PIOTR MIKUSIŃSKI

COMPARISON OF FAMILIES OF PSEUDONORMS

Abstract. Two relations between families of pseudonorms are considered: relation of *being weaker than* and relation of *being sequentially weaker than*. Connections between these concepts are discussed.

1. By a *convergence* G on a set X we mean a mapping from X^N to 2^X ($N = \{1, 2, ...\}$). We say that a sequence x_n is *G-convergent* to x, and write $x_n \to x\ (G)$, iff $x \in G(x_n)$. A real valued function f on X is said to be *G-continuous*, iff $x_n \to x\ (G)$ implies $f(x_n) \to f(x)$.

Let G and H be convergences on X. We write $G \subset H$, iff $G(x_n) \subset \subset H(x_n)$ holds for every sequence x_n of elements from X. We say that G is equal to H, and write $G = H$, if $G \subset H$ and $H \subset G$.

In the following E denotes a linear space over the field of real numbers R. A non-negative functional p defined on E is called a *pseudonorm*, iff

$$p(\lambda x) = |\lambda|\, p(x), \tag{1}$$

$$p(x+y) \leqslant p(x)+p(y) \tag{2}$$

where $x, y \in E$ and $\lambda \in R$.

Let P be any family of pseudonorms defined on E. Then by LP we denote a convergence defined as follows:

$x_n \to x\ (LP)$ *iff* $p(x-x_n)$ *tends to zero for each* $p \in P$.

By MP we denote the family of all LP-continuous pseudonorms on E, i.e., $p \in MP$ iff $p(x-x_n)$ tends to zero for each sequence x_n which is LP-convergent to x. Clearly, $P \subset MP$.

Received November 06, 1979.

AMS (MOS) subject classification (1980). Primary 46A45.

Let P and Q be any families of pseudonorms on E.

LEMMA 1. *If* $P \subset Q$, *then* $LQ \subset LP$.

LEMMA 2. $LP = LMP$.

LEMMA 3. $LQ \subset LP$ iff $MP \subset MQ$.

Easy proofs are omitted.

We say that two families of pseudonorms P and Q are equivalent, if $LP = LQ$.

The definition of convergence originates from [1] and the definition of operator L from [3].

2. Let P and Q be two families of pseudonorms on a linear space E. In [2] the following definition is given.

DEFINITION 1. We say that P is weaker (plus faible) than Q if for each pseudonorm $p \in P$ there exist a pseudonorm $q \in Q$ and a positive number c such that $p(x) \leqslant cq(x)$ for all $x \in E$.

If a family of pseudonorms is used only as a description of a convergence, another definition seems to be more adequate.

DEFINITION 2. We say that P is sequentially weaker than Q, if $LQ \subset LP$.

We discuss in this note some relations between both notions.

THEOREM 1. *If P is weaker than Q, then P is sequentially weaker than Q.*

P r o o f. Assume that $x_n \to x\,(LQ)$ and let $p \in P$. Then $p(x-x_n) \leqslant \leqslant cq(x-x_n)$ for some positive number c and some pseudonorm $q \in Q$, and, consequently $p(x-x_n) \to 0$, because $q(x-x_n) \to 0$.

An example presented in section 3 shows that the converse implication is not true, in general.

THEOREM 2. *If P is sequentially weaker than Q, then P is weaker than MQ.*

P r o o f. This follows from Lemma 3 and from the evident

LEMMA 4. *If* $P \subset Q$, *then P is weaker than Q.*

COROLLARY 1. *The following conditions are equivalent*

1° *P is sequentially weaker than MQ*

2° *P is weaker than MQ*

3° $P \subset MQ$.

P r o o f. This follows from Theorem 1, Theorem 2, Lemma 3 and Lemma 4.

THEOREM 3. *Let* $Q = \{q_1, q_2, \ldots\}$ *be a countable family of pseudonorms and let*

$$Q' = \left\{ \sum_{i=1}^{n} q_i;\ n = 1, 2, \ldots \right\}.$$

If P is sequentially weaker than Q, then P is weaker than Q'.

Proof. Let p be any pseudonorm from the family P. We have to show that there exist a pseudonorm $q \in Q'$ and a positive number c such that

$$p(x) \leqslant cq(x)$$

for all $x \in E$. Assume that it is not true. Then there exists a sequence $x_n \in E$ such that, for $n = 1, 2, \ldots$,

$$p(x_n) > n^2 r_n(x_n)$$

where $r_n(x) = \sum_{i=1}^{n} q_i(x)$. Let

$$y_n = \frac{n x_n}{p(x_n)}.$$

Since the r_n are an increasing sequence of pseudonorms and $r_n(y_n) < 1/n$, then $y_n \to 0\ (LQ')$. Consequently $y_n \to 0\ (LQ)$, because $LQ' = LQ$. But $p(y_n) = n$, and so $y_n \not\to 0\ (LP)$. This contradicts the assumption that P is sequentially weaker than Q.

COROLLARY 2. *If p and q are two pseudonorms on E such that*

$$q(x_n) \to 0 \text{ implies } p(x_n) \to 0 \tag{3}$$

for each sequence $x_n \in E$, then $p(x) \leqslant cq(x)$ for some positive number c and for all $x \in E$.

Proof. Families P and Q from Theorem 3 can consist of one element each: $P = \{p\}$ and $Q = \{q\}$. Then condition (3) means that P is sequentially weaker than Q.

3. Let F be a linear subspace of a linear space E and let P be a family of pseudonorms on E. By $P|_F$ we denote a family of restrictions to the subspace F of pseudonorms from the family P.

THEOREM 4. *Let p be a pseudonorm on the subspace F. If q is a pseudonorm on the space E such that $\{p\}$ is sequentially weaker than $\{q\}|_F$,* i.e.,

$$q(x_n) \to 0 \text{ implies } p(x_n) \to 0 \text{ for each sequence } x_n \in F, \tag{4}$$

then there exists a pseudonorm r on E such that $p(x) = r(x)$ for $x \in F$ and such that $\{r\}$ is sequentially weaker than $\{q\}$, i.e.,

$$q(x_n) \to 0 \text{ implies } r(x_n) \to 0 \text{ for each sequence } x_n \in E. \tag{5}$$

Proof. By Corollary 2, there exists a real number c such that $p(x) \leqslant cq(x)$, for all $x \in F$. It is easy to check that the functional

$$r(x) = \inf\{p(\alpha) + cq(\beta);\ x = \alpha + \beta,\ \alpha \in F\}$$

is a pseudonorm on E, i.e., satisfies conditions (1) and (2).

Since

$$r(x) \leqslant p(0) + cq(x) = cq(x),$$

the pseudonorm r satisfies condition (5). It remains to show that $r(x) = p(x)$ for $x \in F$. If $x \in F$, then

$$r(x) \leqslant p(x) + cq(0) = p(x).$$

On the other hand, if $x, \alpha \in F$, then $\beta = x - \alpha \in F$. Consequently

$$p(x) \leqslant p(\alpha) + p(\beta) \leqslant p(\alpha) + cq(\beta) \tag{6}$$

and hence

$$p(x) \leqslant \inf \{p(\alpha) + cq(\beta);\ x = \alpha + \beta,\ \alpha \in F\} = r(x),$$

because inequalities (6) hold for all x, α, β such that $x = \alpha + \beta$ and $x, \alpha \in F$. We have proved the identity of pseudonorms r and p on F. This completes the proof.

Let p be a pseudonorm on a subspace F of E and let Q be a family of pseudonorms on E. We say that p is continuous on the subspace F, if $p \in M(Q|_F)$. In general, not every continuous pseudonorm on a subspace can be extended to a continuous pseudonorm on whole space. This means that the equality $(MQ)|_F = M(Q|_F)$ is not always true. A simple example is given in [4].

THEOREM 5. *Let Q be a family of pseudonorms on E and let P be a family of pseudonorms on a linear subspace $F \subset E$. If P is weaker than $Q|_F$, then there exists a family S of pseudonorms on E weaker than Q and such that $P = S|_F$.*

P r o o f. Let $p \in P$ and let q be a pseudonorm from Q such that $p(x) \leqslant cq(x)$ for some $c > 0$ and for $x \in F$. Then

$$\bar{p}(x) = \inf \{p(\alpha) + cq(\beta);\ x = \alpha + \beta,\ \alpha \in F\}$$

is an LQ-continuous pseudonorm on E such that $\bar{p}(x) \leqslant cq(x)$ for all $x \in E$. Denote by S the family of such extensions of all pseudonorms from P. The family S satisfies the required conditions.

Now we present an example of a linear space E endowed with a family of pseudonorms Q and a subspace $F \subset E$ endowed with a family of pseudonorms P such that P is sequentially weaker than $Q|_F$ and such that there is no family of pseudonorms S on E satisfying the following two conditions:

(I) S is sequentially weaker than Q,

(II) $S|_F$ and P are equivalent.

EXAMPLE. Let E be the space of all real valued functions on the product $N \times [0, 1]$ such that

$$\sup \{|f(n, t)|;\ n \in N\} < \infty$$

for all $t \in [0, 1]$. Let Q be the family of all pseudonorms

$$q_t(f) = \sup \{|f(n, t)|;\ n \in N\}.$$

Let F be the space of all functions $f \in E$ such that for each fixed $n \in N$ the function $f(n, t)$ is constant on the set $[0,1]\setminus A$, where A is countable subset of $[0, 1]$. By $p_{n,t}$ we denote a pseudonorm

$$p_{n,t}(f) = |f(n, t)|,$$

and by p we denote a pseudonorm

$$p(f) = \sup \{\text{supess} \{|f(n, t)|;\ t \in [0, 1]\};\ n \in N\}.$$

Let

$$P = \{p\} \cup \{p_{n,t};\ n \in N,\ t \in [0, 1]\}.$$

It is easy to check that P is sequentially weaker then $Q|_F$.

Assume, on the contrary, that there exists a family of pseudonorms S on E such that (I) and (II) are satisfied.

The sequence

$$f_i(n, t) = \begin{cases} 0 \text{ if } n < i,\ t \in [0, 1] \\ 1 \text{ if } n \geqslant i,\ t \in [0, 1] \end{cases}$$

is not P-convergent, so, by (II), there exists a pseudonorm $s \in S$ such that $s(g_i) > \varepsilon$ $(i = 1, 2, \ldots)$ for some $\varepsilon > 0$ and for some subsequence g_i of the sequence f_i. We shall construct, by induction, a sequence A_n of subsets of the interval $[0, 1]$. Denote by A_1 the interval $[0, 1/2]$, if the sequence

$$s(g_i X_{N \times [0, 1/2]})$$

contains infinitely many positive elements. Otherwise, by A_1 we denote the interval $(1/2, 1]$.

Assume that we have fixed the interval $A_n = [k/2^n, (k+1)/2^n]$ (or $((k+1)/2^n, (k+2)/2^n])$, where $0 \leqslant k < 2^n - 1$. Then, by A_{n+1} we denote the interval $[k/2^n, (2k+1)/2^{n+1}]$ (or $((k+1)/2^n, (2k+3)/2^{n+1}]$), if the sequence $s(g_i X_{N \times [k/2^n, (2k+1)/2^{n+1}]})$ (or $s(g_i X_{N \times ((k+1)/2^n, (2k+3)/2^{n+1}]})$) contains infinitely many positive elements. Otherwise, by A_{n+1} we denote the interval $((2k+1)/2^{n+1}, (k+1)/2^n]$ (or $((2k+3)/2^{n+1}, (k+2)/2^n]$).

For the sequence of intervals A_n, constructed in this way, there exists a subsequence h_i of the sequence g_i such that

$$s(h_i X_{N \times A_i}) > 0,$$

for $i = 1, 2, \ldots$. Then we put

$$x_i = \frac{h_i X_{N \times A_i}}{s(h_i X_{N \times A_i})}.$$

Now we have to consider two cases.

First case. $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Then the sequence x_i is LQ-convergent to zero, but it is not LS-convergent to zero, because $s(x_i) = 1$ for all i. This is a contradiction.

Second case. $\bigcap_{n=1}^{\infty} A_n = \{t_0\}$ for some $t_0 \in [0, 1]$. Then the sequence

$$y_i = x_i X_{N \times \{t_0\}}$$

is LP-convergent to zero. Consequently $s(y_i)$ tends to zero. Since $s(x_i) = 1$ for all $i \in N$, then $s(x_i - y_i)$ is positive for sufficiently large i. The sequence

$$\frac{x_i - y_i}{s(x_i - y_i)}$$

is LQ-convergent to zero and it is not LS-convergent to zero. But S is equentially weaker than Q. This is a contradiction.

One can give a much simpler example, when P is not a total family of pseudonorms, i.e., when the convergence LP is not Hausdorff.

The presented example shows that the converse implication to that in Theorem 1 is not true. In fact it shows much more: none of the families equivalent to family P has that property. Assume, that some family of pseudonorms P_1 on F, equivalent to P, is weaker than $Q|_F$. Then by Theorem 5, there exists a family S on E such that $P_1 = S|_F$ and such that S is weaker than Q. Hence $LP_1 = LP = L(S|_F)$ and $LQ \subset LS$. However we just have shown that it is impossible.

REFERENCES

[1] A. R. BEDNAREK, J. MIKUSIŃSKI, *Convergence and topology*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 17 (1969), 437—442.

[2] H. G. GARNIR, M. DE WILDE, J. SCHMETS, *Analyse Fonctionnelle*, Tome 1, Birkhäuser Verlag Basel 1968, 26.

[3] J. MIKUSIŃSKI, *Converggence and locally convex spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 23 (1975), 1171—1173.

[4] P. MIKUSIŃSKI, *On pseudonormability of some particular classes of spaces*, (to appear).