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# ON ANALYTIC SOLUTIONS OF THE NON-LINEAR FUNCTIONAL EQUATION

**Abstract.** In the present paper we consider the problem of the existence and uniqueness of local analytic solutions of the equation $\varphi(f(x)) = V(\varphi(x))+U(x)$. This paper is a continuation of [1], [2] and [3].

1. In papers [1] and [2] we considered the problem of the existence and uniqueness of local analytic solutions of equation

$$\varphi(f(x)) = g(x, \varphi(x)), \tag{1}$$

where $\varphi$ is an unknown function, and analytic functions $f, g$ fulfil the following assumptions:

A complex function $f$ is defined and analytic in a neighbourhood of the point zero belonging to the field $\mathbf{C}$ of complex numbers and $f$ has zero of an order $r$ at the origin i.e.

$$f(x) = x^r F(x), \quad |x| < \varrho, \tag{2}$$

where $\varrho$ denotes a certain real number; $F$ is analytic in the disc $\{x \in \mathbf{C} : |x| < \varrho\}$ and such that

$$F(0) \neq 0 \tag{3}$$

whereas $r$ is a positive integer fulfilling the condition

$$r \geqslant 2. \tag{4}$$

A complex function $g$ is defined and analytic in a neighbourhood of the point $(0, 0) \in \mathbf{C} \times \mathbf{C}$ and such that $g(0, 0) = 0$. Thus $g$ has a unique representation of the form

$$g(x, y) = U(x)+V(y)+x \cdot y\, G(x, y), \quad |x| < \varrho_1, |y| < \varrho_2,$$

*Received September 25, 1979.*
AMS (MOS) subject classification (1980). Primary 39B05.

where $\varrho_1, \varrho_2$ denote certain positive real numbers; $U, V$ are analytic functions in the discs $K_1 = \{x \in \mathbf{C} : |x| < \varrho_1\}$ and $K_2 = \{x \in \mathbf{C} : |x| < \varrho_2\}$, respectively, whereas $G$ is an analytic function in the bidisc $K_1 \times K_2$ and

$$G(0, 0) \neq 0.$$

In the paper [3] we assumed that

$$G(0, 0) = 0, \text{ and } G(x, y) \not\equiv 0.$$

In the present paper we shall assume that

$$G(x, y) \equiv 0.$$

The function $g$ is now of the form

$$g(x, y) = U(x) + V(y),$$

and equation (1) is of the form

$$\varphi(f(x)) = U(x) + V(\varphi(x)). \tag{5}$$

In the paper [4] Kuczma considered equation (5) in the case where $U(x) \equiv 0$. We shall deal with equation (5) under such assumptions on the given functions which do not allow to reduce it directly to equation

$$\varphi(x) = h(x, \varphi(f(x))). \tag{E}$$

This equation was solved in [5]. A basic result in that direction is the following:

**THEOREM** (W. Smajdor [5]). *Let $h$ be an analytic function in a neighbourhood of the point $(0, \eta) \in \mathbf{C} \times \mathbf{C}$ and such that $h(0, \eta) = \eta$. Moreover, let $f$ be an analytic function in a neighbourhood of the point $0 \in \mathbf{C}$ and such that $f(0) = 0$ and $|f'(0)| < 1$. If*

$$\varphi(x) = \eta + \sum_{n=1}^{\infty} c_n x^n \tag{6}$$

*is a formal solution of equation* (E), *then it represents an analytic function in a neighbourhood of the point $0 \in \mathbf{C}$. Moreover, if*

$$\left| f'(0)^n \frac{\partial h}{\partial y}(0, \eta) \right| \neq 1, \quad n \in \mathbf{N}^*$$

*then the solution $\varphi$ of the form* (6) *does exist and it is unique.*

By a formal solution we mean a formal power series which satisfies a given equation formally.

We shall assume (2), (3), (4), (5) and the following form of functions $U, V$:

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* In the whole paper $\mathbf{N}$ denotes the set of all positive integers.

$$(7)\qquad \begin{cases} U(x) = x^p\, u(x),\ p \in \mathbf{N},\ p \geqslant 2,\ u(0) \neq 0, \\ V(x) = x^q\, v(x),\ q \in \mathbf{N},\ q \geqslant 2,\ v(0) \neq 0, \end{cases}$$

where $u$ and $v$ are analytic functions in the discs $K_1$ and $K_2$, respectively. (If $q = 1$, then equation (5) can be written in the form (E). $p = 1$ leads to contraction.) For brevity, we adopt the convention that the zero function (and only this) is of order $\infty$ at zero. We shall look for locally analytic solutions $\varphi$ of equation (5) of the form

$$(8)\qquad \varphi(x) = x^a\, \Phi(x)\,, a \in \mathbf{N},$$

where $\Phi$ is an analytic function in the neighbourhood of $0 \in \mathbf{C}$ and

$$(9)\qquad \Phi(0) = \eta \neq 0.$$

Putting (7) into (5) we get

$$(10)\qquad \varphi(f(x)) = x^p\, u(x) + \varphi(x)^q\, v(\varphi(x)).$$

If $p = \infty$ in (10) we obtain the equation

$$\varphi(f(x)) = \varphi(x)^q\, v(\varphi(x)),$$

(see [4]). Applying the substitution (8) to equation (10) and taking (2) into account we get

$$(11)\qquad x^{ra}\, F(x)^a\, \Phi(f(x)) = x^p\, u(x) + x^{qa}\, \Phi(x)^q\, v(x^a\, \Phi(x)).$$

REMARK 1. *If a function $\varphi$ fulfilling conditions* (8) *and* (9) *is a solution of equation* (10), *then the corresponding function $\Phi$* (cf. (8), (9)) *satisfies equation* (11). *If a function $\Phi$ fulfilling condition* (9) *is a solution of equation* (11), *then the function $\varphi$ given by* (8) *satisfies equation* (10).

We omit a simple proof of this remark.

LEMMA 1. *If there exists a formal solution of equation* (11) *of the form* (9) *then one of the following conditions holds*:

$$1^\circ\ ar = p = aq, \qquad 3^\circ\ aq = p < ar,$$
$$2^\circ\ ar = aq < p, \qquad 4^\circ\ ar = p < aq,$$

*and $a = \dfrac{p}{q}$ in the cases $1^\circ$, $3^\circ$, and $a$ is an arbitrary positive integer belonging to the interval $\left[1, \dfrac{p}{q}\right)$ in the case $2^\circ$, $a = \dfrac{p}{r}$ in the case $4^\circ$.*

P r o o f. Suppose that for a positive integer $a$ equation (11) has a formal solution $\Phi$ fulfilling condition (9), and suppose that e.g. $ar < p \leqslant \leqslant aq$ holds. We get from (11)

$$F(x)^a\, \Phi(f(x)) = x^{p-ra} u(x) + x^{qa-ra}\, \Phi(x)^q\, v(x^a\, \Phi(x)),$$

and for $x = 0$ we obtain $F(0)^a\, \eta = 0$ which contradicts (3) and (9). The second part of our assertion results from the form of conditions $1^\circ$—$4^\circ$.

**2.** $r = q$.

THEOREM 1. (i) *For an arbitrary integer* $a \in \left[1, \frac{p}{q}\right)$ *and for every* $\eta \in \mathbf{C}\setminus\{0\}$ *satisfying the equation*

$$\eta^{q-1} v(0) = F(0)^{a}$$

*equation* (10) *has locally exactly one analytic solution* $\varphi$ *of the form*

$$\varphi(x) = x^{a} \Phi(x),$$

*where* $\Phi$ *is an analytic function in a neighbourhood of zero and such that* $\Phi(0) = \eta$.

(ii) *Moreover, if* $\frac{p}{q} \in \mathbf{N}$ *then for every* $\eta \in \mathbf{C}\setminus\{0\}$ *satisfying the equation*

$$\eta^{q} v(0) - F(0)^{\frac{p}{q}} \eta + u(0) = 0$$

*equation* (10) *has locally exactly one analytic solution* $\varphi$ *of the form*

$$\varphi(x) = x^{\frac{p}{q}} \Phi(x),$$

*where* $\Phi$ *is an analytic function in a neighbourhood of zero and such that* $\Phi(0) = \eta$.

(iii) *The equation* (10) *has no other solutions.*

Proof. If $\frac{p}{q} \in \mathbf{N}$ and $a := \frac{p}{q}$ then $ar = aq = p$, and we get case 1° of Lemma 1. Equation (11) is now of the form

$$F(x)^{\frac{p}{q}} \Phi(f(x)) = u(x) + \Phi(x)^{q} v(x^{\frac{p}{q}} \Phi(x)), \tag{12}$$

and for $x = 0$ we obtain

$$F(0)^{\frac{q}{p}} \eta = u(0) + \eta^{q} v(0). \tag{13}$$

Let

$$H(x, y, z) := u(x) - F(x)^{\frac{p}{q}} z + y^{q} v(x^{\frac{p}{q}} y).$$

Then equation (12) is of the form $H(x, \Phi(f(x)), \Phi(x)) = 0$. By (3), (9), (12) and the condition $f(0) = 0$ we obtain the equality

$$H(0, \eta, \eta) = u(0) - F(0)^{\frac{p}{q}} \eta + \eta^{q} v(0) = 0,$$

as a necessary condition of the existence of a solution of equation (12). Since

$$\frac{\partial H}{\partial y}(0, \eta, \eta) = q\eta^{q-1} v(0) \neq 0$$

according to conditions (7) and (9), by means of the implicit function theorem, there exists a neighbourhood of the points $(0, \eta_1), \ldots, (0, \eta_q)$ where

$\eta_1, \ldots, \eta_q$ are the roots of equation (13) in which equation (12) may equivalently be written in the form $\Phi_i(x) = K_i(x, \Phi_i(f(x)))$, where $K_i$ denote certain analytic functions in this neighbourhood and fulfilling the conditions $K_i(0, \eta_i) = \eta_i$, for $i = 1, 2, \ldots, q$. Now, part (ii) of our assertion results from W. Smajdor's Theorem. If $a < \frac{p}{q}$ then $aq = ar < p$, and we get the case 2° of Lemma 1. Let $a \in \left[1, \frac{p}{q}\right)$ be an arbitrary integer. Equation (11) is now of the form

$$F(x)^a \Phi(f(x)) = x^{p-ar} u(x) + \Phi(x)^q v(x^a \Phi(x)), \tag{14}$$

and from the fact $ar < p$ we have for $x = 0$

$$F(0)^a \eta = \eta^q v(0).$$

From (9) we obtain

$$\eta^{q-1} v(0) = F(0)^a. \tag{15}$$

Let

$$H(x, y, z) := x^{p-ar} u(x) + y^q v(x^a y) - F(x)^a z.$$

If $\eta \in \mathbf{C} \setminus \{0\}$ is a solution of equation (15) then $H(0, \eta, \eta) = 0$ and, in view of (7) and (9),

$$\frac{\partial H}{\partial y}(0, \eta, \eta) = q\eta^{q-1} v(0) \neq 0.$$

The implicit function theorem may be applied to equation (14) and it suffices, as previously, to apply W. Smajdor's Theorem. Consequently, the proof of point (i) of our assertion is finished. If $a > \frac{p}{q}$ then $p < aq = ar$. Lemma 1 implies that in this case equation (11) has no formal solutions, and this completes the proof.

3. $r > q$.

THEOREM 2. *If* $\frac{p}{q} \in \mathbf{N}$ *then equation* (11) *has locally exactly* $q$ *analytic solutions. More precisely, for every* $\eta \in \mathbf{C} \setminus \{0\}$ *satisfying the equation*

$$\eta^q v(0) + u(0) = 0$$

*there exists locally one analytic solution* $\varphi$ *of equation* (10). *This solution is of the form*

$$\varphi(x) = x^{\frac{p}{q}} \Phi(x),$$

*where* $\Phi$ *is an analytic function in a neighbourhood of zero and such that* $\Phi(0) = \eta$.

The proof of this theorem is the same as that of Theorem 1 and so, we omit it.

4. $r < q$.

If equation (10) has a formal solution, then Lemma 1 implies that $\alpha r = p < \alpha q$. If $\frac{p}{r} \in \mathrm{N}$, then $\alpha = \frac{p}{r}$. Equation (11) is then of the form

$$F(x)^{\frac{p}{r}}\, \Phi(f(x)) = u(x) + x^{\frac{p}{r}(q-r)}\, \Phi(x)^q\, v(x^{\frac{p}{r}}\, \Phi(x)). \tag{16}$$

Let

$$\frac{p}{r} =: a, \text{ and } s := a\cdot(q-r).$$

The condition $F(0) \neq 0$ implies that there exists a $\bar{\varrho} > 0$ such that $F(x) \neq 0$ whenever $|x| < \bar{\varrho}$. Equation (16) may be written in the form

$$\Phi(f(x)) - \frac{u(x)}{F(x)^a} = x^s\, \Phi(x)^q\, \frac{v(x^a(\Phi(x))}{F(x)^a}. \tag{17}$$

We may take $\bar{\varrho}$ so small that $\bar{\varrho} < \varrho_1$. The function

$$h(x) := \frac{u(x)}{F(x)^a} \tag{18}$$

is analytic for $|x| < \bar{\varrho}$ and satisfies

$$h(0) = \frac{u(0)}{F(0)^a}. \tag{19}$$

Setting (18) into equation (17) we obtain

$$\Phi(f(x)) - h(x) = x^s\, \Phi(x)^q\, \frac{v(x^a(\Phi(x))}{F(x)^a}. \tag{20}$$

For $|x| < \bar{\varrho}$ equation (20) is equivalent to (16). Observe that $s \geqslant 1$. If $\Phi$ is a solution of equation (20) then $\Phi(0) - h(0) = 0$.

From conditions (9), (19) and (2), (4) we get

REMARK 2. *If $r < q$ then*

$$\eta = \frac{u(0)}{F(0)^a} \tag{21}$$

*yields a necessary condition of the existence of a solution of equation* (20).

Suppose that

$$h(x) = \eta + \sum_{n=1}^{\infty} c_n x^n,\quad f(x) = x^{r-1} \sum_{n=1}^{\infty} b_n x^n \text{ for } |x| < \bar{\varrho} \tag{22}$$

and

$$\Phi(x) = \eta + \sum_{n=1}^{\infty} \eta_n x^n, \tag{23}$$

where $\eta$ is given by (21). The function $\Phi(f(x))-h(x)$ may be written in the form

$$\Phi(f(x))-h(x) =$$
$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\sum_{l_1+\dots+l_k=n}\eta_k b_{l_1}\cdot\ldots\cdot b_{l_k}-c_{n+r-1}\right)x^{n+r-1}-\sum_{n=1}^{\infty}c_n x^n.$$

Equation (20) may be written in the form

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\sum_{l_1+\dots+l_k=n}\eta_k b_{l_1}\cdot\ldots\cdot b_{l_k}-c_{n+r-1}\right)x^{n+r-1}-\sum_{n=1}^{\infty}c_n x^n = \tag{24}$$
$$= x^s\,\Phi(x)^q\,\frac{v(x^a\Phi(x))}{F(x)^a}.$$

Put

$$\Phi(x) = \eta+x^\beta\,\psi(x), \tag{25}$$

where $\beta\in\mathbf{N}$, $\eta$ fulfils (21); $\psi$ is an analytic function in a neighbourhood of zero and

$$\psi(0) = \bar{\eta}\neq 0. \tag{26}$$

Putting (25) and (26) into (20) we get

$$x^{r\beta}\,F(x)^\beta\,\psi(f(x))-h(x)+\eta = x^s(\eta+x^\beta\,\psi(x))^q\,\frac{v(\eta+x^\beta\,\psi(x))\cdot x^a}{F(x)^a}. \tag{27}$$

Let

$$\bar{x} := (\eta+x^\beta\,\psi(x))\cdot x^a,\ \ v(x) := \sum_{n=0}^{\infty}v_n x^n \tag{28}$$

and for $x=\bar{x}$ we have

$$v(\bar{x}) = \sum_{n=0}^{\infty}v_n x^{na}\,(\eta+x^\beta\,\psi(x))^n.$$

Equation (27) is now of the form

$$x^{r\beta}\,F(x)^\beta\,\psi(f(x))-h(x)+\eta = x^s\,(\eta+x^\beta\,\psi(x))^q\,v(\bar{x}). \tag{29}$$

Let us consider the following possibilities regarding the exponents $s$ and $r$:
(i) $s<r$; (ii) $s>r$; (iii) $s=r$.
Suppose that case (i) is satisfied. From (24) and (i) we obtain.

REMARK 3. *Conditions* (21) *and*

$$\eta^q\,\frac{v(0)}{F(0)^a}+c_s = 0,\ c_1 = c_2 = \ldots = c_{s-1} = 0 \tag{30}$$

*are necessary for the existence of a solution of equation* (20) *in the case* $s<r$.

The proof of this Remark results easily from (24) and (i).

Let

$$x^t\,T(x) := F(x)^a \sum_{n=s}^{\infty} c_n x^{n-s} - c_s F(0)^a + \sum_{n=1}^{\infty} v_n \eta^{n+q}\, x^{n+a} + [F(0)^a\, c_s + \eta^q\, v(0)]$$

where $T(0) \neq 0$. Equation (27) is now of the form

$$x^{r\beta-s}\, F(x)^{\beta+a}\, \psi(f(x)) =$$

$$= x^t\, T(x) + x^\beta\, \psi(x) \left[ \sum_{i=1}^{q} \binom{q}{i} \eta^{q-i}\, x^{(i-1)}\, \psi(x)^{i-1} \right] v(\bar{x}) + \tag{31}$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{n} v_n \binom{n}{k} \eta^{n-k+q}\, x^{\beta k+na}\, \psi(x)^k.$$

We have the following

THEOREM 3. *Let us suppose that conditions* (21) *and* (30) *are fulfilled and* $s < r$. *Equation* (10) *has only the following solutions:*

(I) *if* $s = r-1$, $t = 1$, $r \geqslant 2$ and

$$F(0)^{1+a} \neq q\eta^{q-1}\, v(0),$$

*then*

$$\varphi(x) = x^a\, (\eta + x\psi(x))$$

*where* $\psi$ *is an analytic function in a neighbourbood of zero and*

$$\psi(0) = \bar{\eta} = T(0)\, (F(0)^{1+a} - q\eta^{q-1}\, v(0))^{-1};$$

(II) *if* $s = r-1$, $t > 1$ *and* $r \geqslant 2$ *and*

$$F(0)^{t+a} = q\eta^{q-1}\, v(0),$$

*then for every* $\bar{\eta} \in \mathbf{C}\backslash\{0\}$

$$\varphi_{\bar{\eta}}(x) = x^a\, (\eta + x\psi_{\bar{\eta}}(x)),$$

*where* $\psi_{\bar{\eta}}$ *is an analytic function in a neighbourhood of zero and* $\psi_{\bar{\eta}}(0) = \bar{\eta} \in \mathbf{C}\backslash\{0\}$ *is an arbitrary constant;*

(III) *if* $r \geqslant 2, t > \dfrac{s}{r-1}$ *then*

$$\varphi(x) = x^a\, (\eta + x^t\, \psi(x)),$$

*where* $\psi$ *is an analytic function in a neighbourhood of zero and*

$$\psi(0) = \bar{\eta} = -T(0)\, (q\eta^{q-1}\, v(0))^{-1}.$$

*In the remaining cases equation* (10) *has no formal solutions.*

Proof. We observe that for every $n, k \in \mathbf{N}$ we have $\beta < k\beta + na$. As a result of equating the orders of zeros at the origin of the functions on both sides of equation (31) (after eliminating those of the inequalities obtained which have no solution in the set of positive integers, and which lead to a contradiction) we get:

$1^\circ$ $r\beta - s = t = \beta$, $2^\circ$ $t = \beta < r\beta - s$, $3^\circ$ $r\beta - s = \beta < t$.

In the case $1^\circ$ we have $s = r-1$, $\beta = 1$, $t = 1$; in the case $2^\circ$ we have $\beta = 1$, $s = r-1$, $t > 1$; in the case $3^\circ$ we have $\beta = t$, $t > \frac{s}{r-1}$.

Let us suppose that $s = r-1$, $t = 1$. Equation (31) is of the form

$$F(x)^{1+a}\psi(f(x)) = T(x) + \psi(x)\left[\sum_{i=1}^{q}\binom{q}{i}\eta^{q-i}x^{(i-1)\beta}\psi(x)^{(i-1)}\right]v(\bar{x}) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} v_n\binom{n}{k}\eta^{n-k+q}x^{k+na-1}\psi(x)^k. \tag{32}$$

Putting $x = 0$ into (32) we have

$$\psi(0)\,[F(0)^{1+a} - q\eta^{(q-1)}v(0)] = T(0).$$

If $F(0)^{1+a} \neq q\eta^{(q-1)}v(0)$ then

$$\psi(0) = \bar{\eta} = T(0)\,[F(0)^{1+a} - q\eta^{(q-1)}v(0)]^{-1}. \tag{33}$$

Let

$$H(x, y, z) := F(x)^{1+a}y - T(x) - z\left[\sum_{i=1}^{q}\binom{q}{i}\eta^{q-1}x^{(i-1)}z^{i-1}\right]v(\bar{x}) - \sum_{n=1}^{\infty}\sum_{k=1}^{n} v_n\binom{n}{k}\eta^{n-k+q}x^{k+na-1}z^k.$$

Then equation (32) is of the form $H(x, \psi(f(x)), \psi(x)) = 0$, and by (33) we obtain $\frac{\partial H}{\partial z}(0, \bar{\eta}, \bar{\eta}) = q\eta^{q-1}v(0) \neq 0$. There exists a neighbourhood of the point $(0, \bar{\eta})$ in which equation (32) may equivalently be written in the form $\psi(x) = K(x, \psi(f(x)))$ where $K$ denotes a certain analytic function in that neighbourhood fulfilling the condition $K(0, \bar{\eta}) = \bar{\eta}$. Now, our assertion results from W. Smajdor's Theorem. The proof of the parts (II) and (III) is the same.

Now, consider cases (ii) and (iii) i.e. $s \geqslant r$. The following remarks result from (24):

REMARK 4. *If the function $\Phi$ given by* (23) *is a formal solution of equation* (20) *then the following conditions holds*:

$$
(\mathrm{H})\begin{cases}
\eta = \dfrac{(0)n}{F(0)^a} \\
c_1 = c_2 = \dots = c_{r-1} = 0, \\
\textit{There exist numbers } \eta_1, \dots, \eta_{s+1-r} \textit{ then for every } i \in \{1, 2, \dots, s-r\} \\
\sum_{k=1}^{i} \sum_{l_1+\dots+l_k=i} \eta_k b_{l_1} \cdot \dots \cdot b_{l_k} = c_{i+r-1}, \\
\sum_{k=1}^{s-r+1} \sum_{l_1+\dots+l_k=s-r+1} \eta_k b_{l_1} \cdot \dots \cdot b_{l_k} - c_s = \eta^a \dfrac{u(0)}{F(0)^a}.
\end{cases}
$$

*Conditions* (H) *are necessary for the existence of an analytic solution of equation* (20) *in the case* (ii) ($s > r$).

REMARK 5. *If the function* $\Phi$ *given by* (23) *is a formal solution of equation* (20) *then the following conditions holds*:

$$
(\mathrm{H}')\begin{cases}
\eta = \dfrac{u(0)}{F(0)^a} \\
c_1 = c_2 = \dots = c_{s-1} = 0, \\
\textit{There exists a number } \eta_1 \textit{ such that} \\
\eta_1 F(0) - c_s = \eta^a \dfrac{u(0)}{F(0)^a}.
\end{cases}
$$

*Conditions* (H′) *are necessary for the existence of an analytic solution of equation* (20) *in the case* (iii) ($s = r$).

Let us suppose that $\eta, \eta_1, \dots, \eta_{s-r+1}$ fulfil conditions (H) or (H′). Then the function $\Phi$ may be written in the form

$$\Phi(x) = R(x) + x^{s-r+2+\delta}\, \psi(x), \tag{35}$$

where

$$R(x) = \sum_{i=0}^{s-r+1} \eta_i x^i,$$

$\psi$ is an analytic function in a neighbourhood of zero and $\psi(0) = \xi \neq 0$, and $\delta \in \mathbf{N} \cup \{0\}$. Let

$$\lambda := s + 2 - r + \delta. \tag{36}$$

Putting $x^a\,(R(x) + x^\lambda\, \psi(x))$ instead of $x$ in (28) we get

$$
\begin{aligned}
& v(x^a(R(x) + x^\lambda\, \psi(x)) = \\
& = v_0 + \sum_{n=1}^{\infty} v_n \sum_{k=1}^{n} \binom{n}{k} R(x)^{n-k} x^{k\lambda + an}\, \psi(x)^k + \sum_{n=1}^{\infty} v_n\, R(x)^n\, x^{an}.
\end{aligned} \tag{37}
$$

Setting (35), (36) and (37) to equation (20) we get

$$x^{r\lambda} F(x)^{a+\lambda} \psi(f(x)) =$$

(38)
$$= x^s \left[\sum_{k=1}^{q} \binom{q}{k} R(x)^{q-k} x^{k\lambda} \psi(x)^k\right] \cdot v\,(x^a (R(x)+x^\lambda \psi(x)))+$$
$$+ x^s R(x)^q \sum_{n=1}^{\infty} v_n R(x)^n x^{an} + x^s \cdot v_0 \cdot R(x)^q - (F(x)^a R(x^r F(x)) - u(x)).$$

Note that (H) and (H′) imply the existence of an $m \in \mathbf{N}$ and an analytic function $M$ such that

$$x^m M(x) := R(x)^q \sum_{n=1}^{\infty} v_n R(x)^n x^{an} + x^z Z(x),$$

where

$$x^{s+z} Z(x) := -F(x)^a R(x^r F(x)) + u(x) + x^s \cdot v_0 \cdot R(x)^q,$$

and $z \in \mathbf{N}$, $M(0) \neq 0$, $Z(0) \neq 0$. Equation (38) is of the form

(39)
$$x^{r\lambda - s} F(x)^{a+\lambda} \psi(f(x)) = \left[\sum_{k=1}^{q} \binom{q}{k} R(x)^{q-k} x^{k\lambda} \psi(x)^k\right] v(x^a (R(x)+x^\lambda \psi(x)))+$$
$$+ R(x)^q \sum_{n=1}^{\infty} v_n \sum_{k=1}^{n} \binom{n}{k} R(x)^{n-k} x^{k\lambda+an} \psi(x)^k + x^m M(x).$$

For every $k, n \in \mathbf{N}$ we have $k\lambda + an > \lambda$ as well as for every $\lambda \in \mathbf{N}$, $r \in [2, s] \cap \mathbf{N}$ one has $\lambda r - s \geqslant \lambda$.

REMARK 6. $\lambda = \min \{k\lambda + an,\ \lambda,\ 2\lambda, \ldots, q\lambda,\ \lambda r - s : k, n \in \mathbf{N},\ r \in [2, s]\}$.
According to Remark 6, we get the following possible equalities regarding the exponents $\lambda$, $\lambda r - s$, $m$, $2\lambda, \ldots, q\lambda$, $k\lambda + an$:
(a) $\lambda = \lambda r - s = m$, (b) $\lambda = \lambda r - s$, (c) $\lambda = m$.
The remaining possibilities regarding these exponents lead to a contradiction. In case (a) we have $m = \lambda = \frac{s}{r-1}$; $\delta = m - s - 2 + r$. In that case equation (39) is of the form

$$F(x)^{a+m} \psi(f(x)) = \left[\sum_{k=1}^{q} \binom{q}{k} R(x)^{q-k} x^{(k-1)m} \psi(x)^k\right] v(x^a (R(x)+x^m \psi(x)))+$$
$$+ R(x)^q \sum_{n=1}^{\infty} v_n \sum_{k=1}^{n} \binom{n}{k} R(x)^{n-k} x^{(k-1)m+an} \psi(x)^k + M(x),$$

hence, for $x = 0$, we obtain

$$F(0)^{a+m} \xi = q\eta^{q-1} v(0) \xi + M(0),$$

and if

$$H(x, y, z) := \left[\sum_{k=1}^{q} \binom{q}{k} R(x)^{q-k} x^{(k-1)m} z^k\right] v(x^a (R(x)+x^m z)) - F(x)^{a+m} \cdot y +$$

$$+ R(x)^q \sum_{n=1}^{\infty} v_n \sum_{k=1}^{n} \binom{n}{k} R(x)^{n-k} x^{(k-1)m+an} z^k + M(x),$$

then

$$\frac{\partial H}{\partial z}(0, \xi, \xi) = q\eta^{q-1} v(0) \neq 0.$$

We have the following

THEOREM 4. *Let us suppose that $s > r$ and conditions* (H) *are fulfilled. Equation* (10) *has only the following solutions:*

(I) *if* $\frac{s}{r-1} \in \mathbf{N}$, $\frac{s}{r-1} = m$ *and if*

$$F(0)^{a+m} \neq q\eta^{q-1} v(0),$$

*then there exists locally exactly one analytic solution of equation* (10). *This solution is of the form*

$$\varphi(x) = x^a (R(x) + x^m \psi(x)),$$

*where $\psi$ is an analytic function in a neighbourhood of zero and such that*

$$\psi(0) = M(0) (F(0)^{a+m} - q\eta^{q-1} v(0))^{-1};$$

(II) *if* $\frac{s}{r-1} \in \mathbf{N}$ *and if* $\frac{s}{r-1} < m$ *and*

$$F(0)^{a+\frac{s}{r-1}} = q\eta^{q-1} v(0),$$

*then for every* $\bar{\eta} \in \mathbf{C} \setminus \{0\}$ *there exists locally exactly one analytic solution of equation* (10). *This solution is of the form*

$$\varphi_{\bar{\eta}}(x) = x^a \left(R(x) + x^{\frac{s}{r-1}} \psi_{\bar{\eta}}(x)\right),$$

*where $\psi_{\bar{\eta}}$ is an analytic function in a neighbourhood of zero and such that* $\psi_{\bar{\eta}}(0) = \bar{\eta} \neq 0$;

(III) *if* $\frac{s}{r-1} \in \mathbf{N}$ *and* $\frac{s}{r-1} > m$, *then there exists locally exactly one analytic solution of equation* (10). *This solution is of the form*

$$\varphi(x) = x^a (R(x) + x^m \psi(x)),$$

*where $\psi$ is an analytic function in a neighbourhood of zero and such that*

$$\psi(0) = -M(0) (q\eta^{q-1} v(0))^{-1}.$$

*In the remaining cases equation* (10) *has no formal solution.*

THEOREM 5. *Let us suppose that* $s = r$ *and conditions* (H′) *are fulfilled. Equation* (10) *has only the following solutions*:
(I) *if* $s = r = m = 2$ *and if* $F(0)^{a+2} \neq q\eta^{q-1} v(0)$, *then there exists locally exactly one analytic solution of equation* (10). *This solution is of the form*

$$\varphi(x) = x^a (\eta + \eta_1 x + x^2 \psi(x)),$$

*where*

$$\psi(0) = M(0) (F(0)^{a+2} - q\eta^{q-1} v(0))^{-1};$$

(II) *if* $s = r = 2 < m$, *and if* $F(0)^{a+1} = q\eta^{q-1} v(0)$, *then for every* $\bar{\eta} \in \mathbf{C} \setminus \{0\}$ *there exists locally exactly one analytic solution of equation* (10). *This solution is of the form*

$$\varphi_{\bar{\eta}}(x) = x^a (\eta + \eta_1 x + x^2 \psi_{\bar{\eta}}(x))$$

*where* $\psi_{\bar{\eta}}(0) = \bar{\eta} \neq 0$;
(III) *if* $r = s$, $m > 1$, *then there exists locally exactly one analytic solution of equation* (10). *This solution is of the form*

$$\varphi(x) = x^a (\eta + \eta_1 x + x^m \psi(x)),$$

*where* $\psi(0) = -M(0) (qn^{q-1} v(0))^{-1}$.
*In the remaining cases equation* (10) *has no formal solution.*

We omit proofs of Theorems 4 and 5.

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