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BOUNDARY VALUES OF THE SOLUTIONS OF THE PARABOLIC EQUATION

Abstract. The paper deals with the problem of the behaviour of a given solution of a quasi-linear parabolic equation near the parabolic boundary. Necessary and sufficient conditions for weak and strong convergence in the Sobolev space $W_p^{1,1}$, $p \geqslant 2$, are given.

1. Introduction. In the theory of partial differential equations the problem of the behaviour of the given solution near the boundary arises in a natural way. A problem arises while determining if the given solution has trace on the boundary. Several function spaces arise as the spaces of traces of solutions of partial differential equations. The purpose of this paper is to obtain conditions giving L^p-traces on the boundary of generalized solutions of a quasi-linear parabolic equation. Section 2 deals with the problem of weak convergence of traces for solutions in the Sobolev space $W_p^{1,1}$, $p \geqslant 2$. Section 3 extends these results to strong convergence. The arguments which we give here are based partially on the references [1], [7] and [8].

2. Weak convergence. Consider the quasi-linear parabolic equation of the form

$$(1) \qquad \sum_{i,j=1}^{n} (a_{ij}(t,x)\, u_{x_j})_{x_i} - b(t,x,u,u_x) - u_t = 0$$

in a cylinder $D = (0,T] \times Q$, where $Q \subset R^n$ is a bounded domain with the boundary ∂Q of the class C^2, $u_{x_i} = D_{x_i} u$, $u_x = (u_{x_1}, \dots, u_{x_n})$. Let us denote $r(x) = \operatorname{dist}(x, \partial Q)$. We make the following assumptions:
(A) There is a positive constant $\gamma > 1$ such that

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$$\gamma^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(t,x)\,\xi_i\xi_j \leqslant \gamma\,|\xi|^2$$

for all $\xi \in R^n$ and $(t,x) \in \bar{D}$.

(B) The coefficients a_{ij} belong to $C^1(\bar{D})$.

(C) The function $b(t,x,u,s)$ is defined for $(t,x,u,s) \in D \times R^{n+1}$ and satisfies the following conditions.

(i) for a.e. $(t,x) \in D$, $b(t,x,\cdot,\cdot)$ is a continuous function on R^{n+1},

(ii) for every fixed $(u,s) \in R^{n+1}$, $b(\cdot,\cdot,u,s)$ is a measurable function on D,

(iii) for all $(t,x,u,s) \in D \times R^{n+1}$

$$|b(t,x,u,s)| \leqslant f(t,x) + L(|u|+|s|),$$

where L is a positive constant and $f: D \to R$ is a non-negative measurable function such that

$$\iint_D f(t,x)^p\, r(x)^\Theta\, dxdt < \infty$$

for some constants p, Θ for which $1 < p < \infty$, $p \leqslant \Theta < 2p-1$.

REMARK 1. Under the assumption (C) the composition $b(t,x,u(t,x), s(t,x))$ is measurable when $u(t,x)$, $s(t,x)$ are measurable and the mapping

$$b(t,x,\cdot,\cdot): L^1_{loc}(D)^{n+1} \to L^1_{loc}(D)$$

is continuous, (see [6]).

In the sequel we use the notion of a generalized solution involving the Sobolev spaces: $W^{1,1}_{loc,p}(D)$, $W^{1,0}_p(D)$, $\dot{W}^{1,0}_p(D)$. We denote by $W^{1,1}_{loc,p}(D)$ the Sobolev space of real functions u such that u and its distributional derivatives $u_{x_1}, \dots, u_{x_n}, u_t$ belong to $L^p_{loc}(D)$ and by $W^{1,0}_p(D)$ the Sobolev space of real functions u such that u and its distributional derivatives $u_{x_1}, \dots, u_{x_n}$ belong to $L^p(D)$. The space of the functions u which belong to $W^{1,0}_p(D)$ and such that supp $u \subset$ Int D we denote by $\dot{W}^{1,0}_p D$.

DEFINITION. A function u is said to be a *weak solution of the equation* (1) *in* D if $u \in W^{1,1}_{loc,p}(D)$ and u satisfies

$$(2)\quad \iint_D \sum_{i,j=1}^{n} a_{ij}(t,x)\, u_{x_i} v_{x_j}\, dxdt + \iint_D b(t,x,u,u_x)\, v\, dxdt + \iint_D u_t v\, dxdt = 0$$

for every $v \in \dot{W}^{1,0}_{p'}(D)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

It follows from the regularity of the boundary ∂Q that there is a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0]$ the domain $Q_\delta = Q \cap \{x : \min_{y \in \partial Q} |x-y| > \delta\}$ with the boundary ∂Q_δ, possesses the following property: to each $x_0 \in \partial Q$ we can assign a unique point $x_\delta(x_0) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The inverse mapping to

$x_0 \to x_\delta(x_0)$ is given by the formula $x_0 = x_\delta + \delta\nu_\delta(x_\delta)$, where $\nu_\delta(x_\delta)$ is the outward normal to ∂Q_δ at x_δ.

Let x_δ denote an arbitrary point of ∂Q_δ. For a fixed $\varepsilon > 0$ introduce the sets

$$A_\varepsilon = \partial Q_\delta \cap \{x : |x - x_\delta| < \varepsilon\}$$

$$B_\varepsilon = \{x : x = \tilde{x}_\delta + \delta\nu_\delta(\tilde{x}_\delta), \tilde{x}_\delta \in \partial Q_\delta \cap \{x : |x - x_\delta| < \varepsilon\}$$

and put

$$\frac{dS_\delta}{dS_0}(x_\delta) = \lim_{\varepsilon \to 0^+} \frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denotes the Lebesgue measure of a set A. It was proved by Michailov [8] that there is a positive number γ_0 such that

$$\gamma_0^{-1} \leqslant \frac{dS_\delta}{dS_0} \leqslant \gamma_0 \tag{3}$$

and

$$\lim_{\delta \to 0^+} \frac{dS_\delta}{dS_0}(x_\delta(x_0)) = 1 \tag{4}$$

uniformly with respect to $x_0 \in \partial Q$.

According to Lemma 1 in [3, p. 382], the distance $r(x)$ belongs to $C^2(\bar{Q} - Q_{\delta_0})$ if δ_0 is sufficiently small. Denote by $\varrho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\varrho(x) = r(x)$ for $x \in \bar{Q} - Q_{\delta_0}$, $\varrho \in C^2(\bar{Q})$, $\varrho(x) \geqslant \frac{3\delta_0}{4}$ in Q_{δ_0}, $\gamma_1^{-1} r(x) \leqslant \varrho(x) \leqslant \gamma_1 r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta = \{x : \varrho(x) = \delta\}$, $|\varrho_x(x)| = 1$ for $x \in \bar{Q} - Q_\delta$, $\delta \in (0, \delta_0]$ and finally $\partial Q = \{x : \varrho(x) = 0\}$, $\varrho(x) > 0$ on Q.

Introduce the surface integral for $\mu, \delta \in (0, \delta_0]$ and $u \in W^{1,1}_{loc, p}(D)$

$$M(\mu, \delta) = \int_\mu^T \int_{\partial Q_\delta} |u(t, x)|^p \, dS_\delta dt + \int_{Q_\delta} |u(\mu, x)|^p (r(x) - \delta) \, dx,$$

where the values of the function $u(t, x)$ on the n-dimensional manifold are understood in the sense of traces, (see [9]).

Let us denote

$$D_\delta^\mu = (\mu, T] \times Q_\delta, \ \partial D_\delta^\mu = (\mu, T] \times \partial Q_\delta \cup \{\mu\} \times Q_\delta,$$

$$\partial D = [0, T] \times \partial Q \cup \{0\} \times Q \text{ and } D_\delta = D_\delta^\delta.$$

Here ∂ means the parabolic boundary.

THEOREM 1. *Let u be a weak solution of* (1) *for fixed* $p \geqslant 2$ *and* $\iint_D u_t^2 |u|^{p-2} t^\beta \, dx dt < \infty$ *for some* $\beta < 1$. *Then the following conditions are equivalent:*

I. *$M(\delta, \mu)$ is bounded on* $(0, \delta_0] \times (0, \delta_0]$,

II. $\iint\limits_D u_x^2 |u|^{p-2} r(x)\, dxdt < \infty$.

P r o o f. Let for $\mu. \delta \in (0, \delta_0]$

$$v(t, x) = \begin{cases} u(t, x)\, |u(t, x)|^{p-2} (\varrho(x) - \delta), & \text{for } (t, x) \in D_\delta^\mu \\ 0, & \text{for } (t, x) \in D - D_\delta^\mu. \end{cases}$$

Using Hölder's inequality and the well known property of weak derivatives $|u|_x = \operatorname{sgn} u \cdot u_x$ it is easy to prove that v is an admissible test function in (2). Substituting v in (2) we obtain

(5)
$$\iint\limits_{D_\delta^\mu} \sum_{i,j=1}^n a_{ij} u_{x_i} (u\, |u|^{p-2})_{x_j} (\varrho - \delta)\, dxdt + \iint\limits_{D_\delta^\mu} \sum_{i,j=1}^n a_{ij} u_{x_i}\, u\, |u|^{p-2} \varrho_{x_j}\, dxdt +$$
$$+ \iint\limits_{D_\delta^\mu} b(t, x, u, u_x)\, u\, |u|^{p-2} (\varrho - \delta)\, dxdt + \iint\limits_{D_\delta^\mu} u_t u\, |u|^{p-2} (\varrho - \delta)\, dxdt = 0.$$

By the Green's formula we have

(6)
$$\left| \iint\limits_{D_\delta^\mu} \sum_{i,j=1}^n a_{ij} u_{x_i}\, u\, |u|^{p-2} \varrho_{x_j}\, dxdt \right| = \left| \frac{1}{p} \iint\limits_{D_\delta^\mu} \sum_{i,j=1}^n (a_{ij}\, |u|^p \varrho_{x_j})_{x_i}\, dxdt + \right.$$
$$\left. + \frac{1}{p} \iint\limits_{D_\delta^\mu} \sum_{i,j=1}^n (a_{ij} \varrho_{x_j})_{x_i} |u|^p\, dxdt \right| = \left| \frac{1}{p} \int\limits_\mu^T \int\limits_{\partial Q_\delta} \sum_{i,j=1}^n a_{ij} \varrho_{x_i} \varrho_{x_j} |u|^p\, dS_\delta\, dt + \right.$$
$$\left. + \frac{1}{p} \iint\limits_{D_\delta^\mu} \sum_{i,j=1}^n (a_{ij} \varrho_{x_j})_{x_i} |u|^p\, dxdt \right| \leq \frac{\gamma}{p} \int\limits_\mu^T \int\limits_{\partial Q_\delta} |u|^p\, dS_\delta\, dt + \frac{C_1}{p} \iint\limits_{D_\delta^\mu} |u|^p\, dxdt,$$

where $C_1 = \max\limits_{(t,x) \in D} \left| \sum\limits_{i,j=1}^n (a_{ij} \varrho_{x_j})_{x_i} \right|$.

Integrating by parts the last integral in (5) we obtain

(7)
$$\iint\limits_{D_\delta^\mu} u_t u\, |u|^{p-2} (\varrho - \delta)\, dxdt = \frac{1}{p} \int\limits_\mu^T \int\limits_{Q_\delta} |u|_t^p (\varrho - \delta)\, dxdt =$$
$$= \frac{1}{p} \int\limits_{Q_\delta} |u(T, x)|^p (\varrho - \delta)\, dx - \frac{1}{p} \int\limits_{Q_\delta} |u(\mu, x)|^p (\varrho - \delta)\, dx.$$

Using the assumption (C) and Young's inequality we have the estimate

(8)
$$\left|\iint\limits_{D_\delta^\mu} bu\,|u|^{p-2}(\varrho-\delta)\,dxdt\right| \leqslant \iint\limits_{D_\delta^\mu} f\,|u|^{p-1}(\varrho-\delta)\,dxdt+$$

$$+L\iint\limits_{D_\delta^\mu} |u|^p(\varrho-\delta)\,dxdt+L\iint\limits_{D_\delta^\mu} |u_x|\,|u|^{p-1}(\varrho-\delta)\,dxdt\leqslant$$

$$\leqslant \iint\limits_{D_\delta^\mu} f^p(\varrho-\delta)^\Theta\,dxdt+\iint\limits_{D_\delta^\mu} |u|^p(\varrho-\delta)^\alpha\,dxdt+L\iint\limits_{D_\delta^\mu} |u|^p(\varrho-\delta)\,dxdt+$$

$$+L\varepsilon\iint\limits_{D_\delta^\mu} u_x^2|u|^{p-2}(\varrho-\delta)\,dxdt+\frac{L}{\varepsilon}\iint\limits_{D_\delta^\mu} |u|^p(\varrho-\delta)\,dxdt,$$

where $\alpha=\dfrac{p-\Theta}{p-1}$ and ε is any positive. The assumption (C) implies that $\alpha>-1$.

The first integral in (5) we can estimate as follows

$$\iint\limits_{D_\delta^\mu}\sum_{i,j=1}^{n} a_{ij}u_{x_i}(u\,|u|^{p-2})_{x_j}(\varrho-\delta)\,dxdt=$$

(9)
$$=(p-1)\iint\limits_{D_\delta^\mu}\sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j}|u|^{p-2}(\varrho-\delta)\,dx\,dt\geqslant$$

$$\geqslant\frac{p-1}{\gamma}\iint\limits_{D_\delta^\mu} u_x^2|u|^{p-2}(\varrho-\delta)\,dxdt.$$

Thus combining (5)—(9) we obtain

$$\frac{p-1}{\gamma}\iint\limits_{D_\delta^\mu} u_x^2\,|u|^{p-2}(\varrho-\delta)\,dxdt+$$

$$+\frac{1}{p}\int\limits_{Q_\delta}|u(T,x)|^p(\varrho-\delta)\,dx\leqslant\frac{\gamma}{p}\int\limits_{\mu}^{T}\int\limits_{\partial Q_\delta}|u|^p\,dS_\delta dt+$$

$$+\frac{1}{p}\int\limits_{Q_\delta}|u(\mu,x)|^p(\varrho-\delta)\,dx+\frac{C_1}{p}\iint\limits_{D_\delta^\mu}|u|^p\,dxdt+\left(L+\frac{L}{\varepsilon}\right)\iint\limits_{D_\delta^\mu}|u|^p(\varrho-\delta)\,dxdt+$$

$$+\iint\limits_{D_\delta^\mu} f^p(\varrho-\delta)^\Theta\,dxdt+\iint\limits_{D_\delta^\mu}|u|^p(\varrho-\delta)^\alpha\,dxdt+L\varepsilon\iint\limits_{D_\delta^\mu}u_x^2|u|^{p-2}(\varrho-\delta)\,dxdt.$$

Choosing ε such that $\frac{p-1}{2\gamma} = L\varepsilon$ and reducing the last term we obtain from this inequality

$$\iint_{D_\delta^\mu} u_x^2 |u|^{p-2} (\varrho-\delta)\, dx dt + C_2 \int_{Q_\delta} |u(T, x)|^p (\varrho-\delta)\, dx \leqslant \tag{10}$$

$$\leqslant C_3 \iint_{D_\delta^\mu} |u|^p (\varrho-\delta)^\alpha\, dx dt + C_4 \iint_D f^p r^\theta\, dx dt + C_5 \iint_D |u|^p\, dx dt + C_6 M(\mu, \delta),$$

where $C_2 = \frac{2\gamma}{p(p-1)}$, $C_3 = C_4 = \frac{2\gamma}{p-1}$, $C_5 = \max\left\{\left(L+\frac{L}{\varepsilon}\right)\frac{2\gamma\gamma_1}{p-1}\operatorname{diam}(Q), \frac{2C_1\gamma}{p(p-1)}\right\}$ and $C_6 = \max\left\{\frac{2\gamma^2}{p(p-1)}, \frac{2\gamma\gamma_1}{p(p-1)}\right\}$.

Let $\alpha \in (-1, 0]$, $\delta \in \left(0, \frac{\delta_0}{2}\right]$, $\mu \in (0, \delta_0]$ and $x \in Q_{\delta_0}$. From the definition of the function ϱ it follows that $(\varrho(x)-\delta)^\alpha \leqslant \left(\frac{\delta_0}{4}\right)^\alpha$ thus we obtain

$$\iint_{D_\delta^\mu} |u|^p (\varrho-\delta)^\alpha\, dx dt = \int_{\delta_0}^{T} \int_{Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha\, dx dt + \int_{\mu}^{T} \int_{Q_\delta - Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha\, dx dt +$$

$$+ \int_{\mu}^{\delta_0} \int_{Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha\, dx dt \leqslant \left(\frac{\delta_0}{4}\right)^\alpha \int_{\delta_0}^{T} \int_{Q_{\delta_0}} |u|^p\, dx dt + \int_{\mu}^{T} dt \int_{\delta}^{\delta_0} (\nu-\delta)^\alpha\, d\nu \int_{\partial Q_\nu} |u|^p\, dS_\nu +$$

$$+ \left(\frac{\delta_0}{4}\right)^{\alpha-1} \int_{\mu}^{\delta_0} \int_{Q_{\delta_0}} |u|^p (\varrho-\delta)\, dx dt \leqslant \left(\frac{\delta_0}{4}\right)^\alpha \int_{\delta_0}^{T} \int_{Q_{\delta_0}} |u|^p\, dx dt +$$

$$+ \frac{\delta_0^{\alpha+1}}{\alpha+1} \sup_{0<\delta\leqslant\delta_0} \int_{\mu}^{T} \int_{\partial Q_\delta} |u|^p\, dS_\delta\, dt + \left(\frac{\delta_0}{4}\right)^{\alpha-1} \delta_0 \sup_{0<\mu\leqslant\delta_0} \int_{Q_\delta} |u(\mu, x)|^p (\varrho-\delta)\, dx.$$

For $\alpha > 0$ we have $(\varrho-\delta)^\alpha \leqslant C_7$, where $C_7 = \max_Q\, [\varrho(x)-\delta]^\alpha$ so we obtain the following estimate

$$\iint_{D_\delta^\mu} |u|^p (\varrho-\delta)^\alpha\, dx dt < C_8 \tag{11}$$

for $\alpha > -1$, $\delta \in \left(0, \frac{\delta_0}{2}\right]$, $\mu \in (0, \delta_0]$ where the constant C_8 is independent of δ and μ.

Now condition (10) implies the estimate

$$\iint_{D_\delta^\mu} u_x^2 |u|^{p-2} (\varrho-\delta)\, dx dt \leqslant C_9 \tag{12}$$

for $\delta \in \left(0, \frac{\delta_0}{2}\right]$ and $\mu \in (0, \delta_0]$ which we can write in the followingg form

$$\iint_D u_x^2 |u|^{p-2} \eta(t, x, \mu, \delta)\, dxdt \leqslant C_9$$

where

$$\eta(t, x, \mu, \delta) = \begin{cases} \varrho(x) - \delta, & \text{for } (t, x) \in D_\delta^\mu \\ 0, & \text{for } (t, x) \in D - D_\delta^\mu. \end{cases}$$

Hence and from the Monotone Convergence Theorem we obtain condition II what proves the implication I $\Rightarrow$ II.

To prove the implication II $\Rightarrow$ I we show first that condition II implies (11). Let $\alpha > -1$, $\delta \in \left(0, \frac{\delta_0}{2}\right]$, $\mu \in (0, \delta_0]$ and

$$(14) \quad \iint_{D_\delta^\mu} |u|^p (\varrho - \delta)^\alpha\, dxdt = \int_\mu^T \int_{Q_{\delta_0}} |u|^p (\varrho - \delta)^\alpha\, dxdt + \int_\mu^T \int_{Q_\delta - Q_{\delta_0}} |u|^p (\varrho - \delta)^\alpha\, dxdt.$$

Integrating by parts we have

$$\int_\mu^T \int_{Q_{\delta_0}} |u|^p (\varrho - \delta)^\alpha\, dxdt = T \int_{Q_{\delta_0}} |u(T, x)|^p (\varrho - \delta)^\alpha dx - \mu \int_{Q_{\delta_0}} |u(\mu, x)|^p (\varrho - \delta)^\alpha\, dx -$$

$$-p \int_\mu^T \int_{Q_{\delta_0}} t\, |u|^{p-2} u u_t (\varrho - \delta)^\alpha\, dxdt \leqslant T \left(\frac{\delta_0}{4}\right)^\alpha \int_{Q_{\delta_0}} |u(T, x)|^p\, dx +$$

$$+ p \sqrt{T} \left[\int_\mu^T \int_{Q_{\delta_0}} u^2 |u|^{p-2} (\varrho - \delta)^\alpha\, dxdt\right]^{\frac{1}{2}} \left(\frac{\delta_0}{4}\right)^{\frac{\alpha}{2}} \left[\int_\mu^T \int_{Q_{\delta_0}} t\, |u|^{p-2} u_t^2\, dxdt\right]^{\frac{1}{2}}$$

thus there is a constant C_{10} such that for every $\delta \in \left(0, \frac{\delta_0}{2}\right]$, $\mu \in (0, \delta_0]$

$$(15) \qquad \int_\mu^T \int_{Q_{\delta_0}} |u|^p (\varrho - \delta)^\alpha\, dxdt < C_{10}.$$

From condition II it follows that

$$\int_0^T \int_{Q_{\delta_0}} \left| |u|_x^p \right| dxdt < \infty$$

because $r(x) \geqslant \frac{3}{4} \delta_0$ for $x \in Q_{\delta_0}$ and thus $|u|^p \in W_p^{1,0}((0, T) \times Q_{\delta_0})$.

It is well known (see [9]), that such function has the trace on the parabolic boundary of $(0, T) \times Q_{\delta_0}$ and

$$\int_0^T \int_{\partial Q_{\delta_0}} |u|^p\, dS_{\delta_0}\, dt < \infty.$$

As $\varrho(x) = \delta_0$ for $x \in \partial Q_{\delta_0}$ thus there is a constant C_{11} such that

(16) $$\int_\mu^T \int_{\partial Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha \, dS_{\delta_0} \, dt < C_{11}$$

for $\delta \in \left(0, \frac{\delta_0}{2}\right]$ and $\mu \in (0, \delta_0]$.

Using the mapping $x \to x_\delta(x)$, (3) and integrating by parts we obtain

$$\int_\mu^T \int_{Q_\delta - Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha \, dx dt = \int_\mu^T dt \int_\delta^{\delta_0} (\nu-\delta)^\alpha \, d\nu \int_{\partial Q_\nu} |u|^p \, dS_\nu \leqslant$$

$$\leqslant \gamma_0 \int_\mu^T dt \int_\delta^{\delta_0} (\nu-\delta)^\alpha \, d\nu \int_{\partial Q} |u(t, x_\nu(x))|^p \, dS =$$

$$= \frac{(\nu-\delta)^{\alpha+1}}{\alpha+1} \gamma_0 \int_\mu^T dt \int_{\partial Q} |u(t, x_\nu(x))|^p \, dS \Bigg|_{\nu=\delta}^{\nu=\delta_0} -$$

$$-\frac{p\gamma_0}{\alpha+1} \int_\mu^T dt \int_\delta^{\delta_0} (\nu-\delta)^{\alpha+1} d\nu \int_{\partial Q} |u(t, x_\nu(x))|^p \, u_x(t, x\;(x)) \, u(t, x_\nu(x)) \frac{\partial x_\nu(x)}{\partial \nu} \, dS \leqslant$$

$$\leqslant \frac{\delta_0^{\alpha+1} \gamma_0^2}{\alpha+1} \int_\mu^T \int_{\partial Q_{\delta_0}} |u|^p \, dS_{\delta_0} \, dt + \frac{p\gamma_0^2}{\alpha+1} \int_\mu^T dt \int_\delta^{\delta_0} (\nu-\delta)^{\alpha+1} \, d\nu \int_{\partial Q_\nu} |u|^{p-1} |u_x| dS_\nu,$$

where we have used $\left| \frac{\partial x_\nu}{\partial \nu} \right| = 1$.

Now using (16) and Hölder's inequality we have

$$\int_\mu^T \int_{Q_\delta - Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha \, dx dt \leqslant \frac{\delta_0^{\alpha+1} \gamma_0^2}{\alpha+1} C_{11} + \frac{\delta_0^{\frac{\alpha+1}{2}} p\gamma_0^2}{\alpha+1} \cdot$$

$$\cdot \left[\int_\mu^T dt \int_\delta^{\delta_0} \int_{\partial Q_\nu} |u|^p (\nu-\delta)^\alpha \, dS_\nu \, d\nu \right]^{\frac{1}{2}} \left[\int_\mu^T dt \int_\delta^{\delta_0} \int_{\partial Q_\nu} |u|^{p-2} u_x^2 (\nu-\delta) \, dS_\nu d\nu \right]^{\frac{1}{2}} \leqslant$$

$$\leqslant \frac{\delta_0^{\alpha+1} \gamma_0^2}{\alpha+1} C_{11} +$$

$$+ \delta_0^{\frac{\alpha+1}{2}} \frac{p\gamma_0^2 \sqrt{\gamma_1}}{\alpha+1} \left[\int_\mu^T \int_{Q_\delta - Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha \, dx dt \right]^{\frac{1}{2}} \left[\iint_D u_x^2 |u|^{p-2} \, r(x) \, dx dt \right]^{\frac{1}{2}}.$$

From the last estimate it follows

(17) $$\int_\mu^T \int_{Q_\delta - Q_{\delta_0}} |u|^p (\varrho-\delta)^\alpha \, dx dt < C_{12}$$

for $\delta \in \left(0, \frac{\delta_0}{2}\right]$ and $\mu \in (0, \delta_0]$, C_{12} being a convenient positive constant.

Now (14), (15) and (17) imply the condition (11).

From the first part of the proof we have the following equality

$$(18)\quad \frac{1}{p}\int_{\mu}^{T}\int_{\partial Q_\delta}\sum_{i,j=1}^{n} a_{ij}\varrho_{x_i}\varrho_{x_j}|u|^p\, dS_\delta\, dt+\frac{1}{p}\int_{Q_\delta}|u(\mu,x)|^p(\varrho-\delta)\,dx=$$

$$=\frac{1}{p}\iint_{D_\delta^\mu}\sum_{i,j=1}^{n}(a_{ij}\varrho_{x_j})_{x_i}|u|^p\,dxdt+\iint_{D_\delta^\mu} b(t,x,u,u_x)\,u|u|^{p-2}(\varrho-\delta)\,dxdt+$$

$$+\frac{1}{p}\int_{Q_\delta}|u(T,x)|^p(\varrho-\delta)\,dx+(p-1)\iint_{D_\delta^\mu}\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}|u|^{p-2}(\varrho-\delta)\,dxdt.$$

Using (A), (B), (C) and the estimate (8) with $\varepsilon=1$ we get

$$\frac{1}{p}\int_{\mu}^{T}\int_{\partial Q_\delta}|u|^p\,dS_\delta\,dt+\frac{1}{p}\int_{Q_\delta}|u(\mu,x)|^p(\varrho-\delta)\,dx\leqslant\frac{C_1}{p}\iint_{D_\delta^\mu}|u|^p\,dxdt+$$

$$+\iint_{D_\delta^\mu}f^p(\varrho-\delta)^\Theta\,dxdt+\iint_{D_\delta^\mu}|u|^p(\varrho-\delta)^\alpha\,dxdt+2L\iint_{D_\delta^\mu}|u|^p(\varrho-\delta)\,dxdt+$$

$$+L\iint_{D_\delta^\mu}u_x^2|u|^{p-2}(\varrho-\delta)\,dxdt+\frac{1}{p}\int_{Q_\delta}|u(T,x)|^p(\varrho-\delta)\,dx+$$

$$+\gamma(p-1)\iint_{D_\delta^\mu}u_x^2|u|^{p-2}(\varrho-\delta)\,dxdt.$$

Condition II and the assumption of the theorem imply

$$\int_Q|u(T,x)|^p\,r(x)\,dx<\infty.$$

Thus from assumption (C), (11), condition I and the last inequality it follows the boundness of the function $M(\mu,\delta)$ on $\left(0,\frac{\delta_0}{2}\right]\times(0,\delta_0]$.

Let now $\delta\in\left(\frac{\delta_0}{2},\delta_0\right]$ and $\mu\in(0,\delta_0]$. A well known property (see [4]) of the traces is that for any function $h\in W_1^1(G)$

$$\|h\|_{L^1(R)}\leqslant K\|h_x\|_{L^1(G)},$$

where R is any submanifold of region G and constant K depends only on region G. Taking advantage of this fact we get

$$\int_{\mu}^{T} \int_{\partial Q_\delta} |u|^p (r-\delta)\, dS_\delta\, dt \leqslant \operatorname{diam}(Q) \int_{\mu}^{T} \int_{\partial Q_\delta} |u|^p\, dS_\delta\, dt \leqslant$$

$$\leqslant \operatorname{diam}(Q)\, K \int_{\mu}^{T} \int_{Q_{\frac{\delta_0}{2}} - Q_{\delta_0}} \big| |u|^p_x \big|\, dx dt \leqslant \operatorname{diam}(Q)\ Kp \int_{0}^{T} \int_{Q_{\frac{\delta_0}{2}} - Q_{\delta_0}} |u|^{p-1} |u_x|\, dx dt \leqslant$$

$$\leqslant \operatorname{diam}(Q)\ Kp \left[\int_{0}^{T} \int_{Q_{\frac{\delta_0}{2}} - Q_{\delta_0}} u_x^2 |u|^{p-2}\, dx dt \right]^{\frac{1}{2}} \left[\int_{0}^{T} \int_{Q_{\frac{\delta_0}{2}} - Q_{\delta_0}} |u|^p\, dx dt \right]^{\frac{1}{2}}.$$

Thus, from condition II and (17) if $a = 0$ and $\delta = \frac{\delta_0}{2}$ we get that the first component of the function $M(\mu, \delta)$ is bounded. For the second component we have the simple estimate

$$\int_{Q_\delta} |u(\mu, x)|^p (r(x) - \delta)\, dx \leqslant \int_{Q_{\frac{\delta_0}{2}}} |u(\mu, x)|^p \left(r(x) - \frac{\delta_0}{2} \right) dx$$

so from the previous case we get that the function $M(\mu, \delta)$ is bounded in the region $(0, \delta_0] \times (0, \delta_0]$ what proves condition I. This ends the proof of Theorem 1.

Let us define the functions $M(\delta) = M(\delta, \delta)$ and

$$\tilde{M}(\delta) = \int_{0}^{T} \int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij} \varrho_{x_i} \varrho_{x_j} |u|^p\, ds_\delta\, dt + \int_{Q_\delta} |u(\delta, x)|^p (\varrho - \delta)\, dx.$$

The assumption (A) implies

(19) $$\frac{1}{\gamma} M(\delta) \leqslant \tilde{M}(\delta) \leqslant \varrho M(\delta).$$

From the results of Gagliardo [2] it follows that if $u \in W^{1,1}_{loc,\, p}(D)$ then the functions $M(\delta)$ and $\tilde{M}(\delta)$ are absolutely continuous on $(0, \delta_0]$, (see [1]).

REMARK 2. Under the assumptions of Theorem 1 condition I can be replaced by

III. $\tilde{M}(\delta)$ is continuous on $[0, \delta_0]$

or

IV. $M(\delta)$ is bounded on $(0, \delta_0]$.

Indeed, condition I follows from III and (19). Using the Dominated and Monotone Convergence Theorems we imply from (18) that there exists $\lim_{\delta \to 0+} \tilde{M}(\delta)$, thus we proved condition III. Condition IV follows from (19).

Let us consider the space $\tilde{L}^p(\partial D)$ of all functions such that

$$\|f\|_p = \left[\int_{0}^{T} \int_{\partial Q} |f(t, x)|^p\, dS dt + \int_{Q} |f(0, x)|^p\, r(x)\, dx \right]^{\frac{1}{p}} < \infty.$$

For $p > 1$ the space L^p with the norm $\|\cdot\|_p$ is a reflexive Banach space and the space $L^{p'}$ is dual to L^p, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover the space L^2 is uniformly convex.

Let us denote

$$u_\delta(t, x) = \begin{cases} u(t, x_\delta(x)), & \text{for } (t, x) \in (0, T] \times \partial Q \\ u(\delta, x), & \text{for } (t, x) \in \{0\} \times Q, \end{cases}$$

where u is a solution of (1), and $\delta \in (0, \delta_0]$. Here the values of the function on the lower-dimensional manifold are understood as its trace on that manifold (see [9]).

THEOREM 2. *Let u be a weak solution of (1) for fixed $p \geqslant 2$ and $\iint_D u_t^2 |u|^{p-2} t^\beta \, dxdt < \infty$ for some $\beta < 1$. Assume one of the conditions I or II holds. Then there is a sequence $\delta_k \to 0$ as $k \to \infty$ and a function $\varphi \in L^p(\partial D)$ such that*

$$\lim_{k \to \infty} \left[\int_0^T \int_{\partial Q} (u(t, x_{\delta_k}(x)) - \varphi(t, x)\, g(t, x))\, dSdt + \right.$$

$$\left. + \int_Q (u(\delta_k, x) - \varphi(0, x))\, g(0, x)\, r(x)\, dx \right] = 0$$

for each $g \in L^{p'}(\partial D)$.

Proof. From condition I of Theorem 1 and (3) we have

$$C_{13} > \int_0^T \int_{\partial Q_\delta} |u(t, x)|^p \, dS_\delta dt + \int_Q |u(\mu, x)|^p \, r(x)\, dx \geqslant$$

$$\geqslant \frac{1}{\gamma_0} \int_0^T \int_{\partial Q} |u(t, x_\delta(x))|^p \, dSdt + \int_Q |u(\mu, x)|^p \, r(x)\, dx$$

for any $\delta, \mu \in (0, \delta_0]$ and some constant C_{13}.

Now taking $\delta = \mu$ we get $\|u_\delta\|_p < C_{13}$ for $\delta \in (0, \delta_0]$. Thus the set $\{u_\delta : \delta \in (0, \delta_0]\}$ is weak compact in $L^p(\partial D)$ and hence the result follows.

We need some lemmas in the following

LEMMA 1. *Let $u \in W^{1,1}_{loc.\,p}(D)$, $\alpha > -1$ and for some constant $\beta < 1$ $\iint_D u_t^2 |u|^{p-2} t^\beta \, dxdt < \infty$. Then there exists constants C_{14} and C_{15} such that*

(20)
$$\iint_D |u|^p (t - \delta)^\alpha \, dxdt < C_{14}$$

and

(21)
$$\int_{Q_\delta} |u(\delta, x)|^p \, dx < C_{15}$$

for $\delta \in \left(0, \frac{\delta_0}{2}\right]$.

P r o o f. Integrating by parts we get

$$\iint\limits_{D_\delta} |u|^p (t-\delta)^a \, dxdt = \frac{(t-\delta)^{a+1}}{a+1} \int\limits_{Q_\delta} |u|^p \, dx \Big|_{t=\delta}^{t=T} - \iint\limits_{D_\delta} \frac{(t-\delta)^{a+1}}{a+1} (|u|^p)_t \, dxdt =$$

$$= \frac{(T-\delta)^{a+1}}{a+1} \int\limits_{Q_\delta} |u(T,x)|^p \, dx - \frac{p}{a+1} \iint\limits_{D_\delta} (t-\delta)^{a+1} |u|^{p-2} u u_t \, dxdt \leqslant$$

$$\leqslant \frac{T^{a+1}}{a+1} \int\limits_{Q} |u(T,x)|^p \, dx + \frac{p}{a+1} T^{\frac{a+1}{2}} \left[\iint\limits_{D_\delta} u_t^2 |u|^{p-2} (t-\delta) \, dxdt \right]^{\frac{1}{2}} \cdot$$

$$\cdot \left[\iint\limits_{D_\delta} |u|^{p-2} u^2 (t-\delta)^a \, dxdt \right]^{\frac{1}{2}} \leqslant \frac{T^{a+1}}{a+1} \int\limits_{Q} |u(T,x)|^p \, dx +$$

$$+ \frac{p}{a+1} T^{\frac{a+1}{2}} \left[\iint\limits_{D} u_t^2 |u|^{p-2} t \, dxdt \right]^{\frac{1}{2}} \left[\iint\limits_{D_\delta} |u|^p (t-\delta)^a \, dxdt \right]^{\frac{1}{2}}$$

which implies (20).

Condition (21) follows from the estimate

$$\int\limits_{Q_\delta} |u(\delta,x)|^p dx = \int\limits_{Q_\delta} |u(T,x)|^p \, dx - \iint\limits_{D_\delta} (|u|^p)_t \, dxdt = \int\limits_{Q_\delta} |u(T,x)|^p \, dx -$$

$$- p \iint\limits_{D_\delta} |u|^{p-2} u u_t \, dxdt \leqslant \int\limits_{Q} |u(T,x)|^p \, dx +$$

$$+ p \left[\iint\limits_{D_\delta} u_t^2 |u|^{p-2} (t-\delta)^\beta \, dxdt \right]^{\frac{1}{2}} \left[\iint\limits_{D_\delta} |u|^p (t-\delta)^{-\beta} \, dxdt \right]^{\frac{1}{2}}$$

at the basis of (20).

LEMMA 2. *Under the assumptions of Theorem* 1 *condition* II *implies*

$$\iint\limits_{D} u_x^2 r \, dxdt < \infty.$$

P r o o f. By Theorem 1 condition II implies the boundedness of the function $\bar{M}(\delta) = \int\limits_{\delta}^{T} \int\limits_{\partial Q_\delta} (u^2+1)^{\frac{p}{2}} dSdt$. Repeating the proof of the implication I $\Rightarrow$ II of Theorem 1 with

$$v(t,x) = \begin{cases} u(u^2+1)^{\frac{p-2}{2}} (\varrho - \delta), & \text{for } (t,x) \in D_\delta \\ 0, & \text{for } (t,x) \notin D_\delta \end{cases}$$

as a test function we obtain

$$\iint\limits_{D} u_x^2 (u^2+1)^{\frac{p-2}{2}} r \, dxdt < \infty$$

and the result follows.

Let us denote by $K(t,x) = \sum\limits_{i,j=1}^{n} a_{ij}(t,x) \, \varrho_{x_i}(x) \, \varrho_{x_j}(x)$. Then we have the following lemma.

LEMMA 3. *Under the assumptions of Theorem 2 the function*

$$G(\delta) = \int_0^T \int_{\partial Q} u(t, x_\delta(x))\, g(t, x)\, K(t, x)\, dS dt + \int_Q u(\delta, x)\, g(0, x)\, \varrho(x)\, dx$$

is continuous on $[0, \delta_0]$ *and*

(22) $$\lim_{\delta\to 0^+} G(\delta) = \int_0^T \int_{\partial Q} \varphi(t, x)\, g(t, x)\, K(t, x)\, dS dt + \int_Q \varphi(0, x)\, g(0, x)\, \varrho(x)\, dx$$

for any function g *in* $L^{\frac{p}{p-1}}(\partial D)$.

Proof. Of course, $G(\delta)$ is continuous on $(0, \delta_0]$ so it suffices to prove continuity at $\delta = 0$. Since $\|u_\delta\|_p < C_{13}$ for $\delta \in (0, \delta_0]$ and elements of $C^1(\bar{D})$ restricted to ∂D are dense in $L^{\frac{p}{p-1}}(\partial D)$ we can assume that there is a $\bar{g} \in C^1(\bar{D})$ such that $\bar{g}|_{\partial Q} = g$. From (2), taking $v = \bar{g}(\varrho - \delta)$ for $(t, x) \in D_\delta$ and $v = 0$ for $(t, x) \notin D_\delta$ as a test function we have

(23) $$\iint_{D_\delta} \Big[\sum_{i,j=1}^n a_{ij} u_{x_i} \bar{g}_{x_j}(\varrho - \delta) + \sum_{i,j=1}^n a_{ij} u_{x_i} \bar{g} \varrho_{x_j} + b\bar{g}(\varrho - \delta) + u_t \bar{g}(\varrho - \delta) \Big] dx dt = 0.$$

By the Green's formula we have

(24) $$\iint_{D_\delta} \sum_{i,j=1}^n a_{ij} u_{x_i} \bar{g} \varrho_{x_j} dx dt = -\int_\delta^T \int_{\partial Q_\delta} \sum_{i,j=1}^n a_{ij} \varrho_{x_i} \varrho_{x_j} u \bar{g}\, dS_\delta\, dt - \iint_{D_\delta} \sum_{i,j=1}^n (a_{ij} \varrho_{x_j} \bar{g})_{x_i} u\, dx dt = -\int_\delta^T \int_{\partial Q} u(t, x_\delta(x))\, g(t, x)\, K(t, x)\, dS dt -$$
$$- \int_\delta^T \int_{\partial Q} u(t, x_\delta(x)) \Big[\frac{dS_\delta}{dS}(x_\delta(x))\, \bar{g}(t, x_\delta(x))\, K(t, x_\delta(x)) - g(t, x)\, K(t, x) \Big] dS dt - \iint_{D_\delta} \sum_{i,j=1}^n (a_{ij} \varrho_{x_j} \bar{g})_{x_i}\, u\, dx dt.$$

Integrating by parts the last term in (23) we get

(25) $$\iint_{D_\delta} u_t \bar{g}(\varrho - \delta)\, dx dt = \int_{Q_\delta} u(T, x)\, \bar{g}(T, x)\, (\varrho(x) - \delta)\, dx - \int_{Q_\delta} u(\delta, x)\, \bar{g}(\delta, x)\, (\varrho(x) - \delta)\, dx - \iint_{D_\delta} u \bar{g}_t (\varrho - \delta)\, dx dt =$$
$$= \int_{Q_\delta} u(T, x)\, \bar{g}(T, x)\, (\varrho(x) - \delta)\, dx - \int_{Q_\delta} u(\delta, x)\, g(0, x)\, (\varrho(x) - \delta)\, dx - \int_{Q_\delta} u(\delta, x)\, (\varrho(x) - \delta)\, [\bar{g}(\delta, x) - g(0, x)]\, dx - \iint_{D_\delta} u \bar{g}_t (\varrho - \delta)\, dx dt.$$

From (23), (24) and (25) we obtain

$$G(\delta) = \int_0^\delta \int_{\partial Q} u(t, x_\delta(x))\, g(t, x)\, K(t, x)\, dSdt + \int_{Q-Q_\delta} u(\delta, x)\, g(0, x)\, \varrho(x)\, dx +$$

$$(26) \qquad + \delta \int_{Q_\delta} u(\delta, x)\, g(0, x)\, dx + \iint_{D_\delta} \sum_{i,j=1}^n a_{ij} u_{x_i} \overline{g}_{x_j} (\varrho - \delta)\, dxdt -$$

$$- \int_\delta^T \int_{\partial Q} u(t, x_\delta(x)) \left[\frac{dS_\delta}{dS} (x_\delta(x))\, \overline{g}(t, x_\delta(x))\, K(t, x_\delta(x)) - g(t, x)\, K(t, x) \right] dSdt -$$

$$- \iint_{D_\delta} \sum_{i,j=1}^n (a_{ij} \varrho_{x_j} \overline{g})_{x_i} u\, dxdt + \iint_{D_\delta} b\overline{g} (\varrho - \delta)\, dxdt + \int_{Q_\delta} u(T, x)\, \overline{g}(T, x)\, (\varrho(x) - \delta)\, dx -$$

$$- \iint_{D_\delta} u\overline{g}_t (\varrho - \delta)\, dxdt - \int_{Q_\delta} u(\delta, x)\, (\varrho(x) - \delta)\, [\overline{g}(\delta, x) - g(0, x)]\, dx.$$

Let us denote the integrals on the right side of (26) respectively by $J_1, J_2, \ldots, J_{10}$. We have the following estimates

$$|J_1| \leqslant \left[\int_0^T \int_{\partial Q} |u(t, x_\delta(x))|^p\, dSdt \right]^{\frac{1}{p}} \left[\int_0^\delta \int_{\partial Q} |gK|^{\frac{p}{p-1}}\, dSdt \right]^{\frac{p-1}{p}}$$

and

$$|J_2| \leqslant \left[\int_Q |u(\delta, x)|^p\, \varrho(x)\, dx \right]^{\frac{1}{p}} \left[\int_{Q-Q_\delta} |g(0, x)|^{\frac{p}{p-1}}\, \varrho(x)\, dx \right]^{\frac{p-1}{p}}$$

so condition I implies

$$\lim_{\delta \to 0^+} J_1 = \lim_{\delta \to 0^+} J_2 = 0.$$

Similarly from (4), Lemma 1 and uniform continuity of the functions K and $\overline{g}$ we get

$$\lim_{\delta \to 0^+} J_3 = \lim_{\delta \to 0^+} J_5 = \lim_{\delta \to 0^+} J_{10} = 0.$$

Continuity at $\delta = 0$ of J_6 follows from the integrability of u.

Applying assumption (C) and the result of Lemma 1 we can easily show that other integrals have the integrable majorants independent of δ and the integrands are continuous for almost all $(t, x) \in D$ or $x \in Q$ respectively, thus from the Monotone and Dominated Convergence Theorems follows their continuity at $\delta = 0$. So we proved the continuity of $G(\delta)$ on $[0, \delta_0]$.

Now, the equality (22) is a simple consequence of Theorem 2.

Let us define the following norm in $L^p(\partial D)$

$$\|f\|_p^1 = \left[\int_0^T \int_{\partial Q} |f(t, x)|^p\, K(t, x)\, dSdt + \int_Q |f(0, x)|^p \varrho(x)\, dx \right]^{\frac{1}{p}}.$$

Since $\gamma^{-1} \leqslant K(t, x) \leqslant \gamma$ and $\gamma_1^{-1} r(x) \leqslant \varrho(x) \leqslant \gamma_1 r(x)$ it follows that the norm $\|\cdot\|_p^1$ is equivalent to the norm $\|\cdot\|_p$ in $L^p(\partial D)$. Thus Lemma 3 implies the following theorem.

THEOREM 3. *Under the assumptions of Theorem 2 u_δ weakly converges in $L^p(\partial)$ to the function φ, as $\delta \to 0^+$, where φ is defined in Theorem 2.*

3. Strong convergence. We begin with a theorem on L^2-convergence.

For $\delta \in (0, \delta_0]$ we can extend the mapping $x_\delta : \partial Q \to \partial Q_\delta$ on $Q - Q_\delta$ in such a way that for $x \in Q - Q_\delta$ we have $x_\delta(x) = x_\delta(x')$, where $x' \in \partial Q$ and $x' - x = \eta\, \nu(x')$ for some $\eta \in (0, \delta]$. Now we can define the mapping $x^\delta : Q \to Q_{\frac{\delta}{2}}$ by

$$x^\delta(x) = \begin{cases} x & , \text{ for } x \in Q_\delta, \\ x_\delta(x) + \frac{1}{2}(x - x_\delta(x)), & \text{for } x \in Q - Q_\delta. \end{cases}$$

Thus $x^\delta(x) = x$ for each $x \in Q_\delta$ and $x^\delta(x) = x_{\frac{\delta}{2}}(x)$ for each $x \in \partial Q$. Moreover $\varrho(x^\delta) \geqslant \frac{\delta}{2}$ and $\gamma_2^{-1} \leqslant |J_{x^\delta}(x)| \leqslant \gamma_2$, where constant γ_2 is independent of δ and $J_{x^\delta}(x)$ is the Jacobian of the mapping $x^\delta(\cdot)$.

Let us denote

$$t^\delta(t) = \begin{cases} t, & \text{for } t \in [\delta, T], \\ \frac{1}{2} t + \frac{1}{2}\delta, & \text{for } t \in [0, \delta]. \end{cases}$$

LEMMA 4. *Let h be a non-negative function in $L^1\left(D_{\frac{\delta}{2}} - D_\delta\right)$. Then*

$$\iint_{D-D_\delta} h(t^\delta, x^\delta)\, dx dt \leqslant \max(2\gamma_2, 2) \iint_{D_{\frac{\delta}{2}} - D_\delta} h(t, x)\, dx dt \tag{27}$$

and if $h \in L^1(D)$ then $\lim\limits_{\delta \to 0^+} \iint\limits_{D-D_\delta} h(t^\delta, x^\delta)\, dx dt = 0.$

P r o o f. By change of variables we get

$$\iint_{D-D_\delta} h(t^\delta, x^\delta)\, dx dt = \int_\delta^T \int_{Q-Q_\delta} h(t, x^\delta(x))\, dx dt + \int_0^\delta \int_{Q-Q_\delta} h(t^\delta, x^\delta)\, dx dt +$$

$$+ \int_0^\delta \int_{Q_\delta} h(t^\delta(t), x)\, dx dt = \int_\delta^T \int_{x^\delta(Q-Q_\delta)} h(t, x)\, J_{x^\delta}^{-1}(x)\, dx dt +$$

$$+ 2 \int_{\frac{\delta}{2}}^\delta \int_{x^\delta(Q-Q_\delta)} h(t, x)\, J_{x^\delta}^{-1}(x)\, dx dt + 2 \int_{\frac{\delta}{2}}^\delta \int_{Q_\delta} h(t, x)\, dx dt \leqslant \gamma_2 \int_\delta^T \int_{Q_{\frac{\delta}{2}} - Q_\delta} h\, dx dt +$$

$$+ 2\gamma_2 \int_{\frac{\delta}{2}}^\delta \int_{Q_{\frac{\delta}{2}} - Q_\delta} h\, dx dt + 2 \int_{\frac{\delta}{2}}^\delta \int_{Q_\delta} h\, dx dt \leqslant \max(2\gamma_2, 2) \iint_{D_{\frac{\delta}{2}} - D_\delta} h\, dx dt.$$

Now the second part of the assertion is obvious by the well known property of integral.

THEOREM 4. *Let u be a weak solution of* (1) *for* $p = 2$, $\iint_D u_t^2 t^\beta \, dxdt < \infty$ *for some* $\beta < 1$ *and let one of conditions* I *or* II *hold for* $p = 2$. *Then there is a function* φ *belonging to* $L^p(\partial D)$ *such that*

$$\lim_{\delta \to 0^+} u_\delta = \varphi \text{ strong in } L^2(\partial D).$$

Proof. As $\|\cdot\|_2$ and $\|\cdot\|_2^1$ are equivalent it suffices to show that there is a $\varphi \in L^2(\partial D)$ such that $\lim_{\delta \to 0^+} \|\varphi - u_\delta\|_2^1 = 0$. By Theorem 3 there is a $\varphi \in L^2(\partial D)$ such that $\lim_{\delta \to 0^+} u_\delta = \varphi$ weakly in L^2. Since $L^2(\partial D)$ is uniformly convex it suffices to show that $\lim_{\delta \to 0^+} \|u_\delta\|_2^1 = \|\varphi\|_2^1$.

Let us denote by $\langle \cdot, \cdot \rangle$ the inner product $L^2(\partial D)$ with the norm $\|\cdot\|_2^1$ and

$$\psi(g) = \sum_{i,j=1}^n a_{ij} u_{x_i} g_{x_j} \varrho - \sum_{i,j=1}^n (a_{ij} \varrho_{x_j} g)_{x_i} u + bg\varrho - ug_t\varrho.$$

Observe that if $u \in W^{1,1}_{loc,p}(D)$ then $u(t^\delta, x^\delta) \in W^{1,1}_p(D)$, thus, as in the proof of Lemma 3 (see [26]), we find that

$$\langle \varphi, g \rangle = \iint_D \psi(g) \, dxdt + \int_Q u(T,x) \, g(T,x) \, \varrho(x) \, dx$$

for any $g \in C^1(\bar{D})$ and hence for any $g \in W^{1,1}_p(D)$.

Taking $g = u(t^\delta, x^\delta)$ we obtain

$$\begin{aligned} (28) \qquad \langle \varphi, u(t^\delta, x^\delta) \rangle = & \iint_{D_\delta} \psi(u(t,x)) \, dxdt + \int_{Q_\delta} u^2(T,x) \, \varrho(x) \, dx + \\ & + \iint_{D-D_\delta} \psi(u(t^\delta, x^\delta(x))) \, dxdt + \int_{Q-Q_\delta} u(T,x) \, u(T, x^\delta(x)) \, \varrho(x) \, dx \end{aligned}$$

as $x^\delta(x) = x$ and $t^\delta(t) = t$ for $x \in Q_\delta$ and $t \in [\delta, T]$.

We show that

$$(29) \qquad \lim_{\delta \to 0^+} \iint_{D-D_\delta} \psi(u(t^\delta, x^\delta)) \, dxdt = 0$$

$$(30) \qquad \lim_{\delta \to 0^+} \int_{Q-Q_\delta} u(T,x) \, u(T, x^\delta(x)) \, \varrho(x) \, dx = 0$$

and

$$(31) \qquad \lim_{\delta \to 0^+} \left[\iint_{D_\delta} \psi(u(t,x)) \, dxdt + \int_{Q_\delta} u^2(T,x) \, \varrho(x) \, dx \right] = \lim_{\delta \to 0^+} (\|u_\delta\|_2^1)^2.$$

From Theorem 3 we have that

$$(\|\varphi\|_2^1)^2 = \lim_{\delta \to 0^+} \langle \varphi, u(t^\delta, x^\delta) \rangle$$

because $x^\delta(x) = x_{\frac{\delta}{2}}(x)$ on ∂Q and $t^\delta(0) = \frac{1}{2}\delta$, so from (28)—(31) it follows that $\lim_{\delta\to 0^+} \|u_\delta\|_2^1 = \|\varphi\|_2^1$ as required.

To prove (29)—(31) set

$$v(t, x) = \begin{cases} u(t, x)(\varrho-\delta), & \text{for } (t, x) \in D_\delta, \\ 0, & \text{for } (t, x) \in D - D_\delta \end{cases}$$

in equation (2) and thus we obtain

$$\text{(32)} \qquad \iint_{D_\delta} \left[\sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} (\varrho-\delta) + \sum_{i,j=1}^{n} a_{ij} u_{x_i} u \varrho_{x_j} + bu(\varrho-\delta) + \right. \\ \left. + u_t u(\varrho-\delta) \right] dx dt = 0.$$

Condition II and equality

$$\lim_{\delta\to 0^+} \iint_{D_\delta} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} (\varrho-\delta)\, dx dt = \iint_{D} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j} \varrho \, dx dt$$

imply

$$\lim_{\delta\to 0^+} \delta \iint_{D_\delta} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{x_j}\, dx dt = 0.$$

Similarly using (11), Hölder's inequality we obtain that

$$\lim_{\delta\to 0^+} \delta \iint_{D_\delta} bu \, dx dt = 0.$$

From the assumption of the theorem and (11) we get

$$\left| \delta \iint_{D_\delta} u u_t \, dx dt \right| \leqslant \delta \iint_{D_\delta} u^2 \, dx dt + \delta \iint_{D_\delta} u_t^2 \, dx dt \leqslant \\ \leqslant \delta \iint_{D} u^2 \, dx dt + \delta^{1-\beta} \iint_{D_\delta} t^\beta u_t^2 \, dx dt$$

thus

$$\lim_{\delta\to 0^+} \delta \iint_{D_\delta} u u_t \, dx dt = 0.$$

Hence and from (32) we have

$$\lim_{\delta\to 0^+} \left[\iint_{D_\delta} \psi(u)\, dx dt + \int_{Q_\delta} u^2(T, x)\, \varrho(x)\, dx \right] =$$

$$= \lim_{\delta\to 0^+} \left| \iint_{D_\delta} \left[- \sum_{i,j=1}^{n} a_{ij} u_{x_i} u \varrho_{x_j} - \sum_{i,j=1}^{n} (a_{ij} \varrho_{x_j} u)_{x_i} u - 2 u u_t (\varrho-\delta) \right] dx dt + \right. \\ \left. + \int_{Q_\delta} u^2(T, x)(\varrho(x)-\delta)\, dx \right| = \lim_{\delta\to 0^+} \left[- \iint_{D} \sum_{i,j=1}^{n} (a_{ij} u_{x_i} \varrho_{x_j} u^2)_{x_i} \, dx dt - \right.$$

$$
\left. - \iint_{D_\delta} (u^2)_t (\varrho - \delta)\, dx dt + \int_{Q_\delta} u^2(T,x)(\varrho(x)-\delta)\, dx \right] =
$$

$$
= \lim_{\delta\to 0^+} \left[\int_\delta^T \int_{\partial Q_\delta} u^2(t,x) K(t,x)\, dS dt + \int_{Q_\delta} u^2(\delta,x)(\varrho(x)-\delta)\, dx \right] =
$$

$$
= \lim_{\delta\to 0^+} \left[\int_0^T \int_{\partial Q} u^2(t, x_\delta(x)) K(t,x)\, dS dt + \int_Q u^2(\delta,x)\varrho(x)\, dx + \right.
$$

$$
+ \int_\delta^T \int_{\partial Q} u^2(t,x_\delta(x))\,[K(t,x_\delta(x)) - K(t,x)]\, dx dt - \int_{Q-Q_\delta} u^2(\delta,x)\varrho(x)\, dx -
$$

$$
\left. - \delta \int_{Q_\delta} u^2(\delta,x)\, dx - \int_0^\delta \int_{\partial Q} u^2(t,x_\delta(x)) K(t,x)\, dS dt \right| = \lim_{\delta\to 0^+} (\|u_\delta\|_{\frac{1}{2}})^2
$$

because the four last terms tend to zero as $\delta \to 0^+$. So we proved (31).

It follows from assumptions (B) and (C) that

$$
(33) \quad \left| \iint_{D-D_\delta} \psi(u(t^\delta, x^\delta))\, dx dt \right| \leqslant C_{16} \iint_{D-D_\delta} \big[|u_x|\, |u_x(t^\delta,x^\delta)|\, \varrho + |u|\, |u(t^\delta,x^\delta)| +
$$

$$
+ |u|\, |u_x(t^\delta,x^\delta)| + f |u(t^\delta,x^\delta)|\, \varrho + |u|\, |u(t^\delta,x^\delta)|\, \varrho + |u_x|\, |u(t^\delta,x^\delta)|\, \varrho +
$$

$$
+ |u|\, |u_t(t^\delta,x^\delta)|\, \varrho \big]\, dx dt
$$

for some positive constant C_{16} independent of δ. Let us denote the integrals on right respectively by $P_1, P_2, \ldots, P_7$.

Since $\varrho(x^\delta(x)) \geqslant \varrho(x)$ for $x \in Q$ we have

$$
P_1^2(\delta) \leqslant \iint_{D-D_\delta} u_x^2 \varrho\, dx dt \iint_{D-D_\delta} u_x^2(t^\delta,x^\delta)\, \varrho(x^\delta)\, dx dt
$$

thus from condition II and Lemma 4 we get that $\lim_{\delta\to 0^+} P_1(\delta) = 0$.

Condition (11) implies $u \in L^2(D)$ thus quite similarly as above we obtain that $\lim_{\delta\to 0^+} P_2(\delta) = 0$ and since $P_5(\delta) \leqslant \sup_{x\in Q} \varrho(x)\, P_2(\delta)$, $\lim_{\delta\to 0^+} P_5(\delta) = 0$, too.

We have the following estimation

$$
\iint_{D-D_\delta} u^2\, dx dt = \int_0^T \int_{Q-Q_\delta} u^2\, dx dt + \int_0^\delta \int_{Q_\delta} u^2\, dx dt =
$$

$$
= \int_0^T dt \int_0^\delta ds \int_{\partial Q_s} u^2(t,x)\, dS dt + \int_0^\delta dt \int_{Q_\delta} u^2(t,x)\, dx \leqslant
$$

$$
(34) \qquad \leqslant \delta \sup_{s\in(0,\delta]} \int_0^T \int_{\partial Q_s} u^2(t,x)\, dS dt + \delta \sup_{t\in(0,\delta]} \int_{Q_\delta} u^2(t,x)\, dx \leqslant
$$

$$
\leqslant \delta \left[\sup_{s\in(0,\delta]} \int_0^T \int_{\partial Q_s} u^2(t,x)\, dS dt + \sup_{t\in(0,\delta]} \int_{Q_t} u^2(t,x)\, dx \right]
$$

by Theorem 2 and conditions (3) and (21) of Lemma 1.

Since $\varrho(x^\delta(x)) \geqslant \frac{\delta}{2}$ we have

$$P_3(\delta) \leqslant \iint\limits_{D-D_\delta} |u_x(t^\delta, x^\delta)|\, \varrho^{\frac{1}{2}}(x^\delta(x)) \frac{|u|}{\sqrt{\delta}} \sqrt{2}\, dxdt$$

thus by Hölder's inequality we get

$$P_3^2(\delta) \leqslant \iint\limits_{D-D_\delta} u_x^2(t^\delta, x^\delta)\, \varrho(x^\delta(x))\, dxdt\, \frac{2}{\delta} \iint\limits_{D-D_\delta} u^2(t, x)\, dxdt$$

so $\lim\limits_{\delta \to 0^+} P_3(\delta) = 0$ by (34) and Lemma 4. Using Hölder's inequality we have

$$P^2(\delta) \leqslant \iint\limits_{D-D_\delta} \frac{u^2(t^\delta, x^\delta)}{\varrho(x)^{\Theta-2}}\, dxdt \iint\limits_{D-D_\delta} f^2 \varrho^\Theta\, dxdt$$

thus $\lim\limits_{\delta \to 0^+} P_4(\delta) = 0$ by assumption (C), and Lemma 4.

In the same way we get

$$P_6^2(\delta) \leqslant \iint\limits_{D-D_\delta} \varrho u_x^2 dxdt \iint\limits_{D-D_\delta} u^2(t^\delta, x^\delta)\, dxdt \sup_{x \in Q} \varrho(x)$$

thus $\lim\limits_{\delta \to 0^+} P_6(\delta) = 0$ by Lemma 4 and condition II.

Since $t^\delta \geqslant \frac{\delta}{2}$ we get

$$P_7^2(\delta) \leqslant \left[\iint\limits_{D-D_\delta} |u_t(t^\delta, x^\delta)|\, (t^\delta)^{\frac{\beta}{2}} |u|\, dxdt \sup_{x \in Q} \varrho(x) \right]^2 \left(\frac{2}{\delta}\right)^\beta \leqslant$$

$$\leqslant \left(\sup_{x \in Q} \varrho(x)\right)^2 \iint\limits_{D-D_\delta} u_t^2(t^\delta, x^\delta)\, (t^\delta)^\beta\, dxdt \left(\frac{2}{\delta}\right)^{\frac{\beta}{2}} \iint\limits_{D-D_\delta} u^2\, dxdt$$

and hence $\lim\limits_{\delta \to 0^+} P_7(\delta) = 0$ by Lemma 4, assumption of this theorem and condition (34), as $1 - \frac{\beta}{2} > 0$. Thus we proved the condition (29).

Condition (30) follows from the estimation

$$\left[\int\limits_{Q-Q_\delta} u(T, x)\, u(T, x^\delta(x))\, \varrho(x)\, dx \right]^2 \leqslant$$

$$\leqslant \left(\sup_{x \in Q} \varrho(x)\right)^2 \int\limits_{Q-Q_\delta} u^2(T, x)\, dx \cdot \gamma_2 \int\limits_{Q_{\frac{\delta}{2}} - Q_\delta} u^2(T, x)\, dx$$

and the fact that $u(T, \cdot) \in L^2(Q)$. This completes the proof of Theorem 4.

In the case $p > 2$ we begin with the following result.

LEMMA 5. *Let u be a weak solution of* (1) *satisfying one of conditions* I *or* II *for a fixed* $p > 2$ *and* $\iint_D u_t^2\,(|u|^{p-2}+1)\,t^\beta\,dxdt < \infty$ *for some* $\beta < 1$. *Then* u_δ *converges to* φ *in* $L^q(\partial D)$ *for each q, where* $0 < q < p$. *The function* φ *is defined in Theorem* 2.

Proof. First we note that u_δ converges weakly to φ in $L^p(\partial D)$ by Theorem 2. We shall show that u_δ converges to φ in $L^2(\partial D)$.

Let $\alpha(\Theta) = 0$ for $p \leqslant \Theta < \frac{3}{2}p$, $\alpha(\Theta) = \frac{\Theta - 1 - p}{p}$ for $\frac{3}{2} \leqslant \Theta < 2p - 1$ and $\chi(\Theta) = \frac{2}{p}\Theta - \alpha(\Theta)$. For Θ and χ such that $p \leqslant \Theta < 2p-1$ and $2 \leqslant \chi < 3$ we have

$$\iint_D f^2\varrho^\chi\,dxdt \leqslant \left[\iint_D f^p\varrho^\Theta\,dxdt\right]^{\frac{2}{p}}\left[\iint_D \varrho^{-\frac{p\alpha}{p-2}}\,dxdt\right]^{\frac{p-2}{p}} < \infty$$

by assumption (C) as $\frac{p\alpha}{p-2} < 1$.

Since $W^{1,1}_{loc,\,p}(D) \subset W^{1,1}_{loc,\,2}(D)$, u is a weak solution of (1) for $p = 2$. By Lemma 2, condition II is fulfilied with $p = 2$. Thus we can use the result of Theorem 4. Hence u_δ converges to some function $\bar\varphi$ in $L^2(\partial D)$ so $\varphi = \bar\varphi$ a.e.

For measurable sets $A \subset \partial D$ and s satisfying $\frac{1}{s} + \frac{q}{p} = 1$ we have

$$\int_A |u_\delta - \varphi|^q \leqslant |A|^s\left[\int_A |u_\delta - \varphi|^p\right]^{\frac{q}{p}} \leqslant |A|^s\left\{\left[\int_{\partial D}|u_\delta|^p\right]^{\frac{1}{p}} + \left[\int_{\partial D}|\varphi|^p\right]^{\frac{1}{p}}\right\}^q \leqslant$$
$$\leqslant |A|^s\,(\|u_\delta\|_p + \|\varphi\|_p)^q.$$

Thus $u_\delta - \varphi$ is equi-absolutely integrable and bounded in $L^q(\partial D)$ so it is compact for δ such that $0 < \delta \leqslant \delta_0$. Now for any sequence $\delta_k \to 0$ there is a subsequence $\delta_k^1 \to 0^+$ with $u_{\delta_k^1} - \varphi \to 0$ a.e. and the result follows.

To prove L^p-convergence we shall need the following theorem on Nemytsky Operators (see [10], p. 155).

THEOREM. *If* $f(t, x, u)$, *defined on* $\partial D \times R$, *satisfies Carathéodory conditions, conditions* (i) *and* (ii) *of assumption* (C) *and*

$$|f(t, x, u)| \leqslant g(t, x) + K|u|^{\frac{s}{t}},$$

where $g \in L^t(\partial D)$, $1 \leqslant s, t < \infty$ *and K is a positive constant, then f generates a continuous operator from* $L^s(\partial D)$ *into* $L^t(\partial D)$ *given by the formula*

$$h : u(\cdot, \cdot) \to f(\cdot, \cdot, u(\cdot, \cdot)).$$

This operator is called the Nemytsky Operator.

We now establish the following L^p-convergence theorem.

THEOREM 5. *Let u be a weak solution of* (1) *satisfying one of conditions* I *or* II *for fixed* $p > 2$, $\iint_D (u_t^2(|u|^{p-2}+1)\, t^\beta\, dxdt$ *for some* $\beta < 1$, *then* u_δ *converges to the function* φ *in* $L^p(\partial D)$.

Proof. Let us denote by u^δ the trace of the composition $u(t^\delta(t), x^\delta(x))$ on ∂D. It is clear that $u^\delta = u_{\frac{\delta}{2}}$ for $\delta \in (0,\delta_0]$.

We begin with the following

REMARK. If u^δ is bounded in $L^p(\partial D)$ and $u^\delta \to \varphi$ in $L^q(\partial D)$ for $q < p$ then $u^\delta|u^\delta|^{p-2} \to \varphi|\varphi|^{p-2}$ weakly in $L^{\frac{p}{p-1}}(\partial D)$. This means that the mapping given by the formula

$$f(t, x, u^\delta) = u^\delta|u^\delta|^{p-2}$$

is continuous from $L^q(\partial D)$ to $L^{\frac{p-1}{q}}(\partial D)$ by Theorem on Nemytsky Operators.

Hence

$$u^\delta|u^\delta|^{p-2} \to \varphi|\varphi|^{p-2} \text{ as } \delta \to 0^+$$

in $L^{\frac{q}{p-1}}(\partial D)$, where we take $\frac{q}{p-1} > 1$. Also $u^\delta|u^\delta|^{p-2}$ is bounded in $L^{\frac{p}{p-1}}(\partial D)$ and so it is weakly compact and the result follows.

The rest of the proof is similar to that of Theorem 4. For every $g \in W^{1,1}_{\frac{p}{p-1}}(D)$ we get

$$\int_0^T \int_{\partial Q} g\, K\, dSdt + \int_Q \varphi(0, x)\, g(0, x)\, \varrho(x)\, dx =$$

$$= \iint_D \psi(g)\, dxdt + \int_Q u(T, x)\, g(T, x)\, \varrho(x)\, dx$$

since $u^\delta \to \varphi$ as $\delta \to 0^+$ weakly in $L^p(\partial D)$ (see the proof of Lemma 3).

Set $g = u(t^\delta, x^\delta)\, |u(t^\delta, x^\delta)|^{p-2}$ in the above equality and noting that $u(t^\delta, x^\delta) = u$ on D_δ and $u^\delta|u^\delta|^{p-2} \to \varphi|\varphi|^{p-2}$ weakly in $L^{\frac{p}{p-1}}(\partial D)$ we obtain

$$(\|\varphi\|_p^1)^p = \lim_{\delta \to 0^+} \Bigg[\int_0^T \int_{\partial Q} \varphi\, u^\delta|u^\delta|^{p-2} K\, dSdt +$$

$$+ \int_Q \varphi(0, x)\, u^\delta(0, x)\, |u^\delta(0, x)|^{p-2} \varrho(x)\, dx = \lim_{\delta \to 0^+} \Bigg[\iint_{D_\delta} \psi(u|u|^{p-2})\, dxdt +$$

$$+ \int_{Q_\delta} |u(T, x)|^p \varrho(x)\, dx \Bigg] + \lim_{\delta \to 0^+} \Bigg[\iint_{D-D_\delta} \psi(u(t^\delta,x^\delta)\, |u(t^\delta, x^\delta)|^{p-2})\, dxdt + \tag{36}$$

$$+ \int_{Q-Q_\delta} u(T, x)\, u(T, x^\delta(x))\, |u(T, x^\delta(x))|^{p-2} \varrho(x)\, dx.$$

Setting

$$v = \begin{cases} u|u|^{p-2}(\varrho-\delta), & \text{for } (t,x)\in D_\delta, \\ 0, & \text{for } (t,x)\in D-D_\delta \end{cases}$$

in (2) we obtain

$$\iint_{D_\delta} \sum_{i,j=1}^{n} a_{ij}u_{x_i}(u|u|^{p-2})_{x_j}(\varrho-\delta)\,dxdt + \iint_{D_\delta} \sum_{i,j=1}^{n} a_{ij}u_{x_i}u|u|^{p-2}\varrho_{x_j}\,dxdt +$$

$$+ \iint_{D_\delta} bu|u|^{p-2}(\varrho-\delta)\,dxdt + \iint_{D_\delta} u_t u|u|^{p-2}(\varrho-\delta)\,dxdt = 0.$$

As in the proof of Theorem 4 it is obvious that

$$\lim_{\delta\to 0^+} \delta \iint_{D_\delta} \left[\sum_{i,j=1}^{n} a_{ij}u_{x_i}(u|u|^{p-2})_{x_j} + bu|u|^{p-2} + u_t u|u|^{p-2} \right] dxdt +$$

$$+ \delta \int_{Q_\delta} |u(T,x)|^p\,dx = 0$$

thus

$$\lim_{\delta\to 0^+} \left[\iint_{D_\delta} \psi\,(u|u|^{p-2})\,dxdt + \int_{Q_\delta} |u(T,x)|^p \varrho(x)\,dx \right] =$$

$$= \lim_{\delta\to 0^+} \left[- \iint_{D_\delta} \sum_{i,j=1}^{n} (a_{ij}\varrho_{x_j}u|u|^p)_{x_i}dxdt - \iint_{D_\delta} (|u|^p)_t(\varrho-\delta)\,dxdt + \right.$$

$$\left. + \int_{Q_\delta} |u(T,x)|^p(\varrho(x)-\delta)\,dx \right] = \lim_{\delta\to 0^+} \left[\int_{\delta}^{T} \int_{\partial Q_\delta} |u|^p K\,dSdt + \right.$$

$$\left. + \int_{Q_\delta} |u(\delta,x)|^p(\varrho(x)-\delta)\,dx = \right. \lim_{\delta\to 0^+} (\|u_\delta\|_p^1)^p.$$

Thus, it suffices to show that second component on the right of (36) tends to zero as $\delta \to 0^+$. It is easily seen that this integrand can be estimated by

$$K\,(|u_x|\,|u_x(\delta)|\,|u(\delta)|^{p-2}\varrho + |u|\,|u(\delta)|^{p-1} + |u|\,|u_x(\delta)|\,|u(\delta)|^{p-2} +$$

$$+ f|u(\delta)|^{p-1}\varrho + |u|\,|u(\delta)|^{p-1}\varrho + |u_x|\,|u(\delta)|^{p-1}\varrho + |u|\,|u_t(\delta)|\,|u(\delta)|^{p-2},$$

where K is a suitable constant and we denote $u(\delta) = u(t^\delta, x^\delta)$.

Estimation of the integrals of the first, second, fourth, fifth and sixth terms is similar to the previous calculations (see the proof of Theorem 4).

We have the following inequality

$$abc^{p-2} \leqslant \text{const}\,(a^p + b^2c^{p-2} + c^p)$$

for each positive a, b, c and $p > 2$.

Set $a = |u|$, $b = |u_x(\delta)|$ or $b = |u_t(\delta)|$ and $c = |u(\delta)|$. Now we can estimate the third and seventh terms analogously as in the proof of Theorem 4. This completes the proof of Theorem 5.

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