ZOFIA MUZYCZKA

ON THE EXISTENCE OF SOLUTIONS OF THE DIFFERENTIAL EQUATION WITH ADVANCED ARGUMENT

Abstract. In this paper the differential-functional equation

$$\varphi'(t) = f(t, \varphi), \; t \in R^+$$
$$\varphi(0) = \eta$$

with an unbounded advanced argument is discovered. Under suitable assumptions, using Schauder Fixed Point Principle, a theorem on existence of solutions in a special functional-space is proved.

1. Introduction. Let E denote a set of functions $\varphi : R^+ \to R^n$, where $R^+ = [0, +\infty)$ and R^n is the euclidean space with the norm $|\cdot|$. Assume that the mapping $f : R^+ \times E \to R^n$ is given and that $\eta \in R^n$. Consider the differential equation

$$\varphi'(t) = f(t, \varphi), \; t \in R^+ \tag{1}$$

with the initial condition

$$\varphi(0) = \eta. \tag{2}$$

In most papers on the existence of a solution of the Cauchy problem (1)—(2) (e.g. see [2], [4], [7]) or the Nicoletti type problem (e.g. see [3], [5], [6]) certain conditions are imposed which bound the advance of the argument by assuming that there exists a function $\delta : R^+ \to R^+$, such that

$$f(t, \varphi) = f(t, \psi) \text{ whenever } \varphi(u) = \psi(u) \text{ for } u \in [0, t+\delta(t)], \tag{3}$$

that is by assuming that the value of the function $f(t, \varphi)$ depends only on the value of the function φ for $u \in [0, t+\delta(t)]$. Boundless advance of the argument of the function φ was admitted in papers [3] and [7] where (under the assumption that the function φ satisfies Lipschitz condition and basing on the Banach Fixed Point Theorem) the existence and uni-

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queness of the solution of the equation under consideration were proved in appropriate classes of functions.

In the present note we prove by means of the Schauder Fixed Point Theorem the existence of a solution of the problem (1)—(2), without assuming the restriction (3). We use for that some ideas of paper [2] as well as of paper [3].

2. Assumptions. Take on the following assumption:

(A) There exists a locally integrable function $L: R^+ \to R^+$ and constants $k > 1$, $\lambda > 0$ such that for arbitrary $(t, \varphi) \in R^+ \times E$ we have

$$|f(t,\varphi)| \leqslant L(t) \sup\left\{|\varphi(s)| \exp\left(-k\int_t^s L(\tau)\,d\tau - \lambda s\right) : s \in R^+,\ s \geqslant t\right\}. \tag{4}$$

Put

$$\|\varphi\| = \sup\left\{|\varphi(t)| \exp\left(-k\int_0^t L(\tau)\,d\tau - \lambda t\right) : t \geqslant 0\right\},\quad \varphi \in E \tag{5}$$

and

$$\Phi = \left\{\varphi \in E : |\varphi(t)| \leqslant a \exp\left(k\int_0^t L(\tau)\,d\tau\right)\right\}, \tag{6}$$

where the constant a satisfies the condition

$$a \geqslant \frac{k|\eta|}{k-1}. \tag{7}$$

Notice that Φ is a non-empty, closed, bounded and convex set.

3. Theorem on existence.

THEOREM. *If the assumption* (A) *is satisfied, then the problem* (1)—(2) *has at least one solution in the class* Φ.

P r o o f. Introduce the transformation T defined for $\varphi \in \Phi$ by the formula

$$(T\varphi)(t) = \eta + \int_0^t f(s,\varphi)\,ds,\quad t \in R^+. \tag{8}$$

We shall show that the transformation T maps the set Φ in itself, i.e. that $T(\Phi) \subset \Phi$. Indeed, let $\varphi \in \Phi$, then following (8), (4), (6), (7) we have the estimations

$$|(T\varphi)(t)| \leqslant |\eta| + \int_0^t |f(s,\varphi)|\,ds \leqslant$$

$$\leqslant |\eta| + \int_0^t L(s) \sup\left\{|\varphi(r)| \exp\left(-k\int_s^r L(\tau)\,d\tau - \lambda r\right) : r \in R^+,\ r \leqslant s\right\} ds \leqslant |\eta| +$$

$$+ \int_0^t L(s) \sup\left\{a \exp\left(k\int_0^r L(\tau)\,d\tau\right) \exp\left(-k\int_s^r L(\tau)\,d\tau - \lambda r\right) : r \in R^+,\ r \geqslant s\right\} ds \leqslant$$

$$\leqslant |\eta| + \int_0^t L(s)\, a \exp\left(k \int_0^s L(\tau)\, d\tau\right) \sup \{\exp(-\lambda r) : r \in R^+,\ r \geqslant s\}\, ds \leqslant$$

$$\leqslant |\eta| + \frac{a}{k} \int_0^t k\, L(s) \exp\left(k \int_0^s L(\tau)\, d\tau\right) ds \leqslant \left(|\eta| + \frac{a}{k}\right) \exp\left(k \int_0^t L(\tau)\, d\tau\right) \leqslant$$

$$\leqslant a \exp\left(k \int_0^t L(\tau)\, d\tau\right).$$

We shall show the complete continuity of the transformation T. Let us choose a number $\varepsilon > 0$ and put

$$\mu(\varepsilon) = \lambda^{-1} \ln \frac{2a}{k\varepsilon}. \tag{9}$$

Then for $t \in (\mu(\varepsilon), +\infty)$, $\varphi_1, \varphi_2 \in \Phi$ in virtue of (8), (4), (6) we obtain the estimations

$$\left|(T\varphi_1)(t) - (T\varphi_2)(t)\right| =$$

$$= \left|\int_0^t f(s, \varphi_1)\, ds - \int_0^t f(s, \psi_2)\, ds\right| \leqslant \int_0^t |f(s, \varphi_1)|\, ds + \int_0^t |f(s, \varphi_2)|\, ds \leqslant$$

$$\leqslant 2 \int_0^t L(s) \sup \left|a \exp\left(k \int_0^s L(\tau)\, d\tau - \lambda r\right) : r \in R^+,\ r \geqslant s\right| ds \leqslant$$

$$\leqslant \frac{2a}{k} \exp\left(k \int_0^t L(\tau)\, d\tau\right).$$

Hence

$$|(T\varphi_1)(t) - (T\varphi_2)(t)| \exp\left(-k \int_0^t L(\tau)\, d\tau - \lambda t\right) \leqslant \frac{2a}{k} \exp(-\lambda t). \tag{10}$$

But according to (9) for $t \in (\mu(\varepsilon), +\infty)$ we get inequality

$$\frac{2a}{k} \exp(-\lambda t) < \varepsilon,$$

which together with (10) gives the estimation

$$|(T\varphi_1)(t) - (T\varphi_2)(t)| \exp\left(-k \int_0^t L(\tau)\, d\tau - \lambda t\right) < \varepsilon,\quad t \in (\mu(\varepsilon), +\infty). \tag{11}$$

Assume now that $t \in [0, \mu(\varepsilon)]$, then

$$|(T\varphi_1)(t) - (T\varphi_2)(t)| \leqslant \int_0^{\mu(\varepsilon)} |f(s, \varphi_1) - f(s, \varphi_2)|\, ds < \varepsilon$$

if only $\|\varphi_1 - \varphi_2\|$ is sufficiently small. Hence

$$|(T\varphi_1)(t) - (T\varphi_2)(t)| \exp\left(-k \int_0^t L(\tau)\, d\tau - \lambda t\right) < \varepsilon,\quad t \in [0, \mu(\varepsilon)]. \tag{12}$$

From the inequalities (11) and (12) it results that $\|T\varphi_1 - T\varphi_2\| \leqslant \varepsilon$ if only $\|\varphi_1 - \varphi_2\|$ is sufficiently small. The last inequality means that T is continuous.

The compactness of the set $T(\Phi)$ remains to be demonstrated. Fix a number β, $0 < \beta < +\infty$, then for arbitrary $t_1, t_2 \in [0, \beta]$, such that $t_1 < t_2$, according to (8), (4), (6) we have

$$|(T\varphi)\,(t_2) - (T\varphi)\,(t_1)| \leqslant \int_{t_1}^{t_2} |f(s, \varphi)|\,\mathrm{d}s \leqslant$$

$$\leqslant \frac{a}{k} \int_{t_1}^{t_2} k\,L(s) \exp\left(k \int_0^s L(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s \leqslant \frac{a}{k} \exp\left(k \int_0^\beta L(\tau)\,\mathrm{d}\tau\right)(t_2 - t_1).$$

Using the above estimation we get for $\varphi \in \Phi$ and $t \in [0, \beta]$,

$$|(T\varphi)\,(t)| \leqslant |(T\varphi)\,(t) - (T\varphi)\,(0)| + |(T\varphi)\,(0)| \leqslant \beta \frac{a}{k} \exp\left(k \int_0^\beta L(\tau)\,\mathrm{d}\tau\right) + |\eta|.$$

From the obtained estimation and from Arzela Theorem for righthand open intervals (see [1]) the compactness of the set $T(\Phi)$ is concluded.

The proposition of the theorem which is being proved results directly from the Schauder Fixed Point Theorem.

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