S(t) b_{ij}, n \geq 0\} = X, b_{ij} \in D_\infty(A), t > 0.$$

If in addition hypothesis H 3 is also satisfied, then (3.15) can be relaxed and $b_{ij} \in X$. Also if $S(t)$ is a group, (3.15) simplifies in this case to the following formula

$$(3.16) \quad \overline{\text{sp}} \{A^n b_{ij}, n \geq 0\} = X.$$

Proof. If the assumptions of the theorem are satisfied, this means for systems SV_p^M and SH_p^M , that $b_{ij} \in \eta_a(A)$ for $i = 0, 1, 2, \dots, M$ and $j = 1, 2, \dots, p$. Hence by [14, Corollary 2.2], our corollary follows.

REMARK 3.2. For brevity of notation the indexes n, i, j , in the above formulas run respectively as follows $n = 0, 1, 2, \dots, i = 0, 1, \dots, M$ and $j = 1, 2, \dots, p$.

REMARK 3.3. The subspace $\text{sp} \{A^n b_{ij}\}$ may be dense in the space X only if the space X is separable and this statement justifies why the space X in the Corollary 3.2 is separable.

REMARK 3.4. It can be observed, that in fact, without loss of generality, all the statements in the above theorems and corollaries remain true if we replace the time interval $[t_0, t_1]$ by $[0, t_1 - t_0]$. It follows immediately from the proof of Theorem 3.1 (see (3.8)).

COROLLARY 3.3. *If $X = R^n$ and $U = R^p$, then if the assumptions about analyticity of the functions $v_i(t)$ are satisfied, the systems SV_p^M and SH_p^M are controllable on $[0, t_1 - t_0]$ if and only if*

$$(3.17) \text{rank } [B_0, B_1, \dots, B_M, AB_0, \dots, AB_M, \dots, A^{n-1}B_0, A^{n-1}B_1, \dots, A^{n-1}B_M] = n.$$

Proof. In this case A and B_i , $i = 0, 1, \dots, M$ are matrices with appropriate dimensions and our corollary follows immediately from Corollary 3.2 and the results given in [13] and [14].

REMARK 3.5. Corollary 3.3 coincides with the results of the paper [1, Th. 3.1 and 3.2] where also the time optimal problems for finite dimensional dynamical systems with constant delays in control have been extensively considered.

4. Approximate relative controllability. For given initial complete state $z_{t_0} = \{x(t_0, u_{t_0})\}$, by some easy manipulation the equality (2.2) can be expressed in the following more convenient form

$$(4.1) \quad x(t, z_{t_0}, u) = x(t, z_{t_0}, 0) + x(t, 0, u)$$

where

$$(4.2) \quad \begin{aligned} x(t, z_{t_0}, 0) = & S(t-t_0) x(t_0) + \sum_{i=0}^{i=m(t)} \int_{v_i(t_0)}^{t_0} S(t-r_i(s)) B_i \dot{r}_i(s) u_{t_0}(s) ds + \\ & + \sum_{i=m(t)+1}^{i=M} \int_{v_i(t_0)}^{v_i(t)} S(t-r_i(s)) B_i \dot{r}_i(s) u_{t_0}(s) ds \end{aligned}$$

$$(4.3) \quad x(t, 0, u) = \sum_{i=0}^{i=m(t)} \int_{t_0}^{v_i(t)} S(t-r_i(s)) B_i \dot{r}_i(s) u(s) ds$$

$$(4.4) \quad m(t) = \begin{cases} m, & \text{for } t \in (r_m(t_0), r_{m+1}(t_0)], m = 0, 1, \dots, M-1, \\ M, & \text{for } t > r_M(t_0). \end{cases}$$

Similarly as in the section 3, without loss of generality, we define the attainable set at time t_1 from the zero initial complete state at time t_0 , $z_{t_0} = \{0, 0\}$, as follows

$$(4.5) \quad \begin{aligned} K_{[t_0, t_1]} &= \{x(t_1, 0, u) \in X : u \in L_1([t_0, t_1], U)\} = \\ &= \left\{ \int_{t_0}^{t_1} \sum_{i=0}^{i=m(t_i)} X(t) \dot{r}_i(t) S(t_1 - r_i(t)) B_i u(t) dt \in X : u \in L_1([t_0, t_1], U) \right\} = \\ &= \left\{ \int_{t_0}^{t_1} G(t) u(t) dt \in X : u \in L_1([t_0, t_1], U) \right\}. \end{aligned}$$

THEOREM 4.1. *The following statements are equivalent:*

- (i) *the system SV_∞^M is approximately relatively controllable on $[t_0, t_1]$,*
- (ii) $K_{[t_0, t_1]} = X$,
- (iii) $x^*(G(t)) = 0$ on $[t_0, t_1]$ for all $x^* \in X^*$, implies $x^* = O_{X^*}$.

Proof. The equivalence (i) \Leftrightarrow (ii) follows immediately from (4.5) and the Definition 2.2. The equivalence (ii) \Leftrightarrow (iii) is obtained by using Proposition 1.1 to the attainable set $K_{[t_0, t_1]}$ and taking into consideration the formula (4.5). Hence the theorem follows.

THEOREM 4.2. *Let $B_i \in L(U, \eta_\alpha(A))$ for $i = 0, 1, \dots, m(t_1)$ and the functions $v_i(t)$ are analytic in $[r_i(t_0), t_1]$, for $i = 0, 1, \dots, m(t_1)$. Then the system SV_∞^M is approximately relatively controllable on $[t_0, t_1]$ if and only if the system without delays in control*

$$(4.6) \quad \dot{x}(t) = Ax(t) + Bw(t),$$

where $B = [B_0, B_1, \dots, B_i, \dots, B_{m(t_1)}]$, $w \in L_1([t_0, t_1], W)$, $W = U^{(m(t_1)+1)}$ is approximately controllable on $[r_{m(t_1)}(t_0), t_1]$.

Proof. Since the functions $v_i(t)$ are strictly increasing and analytic in $[r_i(t_0), t_1]$ then $\dot{r}_i(s) > 0$ and are analytic in $[t_0, v_i(t_1)]$ for $i = 0, 1, \dots, m(t_1)$. Hence, since $B_i \in L(U, \eta_\alpha(A))$, the function $G(s)$ is piecewise analytic in $[t_0, t_1]$ and by (4.5) we have

$$x^*(G(t)) = x^* \left(\sum_{i=0}^{i=m(t_1)} \underset{[t_0, v_i(t_1)]}{X(t)} \quad \dot{r}_i(t) S(t_1 - r_i(t)) B_i \right) = 0 \text{ on } [t_0, t_1] \text{ for all}$$

$x^* \in X^*$, implies $x^* = O_{X^*}$, is equivalent to the following statement:

$x^*(S(t_1 - r_i(t)) B_i) = 0$ on $[t_0, v_i(t_1)]$ for all $x^* \in X^*$, $i = 0, 1, \dots, m(t_1)$, implies $x^* = O_{X^*}$. By analyticity of the functions $S(t_1 - r_i(t)) B_i$ for $i = 0, 1, \dots, m(t_1)$, the above statement is equivalent to the following implication:

$$(4.7) \quad x^*(S(t_1 - t) B_i) = 0 \text{ on } [r_i(t_0), t_1] \text{ for all } x^* \in X^*, i = 0, 1, \dots, m(t_1), \\ \text{implies } x^* = O_{X^*}.$$

But the statement (3.7) is the necessary and sufficient condition for approximate controllability on $[r_{m(t_1)}(t_0), t_1]$ for the system (4.6), [3], [4], [13], [14].

REMARK 4.1. For brevity of notation, in the following corollaries, the indexes n, i, j , run respectively as follows $n = 0, 1, \dots, i = 0, 1, \dots, m(t_1), j = 1, 2, \dots, p$.

COROLLARY 4.1. *Let the assumptions of Theorem 4.2 be satisfied. A sufficient condition for SV_∞^M to be approximately relatively controllable on $[t_0, t_1]$ is given by*

$$(4.8) \quad \overline{\text{sp}} \{A^n B W_\infty\} = X$$

or more generally, by

$$(4.9) \quad \overline{\text{sp}} \{A^n S(t) B W_\infty\} = X, t \in [r_{m(t_1)}(t_0), t_1].$$

If A satisfies also hypothesis H2, then (4.9) can be relaxed as to replace BW_∞ by BW , with arbitrary t in $[r_{m(t_1)}(t_0), t_1]$. Conversely, assume that BW_∞ is dense in BW . Then a necessary condition for SV_∞^M

to be approximately relatively controllable on $[t_0, t_1]$ is given by the following formula

$$(4.10) \quad \overline{\text{sp}} \{A^n S(t) B W_\infty\} = X, \quad t > 0.$$

If in addition the hypothesis H 3 is also satisfied, then in (4.10) BW_∞ can be replaced by BW . Also if $S(t)$ is a group, then (4.10) simplifies in this case to the formula

$$(4.11) \quad \overline{\text{sp}} \{A^n B W\} = X.$$

COROLLARY 4.2. Let X be separable space and the assumption of Theorem 4.2 be satisfied. A sufficient condition for SV_p^M or SH_p^M to be approximately relatively controllable on $[t_0, t_1]$ is given by the following formula

$$(4.12) \quad \overline{\text{sp}} \{A^n b_{ij}, n = 0, 1, \dots, i = 0, 1, \dots, m(t_1), j = 1, 2, \dots, p\} = X, \quad b_{ij} \in D_\infty(A)$$

or more generally, by

$$(4.13) \quad \overline{\text{sp}} \{A^n S(t) b_{ij}\} = X, \quad b_{ij} \in D_\infty(A), \quad t \in [r_{m(t_1)}(t_0), t_1].$$

If A satisfies also hypothesis H 2 then in (4.13) $b_{ij} \in X$ with $t \in [r_{m(t_1)}(t_0), t_1]$. Conversely, a necessary condition for SV_p^M or SH_p^M to be approximately relatively controllable on $[t_0, t_1]$ is given by the following formula

$$(4.14) \quad \overline{\text{sp}} \{A^n S(t) b_{ij}\} = X, \quad b_{ij} \in D_\infty(A), \quad t > 0.$$

If in addition the hypothesis H 3 is satisfied, then in (4.14) we can put $b_{ij} \in X$. If $S(t)$ is a group, then a necessary condition is of the following form

$$(4.15) \quad \overline{\text{sp}} \{A^n b_{ij}\} = X.$$

REMARK 4.2. Proofs of the Corollaries 4.1 and 4.2 are the same as the appropriate proofs given in the paper [14] only with modification concerning the indexes, i and j .

REMARK 4.3. The length of the time interval $[t_0, t_1]$ is important for approximate relative controllability because the number $m(t_1)$ depends on t_1 .

COROLLARY 4.3. If $X = R^n$ and $U = R^p$, then if the assumptions about analyticity of the functions $v_i(t)$ are satisfied, then the systems SV_p^M and SA_p^M are controllable relatively on $[t_0, t_1]$ if and only if the following equality holds

$$(4.16) \quad \text{rank} [B_0, B_1, \dots, B_{m(t_1)}, AB_0, AB_1, \dots, AB_{m(t_1)}, \dots, A^{n-1}B_0, A^{n-1}B_1, \dots, A^{n-1}B_{m(t_1)}] = n.$$

REMARK 4.4. For system SH_p^M we have $r_i(t) = t + h_i$ for $i = 0, 1, \dots, M$.

REMARK 4.5. Corollary 4.3 agrees with the results of the paper [9, Th. 4 and Th. 2]. The similar problem for nonstationary linear systems have been also considered in the paper [6], but only for systems with finite dimensional state space.

REMARK 4.6. From the above results it follows immediately that approximate relative controllability implies approximate controllability.

5. Approximate absolute controllability. In this section it is generally assumed, that $t_0 < v_M(t_1)$, similarly as in Definition 2.3. For given initial complete state $z_{t_0} = \{x(t_0), u_{t_0}\}$ and given the control $u_{t_1} = u(t)$ for $t \in [v_M(t_1), t_1]$ and for $u \in L_1([t_0, v_M(t_1)], U)$ by some easy manipulation equality (2.2) can be expressed in the following more convenient form, (for $t = t_1$):

$$(5.1) \quad x(t_1, z_{t_0}, u) = x(t_1, z_{t_0}, 0) + x(t_1, 0, u_{t_1}) + x(t_1, 0, u),$$

where

$$(5.2) \quad x(t_1, z_{t_0}, 0) = S(t_1 - t_0) x(t_0) + \sum_{i=0}^{i=M} \int_{v_i(t_0)}^{t_0} S(t_1 - r_i(t)) B_i \dot{r}_i(t) u_{t_0}(t) dt,$$

$$(5.3) \quad x(t_1, 0, u_{t_1}) = \sum_{i=0}^{i=M} \int_{v_M(t_1)}^{v_i(t_1)} S(t_1 - r_i(t)) B_i \dot{r}_i(t) u_{t_1}(t) dt,$$

$$(5.4) \quad x(t_1, 0, u) = \int_{t_0}^{v_M(t_1)} \sum_{i=0}^{i=M} S(t_1 - r_i(t)) B_i \dot{r}_i(t) u(t) dt.$$

Similarly as in the preceding sections, let us introduce the set attainable at time $v_M(t_1)$, from the zero initial complete state at time t_0 , i.e. $z_{t_0} = \{0, 0\}$, denoted by $K_{[t_0, v_M(t_1)]}$, and defined in the following way

$$(5.5) \quad K_{[t_0, v_M(t_1)]} = \{x(t_1, 0, u) \in X : u \in L_1([t_0, v_M(t_1)], U)\} = \\ = \left\{ \int_{t_0}^{v_M(t_1)} S(v_M(t_1) - t) \sum_{i=0}^{i=M} S(t_1 - v_M(t_1) + \\ + t - r_i(t)) B_i \dot{r}_i(t) u(t) dt \in X : u \in L_1([t_0, v_M(t_1)], U) \right\}.$$

THEOREM 5.1. *System SV_{∞}^M is approximately absolutely controllable on $[t_0, t_1]$, if and only if dynamical system without delays in control of the following form*

$$(5.6) \quad \dot{x}(t) = Ax(t) + \sum_{i=0}^{i=M} S(t_1 - v_M(t_1) + t - r_i(t)) B_i \dot{r}_i(t) u(t), \quad t \in [t_0, v_M(t_1)]$$

is approximately controllable on $[t_0, v_M(t_1)]$.

Proof. From Definition 2.3 and the formulas (5.1) and (5.5) it follows, that the system SV_{∞}^M is approximately absolutely controllable on

$[t_0, t_1]$ if and only if $K_{[t_0, v_M(t_1)]} = X$. On the other hand it is well known ([3], [4], [13], [14]), that by (5.5) the set $K_{[t_0, v_M(t_1)]}$ is the attainable set for the system (5.6) on the time interval $[t_0, v_M(t_1)]$. Hence, combining these two above statements our theorem follows.

COROLLARY 5.1. *System SH_{∞}^M is approximately absolutely controllable on $[0, t_1]$, if and only if dynamical system without delays in control of the following form*

$$(5.7) \quad \dot{x}(t) = Ax(t) + \sum_{i=0}^{i=M} S(h_M - h_i) B_i u(t), \quad t \in [0, t_1 - h_M]$$

is approximately controllable on $[0, t_1 - h_M]$.

Proof. Since $v_i(t) = t - h_i$ for $i = 0, 1, \dots, M$, then $r_i(t) = t + h_i$ and $\dot{r}_i(t) = 1$. Hence $S(t_1 - v_M(t_1) + t - r_i(t)) = S(h_M - h_i)$ for $i = 0, 1, \dots, M$ and the corollary follows.

COROLLARY 5.2. *If $S(t)$ is a group of bounded linear operators, then system SV_{∞}^M is approximately absolutely controllable on $[t_0, t_1]$, if and only if system without delays*

$$(5.8) \quad \dot{x}(t) = Ax(t) + \sum_{i=0}^{i=M} S(t - r_i(t)) B_i \dot{r}_i(t) u(t), \quad t \in [t_0, v_M(t_1)]$$

is approximately controllable on $[t_0, v_M(t_1)]$.

Proof. If $S(t)$ is a group then $S^{-1}(t) \in L(X)$ for all $t \in \mathbb{R}$. Hence by (5.5) we have

$$(5.9) \quad K_{[t_0, v_M(t_1)]} = \left\{ S(t_1 - v_M(t_1)) \int_{t_0}^{v_M(t_1)} S(v_M(t_1) - t) \cdot \sum_{i=0}^{i=M} S(t - r_i(t)) B_i \dot{r}_i(t) u(t) dt \in X : u \in L_1([t_0, v_M(t_1)], U) \right\} = \\ = \left\{ \int_{t_0}^{v_M(t_1)} S(v_M(t_1) - t) \sum_{i=0}^{i=M} S(t - r_i(t)) B_i \dot{r}_i(t) u(t) dt \in X : u \in L_1([t_0, v_M(t_1)], U) \right\}.$$

But the last equality in formula (5.9) is the attainable set for system of the form (5.8) and hence our corollary follows.

COROLLARY 5.3. *If $S(t)$ is a group of bounded linear operators, then system SH_{∞}^M is approximately absolutely controllable on $[0, t_1]$, if and only if systems without delays*

$$(5.10) \quad \dot{x}(t) = Ax(t) + \sum_{i=0}^{i=M} S(-h_i) B_i u(t), \quad t \in [0, t_1 - h_M]$$

is approximately controllable on $[0, t_1 - h_M]$.

Proof. Since $S(t - r_i(t)) = S(t - t - h_i)$ for $i = 0, 1, \dots, M$ then by Corollary 5.3 our corollary follows.

REMARK 5.1. It is obvious that approximate absolute controllability implies approximate relative controllability which implies approximate controllability.

REMARK 5.2. Corollary 5.3 coincides completely with results given in the paper [9, Cor. 3] for finite dimensional systems with constant multiple delays in control. The results for time-varying delays in control coincide with the results given in the paper [8].

6. Controllability in Hilbert spaces. In this section, using the results obtained in the preceding paragraphs, we shall consider various types of controllability for systems defined in a Hilbert space. Hence, throughout the present section X will be specialized to be a Hilbert space. Moreover, we shall also assume, that the space U is also a Hilbert space.

THEOREM 6.1. *The system SV_{∞}^M is approximately relatively controllable on $[t_0, t_1]$ if and only if the selfadjoint operator $C(t_0, t_1) : X \rightarrow X$, defined as follows*

$$(6.1) \quad C(t_0, t_1) = \sum_{i=0}^{i=m(t_1)-1} \int_{v_{i+1}(t_1)}^{v_i(t_1)} \sum_{k=0}^{k=i} \dot{r}_k(t) S(t_1 - r_k(t)) B_k B_k^* S^*(t_1 - r_k(t)) \dot{r}_k(t) dt + \int_{t_0}^{v_m(t_1)(t_1)} \sum_{k=0}^{k=m(t_1)} \dot{r}_k(t) S(t_1 - r_k(t)) B_k B_k^* S^*(t_1 - r_k(t)) \dot{r}_k(t) dt$$

is positively defined. Moreover the system SV_{∞}^M is approximately absolutely controllable on $[t_0, t_1]$, ($t_0 < v_M(t_1)$) if and only if the selfadjoint operator $C_a(t_0, t_1) : X \rightarrow X$

$$(6.2) \quad C_a(t_0, t_1) = \int_{t_0}^{v_M(t_1)} \sum_{i=0}^{i=M} \dot{r}_i(t) S(t_1 - r_i(t)) B_i \sum_{i=0}^{i=M} B_i^* S^*(t_1 - r_i(t)) \dot{r}_i(t) dt$$

is positively defined.

Proof. Let us observe, that the attainable set $K_{[t_0, t_1]}$ given by the formula (4.5) is in fact the range of linear bounded operator acting on the space of admissible controls and given explicitly by the formula 4.5). Since X and U are Hilbert spaces, then the range of the operator is dense if and only if its adjoint is injective operator and this is equivalent to the condition that $C(t_0, t_1)$ is positively defined. The same method of proof is valid for approximate absolute controllability.

REMARK 6.1. It should be stressed, that Theorem 6.1 is valid without any assumptions on analyticity of the functions $v_i(t)$, $i = 0, 1, \dots, M$, or semigroup $S(t)$.

REMARK 6.2. Theorem 6.1 extends to the case of time-variable delays in control, the results given in the paper [8, Th. 2]. The similar results for systems without delays in control have been obtained in the paper [2].

REMARK 6.3. If the operators $C(t_0, t_1)$, or $C_a(t_0, t_1)$ are positively defined then they have inverse operators, but not necessary bounded.

THEOREM 6.2. Let X be separable Hilbert space. Let A satisfies H 1, H 3 and H 4. Then system SV_p^M is approximately relatively controllable on $[t_0, t_1]$ if and only if

$$(6.3) \quad \text{rank } B_k^{m(t_i)} = \text{rank} \begin{bmatrix} \langle b_{11}, x_{k1} \rangle_x, \dots, \langle b_{1p}, x_{k1} \rangle_x \langle b_{21}, x_{k1} \rangle_x, \\ \langle b_{11}, x_{k2} \rangle_x, \dots, \langle b_{1p}, x_{k2} \rangle_x \langle b_{21}, x_{k2} \rangle_x, \\ \dots \\ \langle b_{11}, x_{k1} \rangle_x, \dots, \langle b_{1p}, x_{k1} \rangle_x \langle b_{21}, x_{k1} \rangle_x, \\ \dots, \langle b_{2p}, x_{k1} \rangle_x, \dots, \langle b_{m(t_i)p}, x_{k1} \rangle_x \\ \dots, \langle b_{2p}, x_{k2} \rangle_x, \dots, \langle b_{m(t_i)p}, x_{k2} \rangle_x \\ \dots \\ \dots, \langle b_{2p}, x_{k1} \rangle_x, \dots, \langle b_{m(t_i)p}, x_{k1} \rangle_x \end{bmatrix} = l_k$$

for $k = 1, 2, 3, \dots$

Proof. The proof of Theorem 6.2 follows from Corollary 4.2 and results of the paper [14, Th. 3.4] (see also e.g. [4], and [11]).

COROLLARY 6.1. If the assumptions of Theorem 6.2 are satisfied, then system SV_p^M is approximately absolutely controllable on $[t_0, t_1]$, ($t_0 < v_M(t_1)$) if and only if

$$(6.4) \quad \text{rank } B_k^M = l_k, \text{ for } k = 1, 2, 3, \dots$$

COROLLARY 6.2. Theorem 6.2 and Corollary 6.1 are valid if the assumption H 3 is replaced by the assumption H 5.

REMARK 6.4. For finite-dimensional systems i.e. $X = R^n$ and $U = R^p$ the analogous results have been derived by using Jordan canonical form of the dynamical systems, but in finite dimensional case this methods of controllability investigation have some disadvantage in contrast to other, simpler methods.

REMARK 6.5. For systems without delays in control, defined in infinite-dimensional Hilbert space, the analogous results have been derived in the papers [4], [11], [14], where several examples, concerning classical boundary problems, are given.

REMARK 6.6. All results given in this paper concern only the approximate type controllability and not exact type controllability. However, it is well known, that in practice, the majority of the dynamical systems are only controllable in the approximate sense, and not controllable in exact sense (see e.g. [2], [3], [4], [5], [10], [11], [12], [13], [14], [15]). Moreover in the papers [13] and [14] it have been proved, that for systems with compact operators B_i $i = 0, 1, \dots, M$ or with compact semigroups $S(t)$ or compact resolvent $R(\lambda, A)$ exact type controllability never occurs.

REMARK 6.7. The results of this paper can be extended to the case of nonautonomous systems, where operators A and B_i $i = 0, 1, \dots, M$ depend explicitly on the time t .

REFERENCES

- [1] H. T. BANKS, M. Q. JACOBS and R. M. LATINA, *The synthesis of optimal controls for linear, time optimal problems with retarded controls*, *J. Optimization Theory Appl.* 8 (1971), 319—360.
- [2] M. C. DELFOUR and S. K. MITTER, *Controllability and observability for infinite-dimensional systems*, *SIAM J. Control Optim.* 10 (1972), 329—333.
- [3] H. O. FATTORINI, *Some remarks on complete controllability*, *SIAM J. Control Optim.* 4 (1966), 686—694.
- [4] H. O. FATTORINI, *Boundary control of temperature distributions in a parallelepiped*, *SIAM J. Control Optim.* 13 (1975), 1—13.
- [5] H. O. FATTORINI and D. L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, *Arch. Rational Mech. Anal.* 43 (1971), 272—292.
- [6] J. KLAMKA, *Controllability of linear systems with time-variable delays in control*, *Informat. J. Control* 24 (1976), 869—878.
- [7] J. KLAMKA, *Absolute controllability of linear systems with time-variable delays in control*, *Systems Sci.* 4 (1978), 43—52.
- [8] J. KLAMKA, *Relative controllability of infinite-dimensional systems with delays in control*, *Systems Sci.* 4 (1978), 43—52.
- [9] A. W. OLBROT, *On the controllability of linear systems with time delays in control*, *IEEE Trans. Automat. Control* 17 (1972), 664—666.
- [10] D. L. RUSSELL, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, *Studies in Appl. Math.* 52 (1973), 189—211.
- [11] Y. SAKAWA, *Controllability for partial differential equations of parabolic type*, *SIAM J. Control Optim.* 12 (1974), 389—400.
- [12] T. I. SEIDMAN, *Observation and prediction for the heat equation. IV: Patch observability and controllability*, *SIAM J. Control Optim.* 15 (1977), 412—427.
- [13] R. TRIGGIANI, *Controllability and observability in Banach space with bounded operators*, *SIAM J. Control Optim.* 13 (1975), 462—491.
- [14] R. TRIGGIANI, *Extensions of rank conditions for controllability and observability to Banach space and unbounded operators*, *SIAM J. Control Optim.* 14 (1976), 313—338.
- [15] R. TRIGGIANI, *On the lack of exact controllability for mild solutions in Banach space*, *J. Math. Anal. Appl.* 50 (1975), 438—446.