KATARZYNA JAKOWSKA-SUWALSKA

ON DEPENDENCE OF LIPSCHITZIAN SOLUTION OF NON-LINEAR FUNCTIONAL EQUATION ON AN ARBITRARY FUNCTION

Abstract. We shall deal with the existence and dependence on an arbitrary function of solutions of the functional equation

(1) $$\varphi(f(x)) = g(x, \varphi(x))$$

in the class of functions fulfilling a Lipschitz condition.

Let f, g be given real valued functions of real variables defined (resp.) in an interval I, and a region $I \times R$, and let $\varphi : I \to R$ be an unknown function. The functions f, g are subjected to the following conditions:

(i) f is continuous and strictly increasing in the interval $I = [0, a]$, $0 < a < \infty$. Moreover $0 < f(x) < x$ in $(0, a]$.

(ii) g is defined in a region $I \times R$.

Hypothesis (i) implies that $f(0) = 0, f(I_1) \subset I_1$ for every interval $I_1 \subset I$, such that $0 \in I_1$ and $\lim_{n \to \infty} f^n(x) = 0$ for every $x \in I$ (cf. [1], p. 20). Here f^n denotes the n-th iteration of the function f. The symbol $Lip(I)$ denotes the set of all functions of real variable fulfilling a Lipschitz condition in the interval I. We adopt the following convention

$$\sum_{i=k}^{k-1} a_i = 0, \; k = 0, 1, \ldots$$

Let $I_i = (f^{i+1}(a), f^i(a)]$ for $i > 0$, $I_0 = [f(a), a]$; then $\bigcup_{i=0}^{\infty} I_i = (0, a]$.

LEMMA. *Let $\varphi_i : I_i \to R$ be given functions such that*

$$|\varphi_i(x) - \varphi_i(\bar{x})| \leqslant v_i |x - \bar{x}| \text{ for } x, \bar{x} \in I_i, i = 0, 1, \ldots$$

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Suppose that $v_i \leqslant v < \infty$, $i = 0, 1, \ldots$, *and the function* $F : (0, a] \to R$ *given by the formula* $F(x) = \varphi_i(x)$, $x \in I_i$, $i = 0, 1, \ldots$ *is correctly defined and continuous. Then*

$$|F(x) - F(\bar{x})| \leqslant v|x - \bar{x}| \text{ for } x, \bar{x} \in (0, a].$$

Moreover, if we put $F(0) := \lim_{x \to 0^+} F(x)$ *then the latter condition holds for all* $x, \bar{x} \in [0, a]$.

THEOREM. *If hypotheses* (i), (ii) *are fulfilled and there exist positive numbers* l, k, s *such that*

$$|g(x, y) - g(\bar{x}, \bar{y})| \leqslant k|x - \bar{x}| + l|y - \bar{y}|, \; x, \bar{x} \in I, \; y, \bar{y} \in R, \tag{2}$$

$$|f^{-1}(x) - f^{-1}(\bar{x})| \leqslant s|x - \bar{x}|, \; x, \bar{x} \in [0, f(a)], \tag{3}$$

and

$$ls < 1, \tag{4}$$

then for arbitrary function $\varphi_0 : I_0 \to R$ *fulfilling the conditions*

$$\varphi_0(f(a)) = g(a, \varphi_0(a)), \; \varphi_0 \in Lip\,(I_0) \tag{5}$$

there exists exactly one function φ *belonging to the class* $Lip(I)$ *satisfying equation* (1) *and fulfilling the condition* $\varphi(x) = \varphi_0$ *for* $x \in I_0$. *Every Lipschizian solution* φ *of equation* (1) *satisfies the condition* $\varphi(0) = \eta$ *where* η *is the only solution of the equation* $\eta = g(0, \eta)$.

Proof. Let $\varphi_0 \in Lip(I_0)$ be a function fulfilling the condition (5). Hence there is a constant $M > 0$ such that

$$|\varphi_0(x) - \varphi_0(\bar{x})| \leqslant M|x - \bar{x}| \text{ for } x, \bar{x} \in I_0. \tag{6}$$

We define a sequence of functions $\{\varphi_i\}$, $i \in \mathbf{N}$, where

$$\varphi_i(x) = g(f^{-1}(x), \varphi_{i-1}(f^{-1}(x))) \text{ for } x \in I_i, \; i \geqslant 1. \tag{7}$$

Let $\bar{\varphi}(x) := \varphi_i(x)$ for $x \in I_i$, $i \geqslant 0$. The function $\bar{\varphi}$ is a solution of equation (1) in $(0, a]$ because

$$\bigwedge_{x \in (0, a]} \bigvee_{k \geqslant 1} \bar{\varphi}(f(x)) = \varphi_k(f(x)) = g(x, \varphi_{k-1}(x)) = g(x, \bar{\varphi}(x)).$$

Now we shall prove that $\bar{\varphi}$ is continuous in $(0, a]$. The function $\bar{\varphi}$ is continuous in every interval $(f^{i+1}(a), f^i(a))$ and we show that

$$\bigwedge_{i \geqslant 1} \lim_{x \to f^i(a)} \bar{\varphi}(x) = \bar{\varphi}(f^i(a)). \tag{8}$$

For $i = 1$ we have

$$\lim_{x \to f(a)^+} \bar{\varphi}(x) = \lim_{x \to f(a)^+} \varphi_0(x) = \varphi_0(f(a)) = g(a, \varphi_0(a)) = \bar{\varphi}(f(a)),$$

$$\lim_{x \to f(a)^-} \bar{\varphi}(x) = \lim_{x \to f(a)^-} \varphi_1(x) = \lim_{x \to f(a)^-} g(f^{-1}(x), \varphi_0(f^{-1}(x))) =$$

$$= g(a, \varphi_0(a)) = \bar{\varphi}(f(a)).$$

Thus (8) holds for $i = 1$. Supposing that (8) is valid for $i = n$ we have by the continuity of f^{-1}, g that $\bar{\varphi}$ is continuous at the point $f^{n+1}(a)$. Induction completes the proof of (8).

Induction leads to the following inequality

$$(9) \qquad |\bar{\varphi}(x)-\bar{\varphi}(\bar{x})| \leqslant \left(ks \sum_{p=0}^{i-1} (ls)^p + (ls)^i M\right) |x-\bar{x}| \text{ for } x, \bar{x} \in I_i, i \geqslant 0.$$

Indeed, for $i = 0$ and $x, \bar{x} \in I_0$ we have by (6)

$$|\bar{\varphi}(x)-\bar{\varphi}(\bar{x})| = |\varphi_0(x)-\varphi_0(\bar{x})| \leqslant M\,|x-\bar{x}|.$$

Thus (9) holds for $i = 0$. Suppose that (9) is valid for $i = n$. Hence, by (2), (3) and (7), we obtain for $x, \bar{x} \in I_{n+1}$

$$|\bar{\varphi}(x)-\bar{\varphi}(\bar{x})| = |\varphi_{n+1}(x)-\varphi_{n+1}(\bar{x})| =$$
$$= |g(f^{-1}(x), \varphi_n(f^{-1}(x)))-g(f^{-1}(\bar{x}), \varphi_n(f^{-1}(\bar{x})))| \leqslant$$
$$\leqslant ks|x-\bar{x}|+l|\varphi_n(f^{-1}(x)-\varphi_n(f^{-1}(\bar{x}))| \leqslant$$
$$\leqslant \left(ks+ls\left(ks\sum_{p=0}^{n-1}(ls)^p+(ls)^n M\right)\right)|x-\bar{x}| = \left(ks\sum_{p=0}^{n}(ls)^p+(ls)^{n+1} M\right)|x-\bar{x}|,$$

and induction completes the proof of (9).

Put

$$v_i = ks \sum_{p=0}^{i-1} (ls)^p + (ls)^i M, \text{ for } i \geqslant 0$$

and

$$v = \frac{ks}{1-ls} + M.$$

It follows from (4) that $v_i \leqslant v$, $i \geqslant 0$, and from our lemma

$$|\bar{\varphi}(x)-\bar{\varphi}(\bar{x})| \leqslant v|x-\bar{x}| \text{ for } x, \bar{x} \in (0, a].$$

Setting in (2) $x = \bar{x} = 0$ we see that

$$\bigwedge_{y, \bar{y} \in R} |g(0, y)-g(0, \bar{y})| \leqslant l|y-\bar{y}|.$$

By (i) we have $s > 1$ and $l < 1$. Applying Banach's principle we obtain the existence of exactly one point $\eta \in R$ such that

$$\eta = g(0, \eta).$$

Define $\eta_\varphi = \lim_{x \to 0^+} \bar{\varphi}(x)$. For $x \in (0, a]$ we have $\bar{\varphi}(x) = g(f^{-1}(x), \bar{\varphi}(f^{-1}(x)))$ and

$$\eta_\varphi = \lim_{x \to 0^+} \bar{\varphi}(x) = \lim_{x \to 0^+} g(f^{-1}(x), \bar{\varphi}(f^{-1}(x))) = g(0, \eta_\varphi).$$

Therefore $\eta_\varphi = \eta$.

Now we shall prove that φ defined by the formula

$$\varphi(x) = \begin{cases} \bar{\varphi}(x), & \text{if } x \in (0, a] \\ \eta, & \text{if } x = 0 \end{cases} \tag{10}$$

for given φ_0 is the unique solution of equation (1). Suppose that ψ_1, ψ_2 are solutions of (1) such that

$$\psi_i(x) = \begin{cases} \varphi_0(x), & \text{if } x \in [f(a), a] \\ \bar{\psi}_i(x), & \text{if } x \in (0, f(a)] \\ \eta, & \text{if } x = 0, \end{cases} \tag{11}$$

$\bar{\psi}_1 \not\equiv \bar{\psi}_2$ and

$$\bar{\psi}_i(x) := g(f^{-1}(x), \bar{\psi}_i(f^{-1}(x))), \quad i = 1, 2, \quad x \in (0, f(a)]. \tag{12}$$

Let $x \in I_j$, $j = 1, 2, \ldots$ We have $f^{-j}(x) \in I_0$ and

$$\begin{aligned} |\psi_1(x) - \psi_2(x)| &= |g(f^{-1}(x), \psi_1(f^{-1}(x))) - g(f^{-1}(x), \psi_2(f^{-1}(x)))| \leqslant \\ &\leqslant l|\psi_1(f^{-1}(x)) - \psi_2(f^{-1}(x))| \leqslant \ldots \leqslant l^j|\psi_1(f^{-j}(x)) - \psi_2(f^{-j}(x))| = \\ &= l^j|\varphi_0(f^{-j}(x)) - \varphi_0(f^{-j}(x))| = 0. \end{aligned}$$

It follows that $\psi_1(x) = \psi_2(x)$ for $x \in I$, and this completes the proof.

REFERENCES

[1] M. KUCZMA, *Functional equations in a single variable*, Monografie Mat. 46, PWN, Warszawa 1968.

[2] J. MATKOWSKI, *On Lipschitzian solution of a functional equation*, Ann. Polon. Math. 28 (1973), 135—139.