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ON A CERTAIN EXTENSIONS OF LINEAR SPACES WITH AN ALGEBRAIC DERIVATION

Abstract. Certain properties of solutions of ordinary linear differential equations have an algebraic character. In this note we are concerned with these properties. Our purpose is to show some connections between the theory of differential rings and the theory of linear spaces equipped with an endomorphism satisfying some additional conditions. It will be shown that some linear spaces may be extended to differential rings.

0. Let V be a commutative ring with an unit element e. Assume that V has no proper divisors of zero. By D we shall denote a transformation in the ring V satisfying the following conditions:

$$(a) \qquad D(x+y) = Dx+Dy$$

$$(\delta) \qquad D(xy) = Dxy+xDy.$$

The operation D is said to be an *algebraic derivative*. The ring V is called a *differential ring*. The elements of V satisfying the equation $Dx = 0$ will be called constants. The unit element and the zero element are constants. The set of constants constitutes a subring $F \subset V$. In many cases F is a field. Very important for applicatios of the theory is the case, when the algebraic derivative D satisfies the following additional condition:

(γ) $Dxy - xDy = 0$ implies that x and y are linearly dependent over F.

By $F[t]$ we shall denote the set of all polynomials of one variable with coefficients from the field F.

Let us consider the equation

$$(0.1) \qquad p(D)\,x = 0,$$

where p is in $F[t]$.

Received July 07, 1980.
AMS (MOS) subject classification (1980). Primary 46E10.

Let $N_{p(D)}$ denote the set of all solutions of equation (0.1) and let $\dim N_{p(D)}$ denote the dimension of $N_{p(D)}$. J. Mikusiński has observed that if an algebraic derivative D satisfies the condition (γ) then

$$\text{(i)} \qquad \dim N_{p(D)} \leqslant n,$$

where n means the degree of p ([5]).

Assume that the ring V has an element t such that $Dt = e$. By T it will be denoted a new operation in V given by the formula $Tx = tx$. It is easy to see that

$$(\tau) \qquad DT - TD = I,$$

where I denotes the identity transformation in V. If F is a field then the ring V can be interpreted as an algebra over F. D and T are linear transformations in V.

It is known that if a linear transformation D mapping a linear space V over a field F with the characteristic zero into itself fulfils the condition (i) and there exists a linear transformation T satisfying the above condition (τ), then

$$\text{(ii)} \qquad \dim N_{p(D)} + \dim N_{q(D)} = \dim N_{pq(D)}$$

for each p and q in $F[t]$ ([6]).

The linear space V has interesting properties, when

(iii) for each x in V there is a polynomial $p \neq 0$ in $F[t]$, such that $p(D)x = 0$.

In this case we say that D is a *locally algebraic linear transformation in* V. We know that if a linear transformation D has properties (i), (ii) and (iii), then there exists a linear transformation T such that equality (τ) holds ([7], [9], [10]). Another condition used in this note reads as follows.

(iv) Every equation $p(D)x = 0$ has a non-trivial solution in V, when $p \in F[t]$ and the degree of p is positive.

After the above preliminaries we give the following definition.

DEFINITION 0.1. A linear space V over a field F with a linear transformation D satisfying the conditions (i), (ii), (iii) and (iv) is said to be *generated by F under D*.

In this note we show that every linear space V over a field F with the characteristic zero generated by F under D may be extended to a differential ring $\mathcal{V}$ with respect to D by extending the space V and the scalar field F.

This note consists of four sections. In the first section we present results concerning the algebraic structure of V in connection with a given linear transformation D admitting properties (i), (ii), (iii) and (iv). The second section of this note is devoted to a construction of a minimal extension $\mathcal{V}$ of the space V by means of an extension of the scalar field F. In the third section we shall deal with transference of linear transformations from V on $\mathcal{V}$. Finally, in the fourth section we shall define a multiplication in $\mathcal{V}$ in such a way that D will be an algebraic derivative in $\mathcal{V}$.

1. Let V be a linear space over a field F with the characteristic zero and D be a linear transformation acting in V. Assume that D satisfies the conditions (i) and (ii). Any arbitrary linear space V over a field F with a fixed linear transformation D may be interpreted as a module over the ring $F[t]$, if we put $px = p(D)x$ for $p \in F[t]$ and $x \in V$. Every invariant with respect to D linear subspace W of V is a submodule of V.

DEFINITION 1.1. An element x in V is said to be *algebraic with respect to D* if it satisfies the equation $p(D)x = 0$ for some $0 \neq p \in F[t]$.

By V_a we shall denote the set of all algebraic elements x in V. It is easy to see that V_a is a submodule of V.

DEFINITION 1.2. A module V over a ring R is said to be *divisible* if for each element y in V and for each scalar $a \neq 0$ of R there exists an element x in V such that $ax = y$.

It is known that an arbitrary module V over the ring $F[t]$, defined above, is divisible only, when the transformation D has properties (i), (ii) and (iii) [9]. We know that a submodule W of an arbitrary module V has no submodule $\overline{W}$ such that V is a direct sum of W and $\overline{W}$. E. M. Levič has shown [4] that if V is a module over a ring $F[t]$, where F is a field with the characteristic zero and W is a divisible submodule of V, then there exists a submodule $\overline{W}$ such that $V = W \oplus \overline{W}$. A combination of the last statements gives the following result.

THEOREM 1.1. *Let D be a linear transformation acting in a linear space V over a field F with the characteristic zero satisfying the conditions* (i) *and* (ii). *Then for the submodule V_a there exists a direct summand V_t,* (i.e. $V = V_a \oplus V_t$).

Now we are going to recall the following lemma which will be used later.

LEMMA 1.1 (J. Mikusiński). *If the polynomials p and q are relatively prime, then we have*

$$N_{p(D)} \cap N_{q(D)} = \{0\}.$$

A proof can be found in [6].

From Lemma 1.1 we obtain immediately the following propositions.

PROPOSITION, 1.1. *Let p be an irreducible polynomial in $F[t]$ and $V_p = \bigcup_{n=1}^{\infty} N_{p^n(D)}$. Then V_a is a direct sum of subspaces V_p.*

PROPOSITION 1.2. *Let A be a set of elements belonging to V_p, linearly independent over F. Then the set $q(D)A$ is linearly independent over F as well, when p and q are relatively prime.*

PROPOSITION 1.3. *Let A be a set elements of V_t linearly independent over F. Then for each polynomial p in $F[t]$ the set $p(D)\,A$ is linearly independent too.*

The following propositions will be connected with the transformation T.

PROPOSITION 1.4. *Let p be an arbitrary irreducible polynomial in $F[t]$ and let A be a linearly independent subset of $N_{p(D)}$. Then the set $L = \bigcup_{n=0}^{\infty} T^n A$ is also linearly independent.*

Proof. Let $x_1, \ldots, x_m$ be in A and let n be a fixed non-negative integer. Consider the equality

$$\sum_{\mu=1}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} T^{\nu} x_{\mu} = 0, \; a_{\mu\nu} \in F. \tag{1.1}$$

Hence, we have

$$p^n(D) \left[\sum_{\mu=1}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} T^{\nu} x_{\mu} \right] = 0. \tag{1.2}$$

In virtue of the following identity

$$p^n(D)\, T^k x = \prod_{\nu=1}^{k} [T\, p(D) + (n-\nu+1)\, p'(D)]\, p^{n-k}(D)\, x \tag{1.3}$$

for $k \leqslant n$, where p' denotes the ordinary derivative of p see [2], equality (1.2) may be written as follows

$$\prod_{\nu=1}^{n} [T\, p(D) + (n-\nu+1)\, p'(D)] \left[\sum_{\mu=1}^{m} a_{\mu\nu} x_{\mu} \right] = 0.$$

This implies that

$$n!\, [p'(D)]^n \left[\sum_{\mu=1}^{m} a_{\mu n} x_{\mu} \right] = 0,$$

because A is included in $N_{p(D)}$. Since the field F has the characteristic zero, thus $(p')^n$ and p^n are relatively prime. By Lemma 1.1 we obtain

$$\sum_{\mu=1}^{m} a_{\mu n} x_{\mu} = 0.$$

This implies that $a_{\mu n} = 0$ for $\mu = 1, \ldots, m$. Thus equality (1.1) reduces to

$$\sum_{\mu=1}^{m} \sum_{\nu=0}^{n-1} a_{\mu\nu} T^{\nu} x_{\mu} = 0. \tag{1.4}$$

Repeating above consideration several times one can show that $a_{\mu\nu} = 0$ for $\mu = 1, ..., m$ and $\nu = 0, ..., n$. This completes the proof of the proposition.

COROLLARY 1.1. *If p is an irreducible polynomial in $F[t]$, the space V with respect to the transformation D has properties* (i), (ii) *and* (iii) *and the set B is a basis of $N_{p(D)}$, then the set*

$$C_n = \bigcup_{\nu=0}^{n-1} T^\nu B$$

is a basis of $N_{p^n(D)}$. Moreover, the set

$$C^0 = \bigcup_{\nu=0}^{\infty} T^\nu B$$

is a basis of V_p.

Proof. Since in this case there exists a linear transformation T we obtain the proposition by Proposition 1.4.

COROLLARY 1.2. *Let B be a basis of $N_{p(D)}$. Under the assumptions of Corollary* 1.1 *the set*

$$C^n = \bigcup_{\nu \geqslant n} p^n(D) T^\nu B \tag{1.5}$$

is linearly independent for $n = 1, 2, \ldots$.

Proof. Let $n > m$. From Corollary 1.1 it follows that the sets C_n and C_m are bases of $N_{p^n(D)}$ and $N_{p^m(D)}$ respectively. Observe first that the space

$$p^m(D)\, N_{p^n(D)}$$

is spanned by the set

$$\bigcup_{\nu=m}^{n-1} p^m(D)\, T^\nu B. \tag{1.6}$$

Since

$$\dim N_{p^n(D)} = \dim N_{p^m(D)} + \dim p^m(D)\, N_{p^n(D)}.$$

The set (1.6) is linearly independent. This implies that the set (1.5) is also linearly independent.

2. Let $\mathcal{F}$ be an arbitrary extension of the field F. Let X_x be the characteristic function of the one-element set $\{x\}$. This means that $X_x(t) = 1$ if $x = t$ and $X_x(t) = 0$ if $t \in V - \{x\}$. By $\mathcal{F}(V)$ we shall denote the set of all linear combinations of functions $X_x, x \in V$, with coefficients from $\mathcal{F}$. The set $\mathcal{F}(V)$ is a linear space over $\mathcal{F}$ with respect to the usual operations of addition and multiplication. The elements X_x constitute a basis of $\mathcal{F}(V)$, when x runs through V. Let $\mathcal{K}$ denote the subspace of $\mathcal{F}(V)$ spanned by the elements of the form

$$X_{\lambda_1 x_1 + \lambda_2 x_2} - \lambda_1 X_{x_1} - \lambda_2 X_{x_2}, \tag{2.1}$$

where λ_1 and λ_2 are in F. To shorten the notation the quotient space $\mathcal{F}(V)/\mathcal{K}$ we denote by $\mathcal{V}$. The natural embedding of V into $\mathcal{F}(V)$ will be denote by φ ($\varphi : x \to X_x$). In addition, by k we shall denote the canonical mapping of $\mathcal{F}(V)$ onto $\mathcal{V}$. It is easy to verify that $\Phi = k \circ \varphi$ is an F-linear mapping from V into $\mathcal{V}$.

THEOREM 2.1. *The space $\mathcal{V}$ is universal in the following sense: for any F-linear mapping P from V into an arbitrary linear space $\mathcal{W}$ over the field $\mathcal{F}$ there exists a $\mathcal{F}$-linear mapping $\mathcal{P}$ from $\mathcal{V}$ into $\mathcal{W}$ such that $P = \mathcal{P} \circ \Phi$.*

Proof. Let Π be a mapping from $\mathcal{F}(V)$ into $\mathcal{W}$ defined as follows: if $f = \sum_{x \in V} a_x X_x$, then we put

$$\Pi_f = \sum_{x \in V} a_x P x. \tag{2.2}$$

It is easy to see that Π is an $\mathcal{F}$-linear mapping, furthermore

$$\mathcal{K} \subset \operatorname{Ker} \Pi. \tag{2.3}$$

Let $[f] \in \mathcal{V}$ be the equivalency class corresponding to f. Now, we introduce a new mapping $\mathcal{P}$ from $\mathcal{V}$ into $\mathcal{W}$ putting

$$\mathcal{P}[f] = \Pi f. \tag{2.4}$$

The definition of the mapping $\mathcal{P}$ is correct as the inclusion (2.3) holds. It is not difficult to verify that

$$(\mathcal{P} \circ \Phi)\, x = \Pi X_x = P x. \tag{2.5}$$

This completes the proof of the theorem.

From this theorem the following corollaries will be derived.

COROLLARY 2.1. *If A is a F-linearly independent set in V, i.e. A is a linearly independent set over F, then $\Phi(A)$ is a $\mathcal{F}$-linearly independent set in $\mathcal{V}$.*

Proof. Let B be a basis of V containing the set A. Now, we define an F-linear transformation P from V into $\mathcal{F}(V)$ as follows: if $x \in A$ we take $Px = X_x$, if $x \in B - A$ we put $Px = 0$. The transformation P is completely defined as an F-linear transformation, because B is a basis of V. By Theorem 2.1 there exists an $\mathcal{F}$-linear transformation $\mathcal{P}$ from $\mathcal{V}$ into $\mathcal{F}(V)$ such that $(\mathcal{P} \circ \Phi)(x) = X_x$ for $x \in A$. Since the elements X_x, $x \in A$ are $\mathcal{F}$-linearly independent therefore the elements $\Phi(x)$, $x \in A$ are $\mathcal{F}$-linearly independent too.

From Corollary 2.1 we obtain immediately the following corollary.

COROLLARY 2.2 *If B is a basis in V, then $\Phi(B)$ is a basis in $\mathcal{V}$.*

COROLLARY 2.3. *Φ is an injective mapping from V into $\mathcal{V}$.*

Proof. Let $\mathcal{F}(B)$ denote the linear space over the field $\mathcal{F}$ which consists of all linear combinations of functions X_x, $x \in B$, whose coefficients are taken from $\mathcal{F}$. Let P be the F-linear mapping from V onto $\mathcal{F}(B)$ defined as in Corollary 2.1, where B is taken in place of A. The

F-linear mapping P establishes an F-linear injective mapping from V into $\mathcal{F}(B$. By Theorem 2.1 there exists an $\mathcal{F}$-linear mapping $\mathcal{P}$ such that $(\mathcal{P}\circ\Phi)(x) = Px$. Thus Φ must be injective.

3. In this section we show that for every F-linear transformation D in V there exists an $\mathcal{F}$-linear transformation $\mathcal{D}$ in $\mathcal{V}$ such that

$$\Phi(Dx) = \mathcal{D}\,\Phi(x). \tag{3.1}$$

In order to prove it we define an $\mathcal{F}$-linear transformation $\mathcal{D}$ in $\mathcal{F}(V)$ by putting

$$\mathcal{D}\, X_x = X_{Dx} \tag{3.2}$$

for $x \in V$. The linear transformation $\mathcal{D}$ is completely determined by (3.2), because the set $\{X_x : x \in V\}$ is a basis of $\mathcal{F}(V)$. One can verify that $\mathcal{K}$ is an invariant subspace with respect to $\mathcal{D}$. Thus the transformation $\mathcal{D}$ can be transmited from $\mathcal{F}(V)$ onto $\mathcal{V}$ by means of the formula

$$\mathcal{D}[f] = [\mathcal{D}f]. \tag{3.3}$$

From this we get

$$\mathcal{D}\,\Phi(x) = \Phi(Dx) \tag{3.4}$$

for $x \in V$. More generally for every polynomial p with coefficients from F we have

$$p(\mathcal{D})\,\Phi(x) = \Phi(p(D)\,x), \tag{3.5}$$

when x is in V.

We are now looking for solutions of the equation

$$p(\mathcal{D})\,z = 0. \tag{3.6}$$

in $\mathcal{V}$, where p is in $F[t]$.

PROPOSITION 3.1. *The element $z = \Phi(x)$ is a solution of equation (3.6) if and only if $p(D)\,x = 0$.*

P r o o f. Let $p(D)\,x = 0$, then $p(\mathcal{D})\,z = 0$, by (3.5). Conversely, let $p(\mathcal{D})\,z = 0$ and $z = \Phi(x)$ for some x in V. By (3.5) we obtain $0 = p(\mathcal{D})\,z = \Phi(p(D)\,x)$. In virtue of Corollary 2.3 we have $p(D)\,x = 0$.

Denote by $\langle\Phi(V_a)\rangle$ and $\langle\Phi(V_t)\rangle$ the subspaces of $\mathcal{V}$ spanned by $\Phi(V_a)$ and $\Phi(V_t)$ respectively.

THEOREM 3.1. *$\mathcal{V}$ is a direct sum of $\langle\Phi(V_a)\rangle$ and $\langle\Phi(V_t)\rangle$. Furthermore, each of these spaces is invariant relatively to $\mathcal{D}$.*

P r o o f. Let $z = \alpha_1\,\Phi(x_1) + \dots + \alpha_n\Phi(x_n)$, $x_i \in V_t$, $\alpha_i \in \mathcal{F}$ for $i = 1, 2, \dots n$. Moreover, assume that x_i are linearly independent over F. Let p be any polynomial in $F[t]$, then we have $p(\mathcal{D})\,z = \alpha_1\,\Phi(p\,D)\,x_1) + \dots + \alpha_n\,\Phi(p(D)x_n)$, by (3.5). In view of Proposition 1.3 and Corollary 2.1 we conclude $\alpha_1 = \dots = \alpha_n = 0$. Thus we have $z = 0$. This means that only the zero element belongs to $\langle\Phi(V_a)\rangle$ and $\langle\Phi(V_t)\rangle$. The invariability of the spaces $\langle\Phi(V_a)\rangle$ and $\langle\Phi(V_t)\rangle$ under D follows from (3.5).

THEOREM 3.2. *The space $\langle\Phi(V_a)\rangle$ is a direct sum of subspaces $\langle\Phi(V_p)\rangle$, when p runs through the set of all irreducible polynomials in $F[t]$.*

Proof. We need only to prove that $\langle\Phi(V_p)\rangle \cap \langle\Phi(V_q) = \{0\}$, when p and q are relatively prime. Let $z = a_1\Phi(x_1) + \ldots + a_n\Phi(x_n)$, $x_i \in V_p$ and $a_i \in \mathcal{F}$ for $i = 1, \ldots, n$. In addition, assume that x_i are linearly independent over F, then we have $q(D)z = a_1\Phi(q(D)x_1) + \ldots + a_n\Phi(q(D)x_n)$ by (3.5). In virtue of Proposition 1.2 and Corollary 2.1 we come to a conclusion that $a_1 = \ldots = a_n = 0$. This completes the proof of the theorem.

From the two last theorems we obtain immediately the following corollary.

COROLLARY 3.1. *Assume that $p \in F[t]$, moreover let $p = p_1^{r_1} \ldots p_k^{r_k}$, where p_i are irreducible polynomials in $F[t]$ for $i = 1, \ldots, k$. If z is a solution of equation (3.6), then there are elements $z_1, \ldots, z_k$ belonging to $\mathcal{V}_{p_i} = \langle\Phi(V_{p_i})\rangle$ such that $z = z_1 + \ldots + z_k$.*

PROPOSITION 3.2. *Let p be an irreducible polynomial in $F[t]$. An element z is a solution of equation*

$$p^m(D)\,x = 0 \tag{3.7}$$

if and only if $z \in \langle\Phi(\bigcup_{\nu=0}^{m-1} T^\nu B)\rangle$, where B is a basis of $N_{p(D)}$.

Proof. The part „if" is a consequence of Proposition 3.1 and identity (1.3). Assume that z is a solution of equation (3.7), then $z \in \mathcal{V}_p$, by Theorem 3.1 and Theorem 3.2. Thus z may be written as follows

$$z = \sum_{\nu=0}^{n} \sum_{\mu=1}^{k} a_{\mu\nu}\Phi(T^\nu x_\mu), \tag{3.8}$$

where $x_\mu \in B$ and $a_{\mu\nu} \in \mathcal{F}$ for $\mu = 1, \ldots, k$ and $\nu = 0, \ldots n$. Without loss of generality we can assume $n \geqslant m$. Multiplying both sides (3.8) by $p^m(D)$ we obtain

$$p^m(D)\,z = \sum_{\nu=0}^{n} \sum_{\mu=1}^{k} a_{\mu\nu}p^m(D)\,\Phi(T^\nu x_\mu) = 0. \tag{3.9}$$

By (3.5) we have

$$\sum_{\nu=0}^{n} \sum_{\mu=1}^{k} a_{\mu\nu}\Phi(p^m(D)\,T^\nu x_\mu) = 0. \tag{3.10}$$

From (1.3) it follows

$$\sum_{\nu=m}^{n} \sum_{\mu=1}^{k} a_{\mu\nu}\Phi(p^m(D)\,T^\nu x) = 0. \tag{3.11}$$

In view of Corollaries 1.2 and 2.1 we get $a_{\mu\nu} = 0$ for $m \leqslant \nu \leqslant n$ and $1 \leqslant \mu \leqslant k$. Thus we have

$$z = \sum_{\nu=0}^{m-1} \sum_{\mu=1}^{k} a_{\mu\nu} \Phi(T^{\nu} x_{\mu}).$$

4. From now on we shall assume that the linear space V over the field F with the characteristic zero is generated by F under D. By $\mathcal{F}$ we shall denote the algebraic closure of F. Let $\mathcal{N}_{p(\mathcal{D})}$ be the kernel of $p(D)$ in $\mathcal{V}$, where $p \in F[t]$. Without loss of generality we can restrict ourselves to polynomials with the leading coefficients which are equal to one; such polynomials will be called normed. Let p be any normed and irreducible polynomial in $F[t]$. By x_p we shall denote an element of V which is a non-zero solution of the equation $p(D)x = 0$, $p(t) = t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$. The elements $x_p, Dx_p, \ldots, D^{n-1} x_p$ constitute a basis of the space $N_{p(D)}$. Put $z_p = \Phi(x_p)$, then the elements z_p, $\mathcal{D}z_p, \ldots, \mathcal{D}^{n-1} z_p$ constitute a basis of $\mathcal{N}_{p(\mathcal{D})}$, by (3.4) and Proposition 3.2. The linear transformation D is represented by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \cdot & \cdot & \cdot \ldots & \cdot \\ -a_0 & -a_1 & -a_2 \ldots & -a_{n-1} \end{bmatrix} \tag{4.1}$$

with respect to this basis, where a_i are the coefficients of p. The roots of p are the eigenvalues of matrix (4.1). Let $\xi_1, \ldots, \xi_n$ be the roots of p. The elements $\xi_1, \ldots, \xi_n$ are different, because the field F has the characteristic zero ([3], p. 200). Let $\gamma_{1i} \ldots \gamma_{ni}$, $i = 1, 2, \ldots, n$ be an eigenvector of the matrix (4.1) corresponding to the eigenvalue ξ_i. Since ξ_i are different for the different subscripts i, thus the matrix

$$\begin{bmatrix} \gamma_{11} \cdots \gamma_{1n} \\ \cdot \cdots \cdot \\ \gamma_{n1} \cdots \gamma_{nn} \end{bmatrix} \tag{4.2}$$

is not singular. This implies that the elements

$$z_{pi} = \gamma_{1i} z_p + \gamma_{2i} z_p + \ldots + \gamma_{ni}{}^{n-1} z_p$$

are $\mathcal{F}$-linearly independent for $i = 1, 2, \ldots, n$. It is not difficult to verify that

$$\mathcal{D}z_{pi} - \xi_i z_{pi} = 0 \text{ for } i = 1, \ldots, n.$$

Let T be a linear transformation in V such that identity (τ) holds. One can transfer the linear transformation T from V onto $\mathcal{V}$ as it was showed in section 3. This new transformation will be denoted by $\mathcal{T}$. The property (τ) is preserved by the transformations $\mathcal{D}$ and $\mathcal{T}$. Let $\mathcal{B}_p = \{\mathcal{T}^k z_{pi} : k = 0, 1,$

$2, \ldots, i = 1, \ldots, n\}$. Let us consider the set $\mathcal{B} = \bigcup_p \mathcal{B}_p$, where p runs through the set of all irreducible polynomials p in $F[t]$. It is easy to see that $\mathcal{B}$ is a basis of $\mathcal{V}$.

Now, we introduce an operation of multiplication in $\mathcal{V}$. Since V is a space generated by the field F and $\mathcal{F}$ is the algebraic closure of F, therefore for each ξ in $\mathcal{F}$ there exists an element z_ξ such that

$$\mathcal{D} z_\xi - \xi z_\xi = 0.$$

For two elements $\mathcal{J}^m z_\xi$ and $\mathcal{J}^n z_\xi$ in $\mathcal{B}$ we take $\mathcal{J}^m z_\xi \cdot \mathcal{J}^n z_\xi = \mathcal{J}^{m+n} z_{\xi+\eta}$. We extend the multiplication operation defined up to now only for the elements belonging to $\mathcal{B}$ on the whole space $\mathcal{V}$ in such a way that it will be an $\mathcal{F}$-bilinear transformation on $\mathcal{V} \times \mathcal{V}$.

It is easy to verify that the linear space $\mathcal{V}$ over the field with the above defined multiplication is an algebra without divisors of zero. An easy check shows that $\mathcal{D}$ has property (δ). The element $\varepsilon = \Phi(e)$ where e is a non-trivial solution of $Dx = 0$ is a unit element of this algebra. It is easy to see that the elements $a\varepsilon$, when a runs through the field $\mathcal{F}$ constitute the field of constants with respect to $\mathcal{D}$. Thus $\mathcal{V}$ can be interpreted as a differential ring.

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