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ON THE TWO-DIMENSIONAL VERSION OF THE SPERNER LEMMA AND BROUWER'S THEOREM

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Abstract. In this work the Brouwer fixed point theorem for a triangle was proved by two methods based on the Sperner Lemma. One of the two proofs of Sperner's Lemma given in the paper was carried out using the so-called index.

1. Introduction

In 1912 Luitzen Brouwer published the famous and remarkable result: on the Euclidean plane each continuous transformation of a closed circle $D = \{(x, y); x^2 + y^2 \leqslant 1\}$ into itself has at least one fixed point (i.e. the $p \in D$ for which $f(p) = p$).

For the interval $I = \langle a, b\rangle$ we have a special case of this theorem: each continuous transformation $f\colon I \to I$ has a fixed point (i.e. the point $p \in I$ such that $f(p) = p$). In this case, it follows immediately from the Darboux property. For higher dimension balls, the proof of the Brouwer theorem required the use of more sophisticated techniques. Emanuel Sperner, giving a simple combinatorial result called the Sperner Lemma in 1928, surprised mathematicians dealing with the subject of fixed points. For it turned out

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that both the Brouwer theorem and other important theorems can be derived from his Lemma. It is worth adding that the beautiful and ingenious the Sperner Lemma is accompanied by an interesting proof based on double counting.

We will present two versions of the Sperner Lemma proof for a triangle. One of them is related to counting of so-called paths that start outside a large triangle with vertices A_1, A_2, A_3 (which has been triangulated), and end in a small triangle with such vertices, and counting paths that not only come out but also come back from outside the large triangle.

The second version is related to the so-called index, which we get from the oriented version of the Sperner Lemma.

In the last part of this work we present two proofs of the Brouwer fixed point theorem for a triangle. The first is based on Lemma 4 (no retraction of a triangle to its boundary). This proof is largely a modification of the proof of Brouwer's theorem continued in [3], where the non-existence of a retraction of a circle on its boundary was used. On the other hand, in the second proof, we use the Sperner lemma by first determining the so-called Sperner numbering (using the vector $f(p)-p$ from p to $f(p)$, for a point p belonging to the triangle Δ, where $f\colon \Delta \to \Delta$ is continuous). The proof is a slight modification of the relevant proof from [4].

Brouwer's theorem in the general version applies to a simplex of any dimension, and hence (using the retraction of a simplex onto a closed ball contained in this simplex) we obtain from Fact 1 Brouwer's theorem for a closed ball in n-dimensional space $\mathbb{R}^n$.

Currently, the Brouwer theorem has a number of proofs using a variety of topological and combinatorial techniques. J. Milnor in [2] (compare [1]) proved this theorem for balls using analytical methods.

It is possible show that every compact map $F\colon B \to B$ (i.e. a continuous map such that $\overline{F(B)}$ is compact) of the ball B of any normed linear space has a fixed point.
We owe this result (from 1930), extending Brouwer's theorem to infinite dimensional spaces, to Juliusz Schander.

2. The Sperner Lemma

Division of a triangle into a finite number of smaller triangles in such a way that the cross-section of each two of the divisions triangles be their common side, common vertex or empty set, we will call triangulation or simplicial division.

Lemma 1 (E. Sperner 1928). *Divide the triangle $A_1A_2A_3$ simplicialy. Let the vertices of the triangles of division be numered $1, 2, 3$ so that*

(∗) *the vertex A_i has the number i, a vertex lying on the side $A_iA_j$has the number i or j, the numbers of the other vertices are arbitrary.*

Then among the triangles of division there is one whose vertices are numbered $1, 2, 3$ (Figure 1). (The numbering that satisfies the condition (∗) is called the Sperner's labeling.)

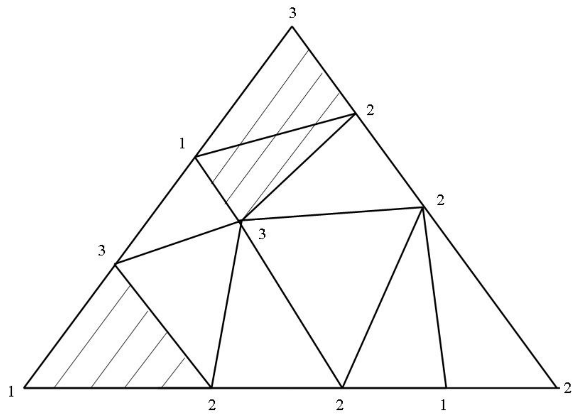

Figure 1. Sperner's lemma

The oriented version of Sperner's Lemma gives the relationship between what is happening inside the triangulated polygon (we say here the so-called content of this polygon) and what is happening on its boundary (we are then talking about the so-called index).

A triangulation of a polygon W is a division of W into a finite number of triangles so that each edge on the boundary of W belongs to just one triangle of the subdivision, and each edge in the interior of W is shared by exactly two triangles of the subdivision.

In order to formulate and prove this version of Lemma, let us denote:

- by T_+ the number of complete triangles, i.e. those whose vertices have the colors-numbers from the set $\{1, 2, 3\}$ and are directed positively, which means that moving along the perimeter of the triangle we see numbers in the order 1, 2, 3,
- by T_- the number of these complete triangles whose vertices appear in the order 1, 2, 3 when we move clockwise around the circumference.

We will deal with the edges at the boundary of the polygon. We will highlight the edges that have colors 1 and 2. Edges marked with 12 (read as one, two) will be called positively oriented, which means that moving along the perimeter of the polygon counter-clockwise we see the vertices of the edges in order 1, 2; the edges 21 (read two, one) will be "negatively oriented". We denote the numbers of these edges as E_+ and E_-, respectively. We now have the oriented version of the Sperner Lemma:

LEMMA 2. $E_+ - E_- = T_+ - T_-$.

PROOF. Crossing each of the edges 12 and 21 with a perpendicular vector (in the direction as shown in Figure 2) we get a graph made of these vectors. It consists of directed paths. Each path starts in a negatively oriented triangle

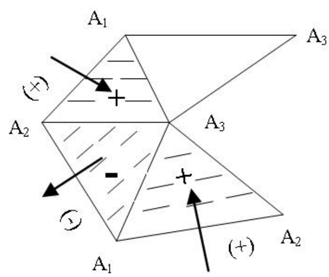

Figure 2. Oriented version of the Sperner Lemma

or on a positive oriented segment, and ends in a positive triangle or negatively oriented segment (Figure 2). The number of path starts $(T_- + E_+)$ and the number of path ends $(T_+ + E_-)$ must be equal, which ends the proof. □

DEFINITION 1. *Index* I is the number of edges A_1A_2 on the boundary of a polygon, calculated according to the rule

$$E_+ - E_-.$$

DEFINITION 2. The *content of* C is the number of complete triangles, calculated according to the rule

$$T_+ - T_-.$$

The oriented version of Sperner's Lemma now takes the form of equality

$$I = C.$$

THE FIRST PROOF OF THE SPERNER LEMMA. It is enough to show that the index is odd.

Let

b – number of complete sections (i.e. with the ends 1, 2) on A_1A_2,

a – number of sections A_1A_1 (on A_1A_2),
q – number of vertices A_1 inside side A_1A_2.
So we have, the total number of vertices A_1 on A_1A_2 is $q+1$. Since each segment A_1A_1 contains two vertices A_1, while the inner segment A_1A_2 has only one such vertex, so we have

$$2a + b = 2q + 1,$$

because A_1 is a common vertex (Figure 1). Therefore b is an odd number. □

Later in the article, we will call the original triangle *large*, and we will call singles triangles of triangulation *small*. By the door we will understand each edge with the ends 1, 2. A small triangle with vertices 1, 2, 3 we will call complete.

The second proof of the Sperner Lemma. Let us first consider the special situation illustrated in Figure 1. We walk in this situation from a small complete triangle to another small triangle of this type through a door (i.e. a common side with ends 1 and 2). Of course, we then get an even number of such triangles, which will not affect the kind of the total number of them, when it will be odd or even.

Now let us consider the path through the triangle, which begins at certain point beyond the triangle, and after passing through the next door it ends in one of two ways:

(1) exit beyond the triangle, or
(2) entry into the complete little triangle.

Moreover, such a path after entering is determined (since no triangle has three edges with ends 1 and 2).

Now note that each path that ends outside the triangle is determined by a pair of doors on its boundary, while the path entering a small complete triangle is determined by only one door. Because on the boundary of the triangle there is an odd number of doors (indeed, going along the large side from A_1 to A_2 and numbering the consecutive vertices of the division, we must an odd number of times change 1 to 2 or vice versa), so some path must enter a small complete triangle, which proves the truth of the Lemma. □

3. Brouwer's theorem as a consequence of Sperner's lemma

LEMMA 3. *Let Δ be a triangle with sides K_1, K_2, K_3. Let C_1, C_2, C_3 be closed sets such that $K_1 \subset C_1, K_2 \subset C_2, K_3 \subset C_3$ and $\Delta \subset C_1 \cup C_2 \cup C_3$. Then $C_1 \cap C_2 \cap C_3 \neq \varnothing$.*

PROOF. The assertion of the lemma is satisfied if the triangle Δ is contained in the union of two sets of the family $\{C_1, C_2, C_3\}$. Let us assume that the triangle Δ is not contained in the union of the sets of this family. For each $n \geqslant 2$ we divide the sides of the triangle into n equal parts, and by connecting them with lines parallel to the sides of the triangle, we get a n-th order net. Let each vertex of the net in K_1 be the number 1 or 2, the numbers of the next vertices are: 2 or 3 in K_2, 3 or 1 in K_3. On the basis of the Sperner Lemma, in each such net there is $\alpha_n\beta_n\gamma_n$ triangle whose vertices are numered 1, 2, 3. The Bolzano–Weierstrass theorem implies the existence of a convergent subsequence $\alpha_{n_i} \to p$, and since the diameters of the triangles of successive nets tends to zero, $\beta_{n_i} \to p$, $\gamma_{n_i} \to p$. Moreover, $\alpha_{n_i} \in C_1, \beta_{n_i} \in C_2$ and $\gamma_{n_i} \in C_3$, therefore (since the sets C_1, C_2, C_3 are closed) we get $p \in C_1 \cap C_2 \cap C_3$. □

In order to formulate Lemma 4, we will need the concept of retract and retraction.

The subset A of the metric (generally topological) space X is a *retract* of X, if there is a continuous mapping (*retraction*) $r\colon X \to A$ such that $r(x) = x$ for all $x \in A$.

LEMMA 4. *There is no retraction of triangle Δ with sides K_1, K_2, K_3 to its boundary $\partial\Delta = K_1 \cup K_2 \cup K_3$.*

PROOF. Suppose $f\colon \Delta \to \partial\Delta$ is a continuous transformation such that $f(x) = x$ for $x \in \partial\Delta$. Then $f(K_i) \subset K_i,\ i = 1, 2, 3$.

Now let $C_i = f^{-1}(K_i),\ i = 1, 2, 3$. The sets K_i $(i = 1, 2, 3)$ are closed, so due to the continuity of f the sets C_i are also closed. Moreover, $K_1 \subset C_1, K_2 \subset C_2$ and $K_3 \subset C_3$. For every $x \in \Delta$, $f(x) \in \partial\Delta = K_1 \cup K_2 \cup K_3$, and hence $C_1 \cup C_2 \cup C_3 = \Delta$. Section $K_1 \cap K_2 \cap K_3$ is empty, so $C_1 \cap C_2 \cap C_3 = \varnothing$. So we got a contradiction with Lemma 3. □

THEOREM 1 (L. Brouwer, 1912). *Each continuous transformation $f\colon \Delta \to \Delta$ (of a closed triangle Δ) has at least one fixed point, that is, a point $x \in \Delta$ for which $f(x) = x$.*

Below we present two proofs of the Brouwer theorem.

As we mentioned in the introduction, the first one is largely a modification of Shaskin's proof in [3] of this theorem for a circle. On the other hand, the second proof in which we use a certain "vector field given by $f(P) - P$" was inspired by [4].

The first proof of the Brouwer theorem (based on Lemma 4). Suppose there is continuous transformation $f\colon \Delta \to \Delta$ such that $f(p) \neq p$ for any $p \in \Delta$. For each such p by $g(p)$ let us denote the intersection point of the half-line originating from $f(p)$ and passing through p with the boundary $\partial\Delta$ of the triangle Δ (Figure 3).

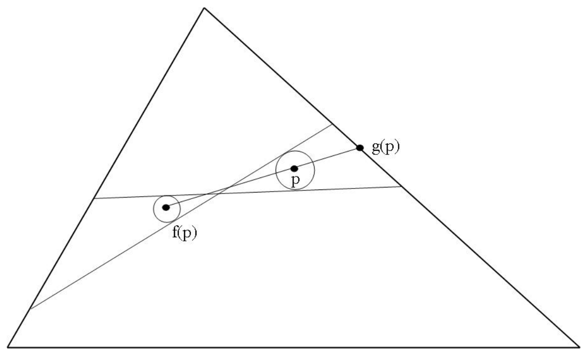

Figure 3. The first proof of the Brouwer theorem

So we have a transformation $g\colon \Delta \to \partial\Delta$ such that $g(p) = p$ for $p \in \partial\Delta$. We will prove the continuity of transformation g using open balls (for example, $B(p, \delta)$ means an open ball, which here is an open circle centrated at point p and radius δ called a neighborhood of the point p).

Let us take any $\varepsilon > 0$ and consider the neighborhood of the point $g(p)$ on the boundary $\partial\Delta$ with an ε radius. On this neighborhood we build a cone with a vertex on the segment joining points p and $f(p)$. Then we choose number $\varepsilon' > 0$ such that the neighborhood $B(f(p), \varepsilon')$ is contained in the cone. In view of the continuity of transformation f there is $\delta > 0$ such that $f(B(p, \delta)) \subset B(f(p), \varepsilon')$ and $B(p, \delta)$ is contained in the cone. So we have $g(B(p, \delta)) \subset B(g(p), \varepsilon)$, which means the continuity of transformation g. We got a contradiction with Lemma 4. □

The second proof of the Brouwer theorem. For simplicity, position of the triangle Δ is such that its base is horizontal and the third vertex is above the base. We will consider a sequence of triangulation τ_n of Δ with diameters $\delta(\tau_n) \to 0$ (as $n \to \infty$). We will associate each such triangulation with the Sperner's labeling. Let for every $P \in \Delta$, $w(P) = f(P) - P$.

A vertex P of the triangulation receives a label A_1 if $w(P)$ points in the northeast direction, a label A_2 if $w(P)$ points in the northwest direction, a label A_3 if $w(P)$ points any other direction.

We assume that $f(P) \neq P$. We then have the triangulation τ_n to ensure the existence of the complete triangle Δ_n. Now consider the sequences of vertices B_n, C_n and D_n of consecutive such triangles Δ_n. Based on the Bolzano-Weiertstrass theorem one can choose from them convergent subsequences $B_{n_k}, C_{n_k}, D_{n_k}$ such that $\lim_k B_{n_k} = \lim_k C_{n_k} = \lim_k D_{n_k} = P$, because $\delta(\Delta_n) \to 0$. By continuity of f, $w(P) = 0$. Consequently, $f(P) = P$. $\square$

In order to clarify some of the observations at the end of the paper regarding retraction and those concerning homeomorphisms, we will present below two facts related to these concepts.

Let us recall that the subspace A of the topological (metric) space X is a retract of X, if there is a continuous transformation (retraction) $r\colon X \to A$ such that $r(x) = x$ for all $x \in A$. Let us add that a topological (metric) space X has the *fixed-point property* if every continuous transformation $f\colon X \to X$ has a fixed point.

Below we will give two known facts with their justifications.

FACT 1. *All retractions retain the fixed-point property.*

Indeed, assume that $A \subset X$ is a retract of X, and $f\colon A \to A$ continuous transformation. Taking any retraction $r\colon X \to A$ we get a continuous transformation $g = f \circ r\colon X \to X$ having a fixed point $x_0 \in X$, i.e. $f(r(x_0)) = x_0$. Since the values of f belong to A, $x_0 \in A$; hence $r(x_0) = x_0$. Therefore $f(x_0) = x_0$, i.e. x_0 is the fixed point of transformation $f\colon A \to A$.

Let us add that the property of the fixed point is a topological invariant. This is clarified by the following

FACT 2. *Let X, Y be homeomorphic topological (metric) spaces. If the space X has the fixed-point property, then the space Y also has the fixed-point property.*

Indeed, let $g\colon Y \to X$, $g^{-1}\colon X \to Y$ be continuous mappings establishing homeomorphism of the space X with the space Y. Such mappings exist because X and Y are homeomorphic. Consider the continuous mapping $F\colon Y \to Y$. We will prove that F has a fixed point. Superposition $g \circ F \circ g^{-1}$ is a continuous mapping of the space X into itself, so by assumption it has a fixed point $x_0 \in X$:

$$(g \circ F \circ g^{-1})(x_0) = x_0,\ (F \circ g^{-1})(x_0) = g^{-1}(x_0).$$

Therefore $y_0 = g^{-1}(x_0) \in Y$ is a fixed point of the mapping F.

It can be shown that the triangle Δ and the circle, i.e. the closed two-dimensional ball B^2, are homeomorphic, so on the basis of Fact 2, Brouwer's theorem holds for B^2. This result can also be obtained from Fact 1 by considering the retraction of the triangle Δ (two-dimensional simplex) on B^2.

Brouwer's theorem remains true in Euclidean spaces $\mathbb{R}^n$, most often it is formulated in the form (for a closed ball $B^n \subset \mathbb{R}^n$):

Every continuous $f\colon B^n \to B^n$ has at least one fixed point.

The Brouwer theorem in this form can be obtained from Brouwer's theorem for a simplex using the existence of a retraction an n-dimensional simplex onto a ball B^n (contained in it) and using Fact 1.

Remark 1. Karol Borsuk proved in 1931 that the $S^{n-1} = \partial B^n$ sphere is not a retract of the ball B^n. Since it can be shown that this theorem implies the Brouwer theorem, then if we demonstrate (below) that the Borsuk theorem follows from the Brouwer theorem, we will obtain equivalent results. Well, assuming that there is a retraction r of the ball B^n on S^{n-1} and considering the transformation $g\colon B^n \to B^n$ given the formula $g(x) = -r(x)$ for $x \in B^n$ we conclude that g has no fixed point. The obtained contradiction with the Brouwer theorem guarantees the truth of the aforementioned equivalence. Let us add that Borsuk's result is intuitively more obvious.

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