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ON QUATERNION GAUSSIAN BRONZE FIBONACCI NUMBERS

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Abstract. In the present work, a new sequence of quaternions related to the Gaussian Bronze numbers is defined and studied. Binet's formula, generating function and certain properties and identities are provided. Tridiagonal matrices are considered to determine the general term of this sequence.

1. Introduction and background

Recently, the subject of several studies by many researchers is related to numerical sequences. The well-known Fibonacci sequence and also the Lucas sequence are two of several examples of numerical sequences that great emphasis has been given. However, many other numerical sequences have contributed to the increase of researchers interested in this area and in its several applications. Sequences of numbers, polynomials, quaternions, octonions, sedenions, etc., are the topic of a vast literature. For instance, one can consult the works of Catarino ([7]) where the sequence of $k-$Pell hybrid numbers was introduced, of Kizilates ([25]) with the study of another sequence of hybrid

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numbers (Fibonacci and Lucas), of Koshy ([26]) with the study of some applications of the Fibonacci and Lucas sequences, of Catarino and Borges ([9]) with the introduction of the numerical sequence constituted by Leonardo's numbers and in [8] of the incomplete version of these types of numbers, of Catarino and Vasco ([11]) with the study of the $k-$Pell generalized numbers of order m, where m is a non negative integer. Many identities and properties of these sequences come from the use of the so-called Binet's formula for each sequence deduced by Levesque in 1985 ([27]). With this formula we have the possibility to determine the general term of a sequence without having to resort to other terms of the sequence.

The study of quaternions sequences is also a great topic of research. For instance in the work of Catarino ([6]) the sequence of bicomplex quaternions whose terms are the k-Pell numbers was introduced, among many other works. Also, in [1], [28], [33] and [34], a split version of quaternions is considered. Another kind of Pell and Pell–Lucas quaternions are presented in [4], where some identities satisfied by these sequences of quaternions and their respective generating functions are stated. One can consult the works [5], [12], [16], [31] and [32], where also some properties of these types of quaternion sequences are established.

A quaternion q is a hyper-complex number, written as

$$q = q_0 1 + q_1 i + q_2 j + q_3 k,$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$; 1 is the multiplication identity (thus $q_0 1 = q_0$) and i, j, k are imaginary units, satisfying the famous multiplication rules defined by Hamilton in 1866 ([18]):

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{1.1}$$

The set of quaternions, which we will denote by $\mathbb{H}$, form a four-dimensional associative and noncommutative algebra over the set of real numbers. For any $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$, elements of $\mathbb{H}$, the quaternions addition is obtained as

$$p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k,$$

and the quaternions multiplication (known as the Hamilton product) occurs, accordingly with the distributive law and (1.1), as

$$\begin{aligned} pq = {} & (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\, i \\ & + (p_0q_2 + p_2q_0 + p_3q_1 - p_1q_3)\, j + (p_0q_3 + p_1q_2 + p_3q_0 - p_2q_1)\, k. \end{aligned}$$

We consider the Euclidean vector space $\mathbb{R}^4$, with the orthonormal basis $\{1, i, j, k\}$, and identify the element $(q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ with the element $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, thus embedding $\mathbb{R}^4$ in $\mathbb{H}$. Based on this, the quaternion q can also be defined as a four-tuple:

$$q = q_0 1 + q_1 i + q_2 j + q_3 k = (q_0, q_1, q_2, q_3),$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$.

Quaternions extend the complex numbers to four-dimensional space. The conjugate $\overline{q}$ of the quaternion q is given by $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$ and the norm and absolute value of q are $||q|| = q\overline{q} = \overline{q}q$ and $|q| = \sqrt{||q||}$, respectively. For any quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$, the scalar part and vector part of q is given by $\mathrm{Sc}(q) = q_0$ and $\mathrm{Vec}(q) = q_1 i + q_2 j + q_3 k$, respectively.

The center of $\mathbb{H}$ is the set of real quaternions, that is, the set of quaternions whose vector part is zero:

$$\{q \in \mathbb{H} : qp = pq,\ \forall p \in \mathbb{H}\} = \{(q_0, 0, 0, 0) : q_0 \in \mathbb{R}\} \subset \mathbb{H},$$

which is clearly isomorphic to $\mathbb{R}$.

In this work, it is our purpose to introduce and study a new quaternion sequence with Gaussian integers and present some of its properties. A Gaussian integer $x + iy$ is a complex number whose real and imaginary parts are both integers, i.e. $x, y \in \mathbb{Z}$. The conjugate of $p = x + iy$ is $\overline{p} = x - iy$. The set of all Gaussian integers with the usual addition and multiplication of complex numbers forms an integral domain.

The use of numerical sequences defined recursively together with the Gaussian type integers, gives rise to a new sequence of complex numbers, such that, for instance, the sequences of Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell–Lucas, Gaussian Jacobsthal and Gaussian Bronze Fibonacci numbers. There are several studies dedicated to these sequences of Gaussian numbers such as the works in [2], [3], [10], [17], [19], [21] and [22], among others.

The sequence that will serve as the basis for the study that we will present in this article is the sequence of Bronze Fibonacci numbers that was extended to the Gaussian Bronze Fibonacci numbers by Kartal in [22].

The sequence $\{BF_n\}_{n\geq 0}$ of Bronze Fibonacci numbers is also called as 3-Fibonacci sequence, listed in The On-line Encyclopedia of Integer Sequences ([29]) as the sequence A006190 and defined by the following recurrence relation

$$BF_{n+2} = 3BF_{n+1} + BF_n, \tag{1.2}$$

with the initial conditions $BF_0 = 0$ and $BF_1 = 1$.

In Table 1, we present a few Bronze Fibonacci numbers.

Table 1. The Bronze Fibonacci numbers BF_n, for $0 \leq n \leq 7$

BF_0	BF_1	BF_2	BF_3	BF_4	BF_5	BF_6	BF_7
0	1	3	10	33	109	360	1189

The characteristic equation associated with the recurrence relation (1.2) is $r^2 - 3r - 1 = 0$, which has two roots $r_1 = \frac{3+\sqrt{13}}{2}$, called the Bronze number, and $r_2 = \frac{3-\sqrt{13}}{2}$. Note that $r_1 + r_2 = 3$, $r_1 - r_2 = \sqrt{13}$, $r_1 r_2 = -1$, $\frac{r_1}{r_2} = -r_1^2$ and $\frac{r_2}{r_1} = -r_2^2$. The Binet's formula of the sequence $\{BF_n\}_{n \geq 0}$ is given by

$$BF_n = \frac{r_1^n - r_2^n}{r_1 - r_2}. \tag{1.3}$$

The next result is known as the Convolution Identity and it's one more identity that is satisfied by the sequence $\{BF_n\}_{n \geq 0}$. This identity will be used later in this article when studying the Cassini's identity for the new sequence of quaternions that we are going to introduce.

Lemma 1.1. *For m, n integers such that $m, n \geq 1$, the following identity is satisfied by the Bronze Fibonacci sequence*

$$BF_{m+n} = BF_{m-1} BF_n + BF_m BF_{n+1}.$$

Proof. We fix m and proceed by induction on n. If $n = 1$ then $BF_{m+1} = BF_{m-1} BF_1 + BF_m BF_2 = BF_{m-1}.1 + BF_m.3$, which is true by the recurrence relation (1.2). Now let us suppose that

$$BF_{m+l} = BF_{m-1} BF_l + BF_m BF_{l+1}, \quad l \geq 1. \tag{1.4}$$

Once more by the recurrence relation (1.2) and the induction hypothesis (1.4), we obtain

$$\begin{aligned} BF_{m+l+1} &= 3\left(BF_{m-1} BF_l + BF_m BF_{l+1}\right) + \left(BF_{m-1} BF_{l-1} + BF_m BF_l\right) \\ &= BF_{m-1}\left(3BF_l + BF_{l-1}\right) + BF_m\left(3BF_{l+1} + BF_l\right) \\ &= BF_{m-1} BF_{l+1} + BF_m BF_{l+2}, \end{aligned}$$

which proves the claim. $\square$

The particular cases of $m = n$ and $m = n + 1$ give the expressions of the general term of Bronze Fibonacci numbers with index even and odd, respectively, and they are given as

$$BF_{2n} = BF_{n-1}BF_n + BF_nBF_{n+1} \tag{1.5}$$

and

$$BF_{2n+1} = BF_n^2 + BF_{n+1}^2. \tag{1.6}$$

Extending the sequence $\{BF_n\}_{n\geq 0}$ to the Gaussian Bronze Fibonacci sequence $\{GBF_n\}_{n\geq 0}$, the recurrence relation satisfied by the last sequence is similar to the first one and it is given as follows:

$$GBF_{n+2} = 3GBF_{n+1} + GBF_n, \tag{1.7}$$

with the initial conditions $GBF_0 = i$ and $GBF_1 = 1$ and GBF_n is a complex number given by $GBF_n = BF_n + iBF_{n-1}$, where BF_j is the j^{th} Bronze-Fibonacci number.

In Table 2, we present a few Gaussian Bronze Fibonacci numbers.

Table 2. The Gaussian Bronze Fibonacci numbers GBF_n, for $0 \leq n \leq 5$

GBF_0	GBF_1	GBF_2	GBF_3	GBF_4	GBF_5
i	1	$3+i$	$10+3i$	$33+10i$	$109+33i$

According to Theorems 2 and 3 stated in [22], the generating function and the Binet's formula, for the sequence $\{GBF_n\}_{n\geq 0}$, are given, respectively, by

$$g(z) = \sum_{n=0}^{\infty} GBF_n z^n = \frac{z + i\,(1-3z)}{1-3z-z^2}$$

and

$$GBF_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}, \tag{1.8}$$

where α and β are the roots of the characteristic equation associated with the recurrence relation (1.7).

In [14], Halici and in [15], Halici and Cerda-Morales studied properties of quaternions Gaussian Lucas and quaternions Gaussian Fibonacci numbers, respectively, where some basic identities are stated and with the use of the Binet's formula, some fundamental relations between these numbers are presented.

Motivated essentially by the recent works [14], [15] and [22], in this paper, we introduce the quaternion-Gaussian Bronze Fibonacci sequence and we study some properties, including the Binet's formula, the generating function and some other interesting identities.

The structure of the article is as follows: in the next section, we introduce the formal definition of the quaternionic Gaussian Bronze Fibonacci sequence and we give the Binet's formula and the generating function. Section 3 is dedicated to the study of some identities satisfied by this sequence, such as the sum of terms and the norm value. Section 4 is devoted to results involving some special tridiagonal matrices and their determinants. In particular, we give alternative ways to determine the general term of the Bronze Fibonacci sequence, of the Gaussian Bronze Fibonacci sequence and of the quaternion Gaussian Bronze Fibonacci sequence. Finally, some perspectives of future work are given in the last section of this paper.

2. Quaternion-Gaussian Bronze Fibonacci Numbers, Binet's formula and Generating function

In this section we introduce the sequence of quaternions whose terms are Gaussian Bronze Fibonacci numbers. We begin with a formal definition of the general term in the quaternion-Gaussian Bronze Fibonacci number sequence.

Definition 2.1. For a non negative integer n, let us denote the sequence of quaternion-Gaussian Bronze Fibonacci by $\{GBFQ_n\}_{n\geq 0}$. The general term is given by

$$\begin{aligned} GBFQ_n &= GBF_n 1 + GBF_{n+1} i + GBF_{n+2} j + GBF_{n+3} k \\ &= (GBF_n, GBF_{n+1}, GBF_{n+2}, GBF_{n+3}) \end{aligned}$$

where GBF_j is the j^{th} term of the sequence of Gaussian Bronze Fibonacci numbers and $1, i, j$ and k are the four base elements that satisfy the rules (1.1).

According to the recurrence relation (1.7) (see also [22]), for the sequence of quaternion-Gaussian Bronze Fibonacci, we can write the following recurrence relation:

$$GBFQ_{n+2} = 3GBFQ_{n+1} + GBFQ_n, \quad n \geq 0. \tag{2.1}$$

Note that the initial conditions are

$$GBFQ_0 = (i, 1, 3 + i, 10 + 3i) = 2i + 11k, \tag{2.2}$$

$$GBFQ_1 = (1, 3 + i, 10 + 3i, 33 + 10i) = 3(i + 12k). \tag{2.3}$$

The next result shows the Binet's formula, which gives the term of order n of the sequence $\{GBFQ_n\}_{n\geq 0}$ without having to resort to other terms of the sequence, as we have mentioned before. To get it we will use a result from [20]. Here the authors found the solutions for quaternionic quadratic equations of the special form $x^2+bx+c = 0$, where $b, c \in \mathbb{H}$. In the particular case in which $b, c \in \mathbb{R}$ and $b^2 \geq 4c$, it was proven that the quaternionic quadratic equation has exactly two distinct solutions lying in the centre of $\mathbb{H}$, that is, in the set of real numbers: $x_1 = \frac{-b+\sqrt{b^2-4c}}{2}$ and $x_2 = \frac{-b-\sqrt{b^2-4c}}{2}$ (see Theorem 2.3, case 2, and Corollary 2.9 in reference [20]).

Theorem 2.2. *For $n \geq 0$, the n^{th} term of the sequence $\{GBFQ_n\}_{n\geq 0}$ is given as follows*

$$GBFQ_n = \frac{1}{s_1 - s_2}\left(GBFQ_1\left(s_1^n - s_2^n\right) + GBFQ_0\left(s_1 s_2^n - s_2 s_1^n\right)\right),$$

where s_1 and s_2 are the roots of the characteristic equation associated with the recurrence relation (2.1).

Proof. The characteristic equation associated with the recurrence relation (2.1) is $s^2 - 3s - 1 = 0$. It was proven by Huang and So in [20] that this quadratic equation, with quaternion indeterminate s, has exactly two distinct solutions which lie in the centre of $\mathbb{H}$, that is, in the set of real numbers:

$$s_1 = \frac{3+\sqrt{13}}{2} \quad \text{and} \quad s_2 = \frac{3-\sqrt{13}}{2}.$$

Therefore, $GBFQ_n = As_1^n + Bs_2^n$ is the general solution of equation (2.1). Considering $n = 0$ and $n = 1$ in this identity and solving this system of linear equations, we obtain unique values for A and for B, which are, in this case, $A = \frac{GBFQ_1 - GBFQ_0 s_2}{s_1 - s_2}$ and $B = \frac{GBFQ_0 s_1 - GBFQ_1}{s_1 - s_2}$. So, using these values in the expression of $GBFQ_n$ stated before and performing some calculations, we get the required result. □

Corollary 2.3. *For $n \geq 1$, the n^{th} term of the sequence $\{GBFQ_n\}_{n\geq 0}$ is given as follows*

$$GBFQ_n = BF_n\, GBFQ_1 + BF_{n-1}\, GBFQ_0. \tag{2.4}$$

Proof. Since the roots s_1 and s_2 of the quaternionic characteristic equation associated with the recurrence relation (2.1) lie in $\mathbb{R}$, the center of $\mathbb{H}$, we have $s_1 = r_1$ and $s_2 = r_2$, with r_1 and r_2 as in (1.3). We get $s_1 s_2^n - s_2 s_1^n = r_1 r_2^n - r_2 r_1^n = r_1 r_2 (r_2^{n-1} - r_1^{n-1}) = r_1^{n-1} - r_2^{n-1}$ and the result follows. □

In the following result we give the generating function for the sequence $\{GBFQ_n\}_{n\geq 0}$.

Theorem 2.4. *The generating function for the sequence $\{GBFQ_n\}_{n\geq 0}$ is given by*

$$\begin{aligned} g_{GBFQ_n}(t) = \sum_{n=0}^{\infty} GBFQ_n t^n &= \frac{GBFQ_0\,(1-3t) + GBFQ_1 t}{1-3t-t^2} \\ &= \frac{(2i+11k)+(-3i+3k)t}{1-3t-t^2}. \end{aligned}$$

Proof. Using the definition of generating function, we have

$$g_{GBFQ_n}(t) = GFBQ_0 + GFBQ_1 t + GFBQ_2 t^2 + \ldots + GFBQ_n t^n + \ldots$$

and multiplying both sides of this identity by $-3t$ and $-t^2$, from the recurrence relation (2.1), we obtain that

$$\left(1-3t-t^2\right) g_{GBFQ_n}(t) = GFBQ_0\,(1-3t) + GFBQ_1 t$$

and the result follows. □

3. Some identities involving the quaternionic sequence of Gaussian Bronze Fibonacci

In this section we present some identities that are satisfied by the quaternionic sequence of Gaussian Bronze Fibonacci. We start by the statement of the expression of the sum of the first $n+1$ elements of the sequence $\{GBFQ_j\}_{j\geq 0}$.

THEOREM 3.1. *The sum of the first* $n+1$ *terms of the sequence* $\{GBFQ_j\}_{j\geq 0}$ *is given by*

$$\begin{aligned}\sum_{l=0}^{n} GBFQ_l &= \frac{1}{3}\left(GBFQ_n + GBFQ_{n+1} + 2GBFQ_0 - GBFQ_1\right)\\ &= \frac{1}{3}\left(GBFQ_n + GBFQ_{n+1} + (i-14k)\right).\end{aligned}$$

PROOF. Taking into account the recurrence relation (2.1), we can equivalently consider that $\frac{1}{3}\left(GBFQ_{n+2} - GBFQ_n\right) = GBFQ_{n+1}$ and then

$$\begin{aligned}GBFQ_1 &= \frac{1}{3}\left(GBFQ_2 - GBFQ_0\right),\\ GBFQ_2 &= \frac{1}{3}\left(GBFQ_3 - GBFQ_1\right),\\ GBFQ_3 &= \frac{1}{3}\left(GBFQ_4 - GBFQ_2\right),\\ &\vdots\\ GBFQ_{n-2} &= \frac{1}{3}\left(GBFQ_{n-1} - GBFQ_{n-3}\right),\\ GBFQ_{n-1} &= \frac{1}{3}\left(GBFQ_n - GBFQ_{n-2}\right),\\ GBFQ_n &= \frac{1}{3}\left(GBFQ_{n+1} - GBFQ_{n-1}\right).\end{aligned}$$

It follows that

$$\sum_{l=1}^{n} GBFQ_l = \frac{1}{3}\left(GBFQ_n + GBFQ_{n+1} - GBFQ_0 - GBFQ_1\right)$$

and

$$\sum_{l=0}^{n} GBFQ_l = GBFQ_0 + \frac{1}{3}\left(GBFQ_n + GBFQ_{n+1} - GBFQ_0 - GBFQ_1\right)$$

and after some calculations, we get the required result. □

The next two results are related to the sum of terms of the sequence $\{GBFQ_j\}_{j\geq 0}$ with even and odd indexes. For these results the respective proofs are omitted as it can be easily made in a similar way to the proof of the previous theorem.

Theorem 3.2. *The sum of the even indexed quaternion Gaussian Bronze Fibonacci numbers is given by*

$$\sum_{l=1}^{n} GBFQ_{2l} = \frac{1}{3}\left(GBFQ_{2n+1} - GBFQ_1\right)$$
$$= \frac{1}{3}GBFQ_{2n+1} - (i + 12k).$$

Now for the odd indexes we have

Theorem 3.3. *The sum of the odd indexed quaternion Gaussian Bronze Fibonacci numbers is given by*

$$\sum_{l=0}^{n} GBFQ_{2l+1} = \frac{1}{3}\left(GBFQ_{2n+2} - GBFQ_0\right)$$
$$= \frac{1}{3}\left(GBFQ_{2n+2} - (2i + 11k)\right).$$

Our next result establishes an identity involving the sum of squares of two consecutive terms.

Proposition 3.4. *Let n be a non negative integer. Then the following identity follows*

$$GBFQ_n^2 + GBFQ_{n+1}^2 = -143\, BF_{2n+3},$$

where BF_n is the n^{th} Bronze Fibonacci number.

Proof. From equality (2.4) we get

$$GBFQ_n^2 = BF_n^2 GBFQ_1^2 + BF_{n-1}^2 GBFQ_0^2$$
$$+ BF_n BF_{n-1}(GBFQ_1 GBFQ_0 + GBFQ_0 GBFQ_1).$$

Since (see (2.2) and (2.3))

$$GBFQ_0^2 = (2i + 11k)(2i + 11k) = -125,$$

$$GBFQ_1^2 = 9(i + 12k)(i + 12k) = -1305,$$

$$GBFQ_1 GBFQ_0 = 3(i + 12k)(2i + 11k) = 3(-134 + 13j),$$

$$GBFQ_0 GBFQ_1 = 3(2i + 11k)(i + 12k) = 3(-134 - 13j),$$

it follows

$$GBFQ_n^2 = -(1305BF_n^2 + 804BF_nBF_{n-1} + 125BF_{n-1}^2).$$

Similarly,

$$GBFQ_{n+1}^2 = -(1305BF_{n+1}^2 + 804BF_{n+1}BF_n + 125BF_n^2).$$

Therefore

$$\begin{aligned} GBFQ_n^2 + GBFQ_{n+1}^2 = &-1305(BF_n^2 + BF_{n+1}^2) - 125(BF_{n-1}^2 + BF_n^2) \\ &- 804(BF_nBF_{n-1} + BF_{n+1}BF_n), \end{aligned}$$

and, taking into account equalities (1.5) and (1.6),

$$\begin{aligned} GBFQ_n^2 + GBFQ_{n+1}^2 &= -(1305BF_{2n+1} + 125BF_{2n-1} + 804BF_{2n}) \\ &= -(1305(3BF_{2n} + BF_{2n-1}) + 125BF_{2n-1} + 804BF_{2n}) \\ &= -(4719BF_{2n} + 1430BF_{2n-1}) \\ &= -(429BF_{2n} + 1430BF_{2n+1}) \\ &= -(429BF_{2n+2} + 143BF_{2n+1}) \\ &= -143BF_{2n+3}. \end{aligned}$$

□

The next result concerns the norm of quaternion-Gaussian Bronze Fibonacci numbers. We observe that $GBFQ_n$ is a quaternion with complex entries (see Definition 2.1), whose norm is obtained as

$$\|GBFQ_n\| = GBFQ_n\overline{GBFQ_n} = GBF_n^2 + GBF_{n+1}^2 + GBF_{n+2}^2 + GBF_{n+3}^2,$$

where $\overline{GBFQ_n} = GBF_n - GBF_{n+1}i - GBF_{n+2}j - GBF_{n+3}k$ is the 'Hamiltonian' or quaternion conjugate (see, for instance, [30]).

Theorem 3.5. *The norm value of $GBFQ_n$ is*

$$\|GBFQ_n\| = 11(3 + 2i)BF_{2n+2},$$

where BF_n is the n^{th} Bronze Fibonacci number.

PROOF. We start by proving two claims.
CLAIM 1. For $n \geq 0$,

$$GBF_n^2 = \frac{(1-i\beta)^2\alpha^{2n} - 2(-1)^n(2-3i) + (1-i\alpha)^2\beta^{2n}}{(\alpha-\beta)^2},$$

where $\alpha = \frac{3+\sqrt{13}}{2}$ and $\beta = \frac{3-\sqrt{13}}{2}$ are the roots of the characteristic equation associated with the recurrence relation (1.7).

PROOF OF CLAIM 1. The Binet's formula for Gaussian Bronze Fibonacci sequence is given by (see (1.8))

$$GBF_n = \frac{\alpha^n - \beta^n}{\alpha-\beta} + i\frac{\alpha\beta^n - \beta\alpha^n}{\alpha-\beta},$$

where α and β are the roots of the characteristic equation associated with the recurrence relation (1.7). We re-write

$$GBF_n = \frac{1}{\alpha-\beta}\left[(1-i\beta)\alpha^n - (1-i\alpha)\beta^n\right].$$

Therefore

$$(\alpha-\beta)^2 GBF_n^2 = (1-i\beta)^2\alpha^{2n} - 2(\alpha\beta)^n\left[1 - i(\alpha+\beta) - \alpha\beta\right] + (1-i\alpha)^2\beta^{2n}.$$

Since $\alpha+\beta = 3$ and $\alpha\beta = -1$ we obtain the desired result.
CLAIM 2.

$$\|GBFQ_n\| = 11(3+2i)\frac{\alpha^{2n-1}(36\alpha+11) + \beta^{2n-1}(36\beta+11)}{(\alpha-\beta)^2}.$$

PROOF OF CLAIM 2. By Claim 1,

$$\begin{aligned}
(\alpha-\beta)^2\|GBFQ_n\| &= (1-i\beta)^2\alpha^{2n} - 2(-1)^n(2-3i) + (1-i\alpha)^2\beta^{2n} \\
&\quad + (1-i\beta)^2\alpha^{2n+2} - 2(-1)^{n+1}(2-3i) + (1-i\alpha)^2\beta^{2n+2} \\
&\quad + (1-i\beta)^2\alpha^{2n+4} - 2(-1)^{n+2}(2-3i) + (1-i\alpha)^2\beta^{2n+4} \\
&\quad + (1-i\beta)^2\alpha^{2n+6} - 2(-1)^{n+3}(2-3i) + (1-i\alpha)^2\beta^{2n+6} \\
&= (1-i\beta)^2\alpha^{2n}(1+\alpha^2+\alpha^4+\alpha^6) \\
&\quad + (1-i\alpha)^2(1+\beta^2+\beta^4+\beta^6).
\end{aligned}$$

Since α and β are the roots of the characteristic equation $r^2 - 3r - 1 = 0$, the repeated relations of α and β satisfy the equalities

$$\alpha^n = BF_n\alpha + BF_{n-1} \qquad \text{and} \qquad \beta^n = BF_n\beta + BF_{n-1},$$

thus yielding

$$1 + \alpha^2 + \alpha^4 + \alpha^6 = 396\alpha + 121 \quad \text{and} \quad 1 + \beta^2 + \beta^4 + \beta^6 = 396\beta + 121.$$

Therefore

$$\begin{aligned}
(\alpha - \beta)^2 \|GBFQ_n\| &= (1 - i\beta)^2\alpha^{2n}(396\alpha + 121) + (1 - i\alpha)^2\beta^{2n}(396\beta + 121) \\
&= 11\left[(1 - \beta^2)\alpha^{2n}(36\alpha + 11) + (1 - \alpha^2)\beta^{2n}(36\beta + 11)\right] \\
&\quad + 22i\left[\alpha^{2n-1}(36\alpha + 11) + \beta^{2n-1}(36\beta + 11)\right].
\end{aligned}$$

Notice that $1 - \alpha^2 = -3\alpha$ and $1 - \beta^2 = -3\beta$. Then

$$\begin{aligned}
(\alpha - \beta)^2 \|GBFQ_n\| &= 11\left[-3\beta\alpha^{2n}(36\alpha + 11) - 3\alpha\beta^{2n}(36\beta + 11)\right] \\
&\quad + 22i\left[\alpha^{2n-1}(36\alpha + 11) + \beta^{2n-1}(36\beta + 11)\right] \\
&= 11\left[3\alpha^{2n-1}(36\alpha + 11) + 3\beta^{2n-1}(36\beta + 11)\right] \\
&\quad + 22i\left[\alpha^{2n-1}(36\alpha + 11) + \beta^{2n-1}(36\beta + 11)\right] \\
&= 11(3 + 2i)\left[\alpha^{2n-1}(36\alpha + 11) + \beta^{2n-1}(36\beta + 11)\right],
\end{aligned}$$

thus proving Claim 2.

It only remains to show that

$$\frac{\alpha^{2n-1}(36\alpha + 11) + \beta^{2n-1}(36\beta + 11)}{(\alpha - \beta)^2} = BF_{2n+2},$$

or, equivalently, that

$$BF_{2n+2}(\alpha - \beta) = \frac{\alpha^{2n-1}(36\alpha + 11) + \beta^{2n-1}(36\beta + 11)}{\alpha - \beta}.$$

Notice that, by the Binet's formula of the sequence $\{BF_n\}$ (see (1.3)), we obtain

$$\begin{aligned} BF_{2n+2}(\alpha-\beta) &= \frac{\alpha^{2n+3}-\alpha^{2n+2}\beta-\beta^{2n+2}\alpha+\beta^{2n+3}}{\alpha-\beta} \\ &= \frac{\alpha^{2n+3}+\alpha^{2n+1}+\beta^{2n+1}+\beta^{2n+3}}{\alpha-\beta} \\ &= \frac{\alpha^{2n-1}(\alpha^4+\alpha^2)+\beta^{2n-1}(\beta^4+\beta^2)}{\alpha-\beta} \\ &= \frac{\alpha^{2n-1}(36\alpha+11)+\beta^{2n-1}(36\beta+11)}{\alpha-\beta}. \qquad \square \end{aligned}$$

4. Some tridiagonal matrices

In this section, we consider some special tridiagonal matrices which will allow us to further establish properties of quaternion-Gaussian Bronze Fibonacci numbers. We start by introducing a tridiagonal matrix A_n which allows us to find the scalar part of $GBFQ_n$. Then we find a matrices identity giving the components of $GBFQ_n$ and the Cassini's identity is also established. Finally, motivated by the works of [13] and [24], we give alternative ways of finding the general terms of the sequences $\{BF_n\}, \{GBF_n\}$ and $\{GBFQ_n\}$.

We define the following tridiagonal matrix with complex coefficients and order $n \times n$

$$A_n = \begin{bmatrix} 1 & i & 0 & \dots & \dots & \dots & 0 \\ -1 & 3 & 1 & \ddots & & & \vdots \\ 0 & -1 & 3 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 3 & 1 & 0 \\ \vdots & & & \ddots & -1 & 3 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 3 \end{bmatrix}. \tag{4.1}$$

With this matrix we can determine the scalar part of the general term of the sequence $\{GBFQ_j\}_{j\geq 0}$, by the use of the value of the respective determinant. At this point, it is important to remark that, due to the noncommutative multiplication of the quaternion elements, whenever we calculate the determinant of a square matrix with quaternions entries using the Laplace expansion, we will always start with the entries of the last column, i.e., for any square matrix $X = [x_{ij}]_{n\times n}$ the determinant is given by

$$\det X = \sum_{i=1}^{n} c_{in} x_{in},$$

with

$$c_{in} = (-1)^{i+n} \det Y_{in},$$

where $\det Y_{in}$ is the i, n minor of X.

Proposition 4.1. *The scalar part GBF_j of the general term of the sequence $\{GBFQ_j\}_{j\geq 0}$ is given by*

$$\det(A_j) = GBF_j = Sc(GBFQ_j).$$

Proof. We proceed by induction on j. For $j = 1$ and $j = 2$, we have respectively, $\det(A_1) = 1 = GBF_1 = \mathrm{Sc}(GBFQ_1)$, $\det(A_2) = 3+i = GBF_2 = \mathrm{Sc}(GBFQ_2)$, which is true by Definition 2.1. Let us assume the claim for $j-1$ and $j-2$. So, $\det(A_{j-1}) = GBF_{j-1} = \mathrm{Sc}(GBFQ_{j-1})$ and $\det(A_{j-2}) = GBF_{j-2} = \mathrm{Sc}(GBFQ_{j-2})$. Then, the determinant of the matrix A_j can be computed by using the Laplace expansion along the last column and so:

$$\det(A_j) = 3\,(-1)^{2j}\det(A_{j-1}) + 1\,(-1)^{2j-1}\det\begin{bmatrix} 1 & i & 0 & \dots & \dots & 0 \\ -1 & 3 & 1 & \ddots & & \vdots \\ 0 & -1 & 3 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & -1 & 3 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 \end{bmatrix}.$$

By the use of properties of determinants we know that

$$\det\begin{bmatrix} 1 & i & 0 & \dots & \dots & 0 \\ -1 & 3 & 1 & \ddots & & \vdots \\ 0 & -1 & 3 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & -1 & 3 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 \end{bmatrix} = \det\begin{bmatrix} 1 & i & 0 & \dots & \dots & 0 \\ -1 & 3 & 1 & \ddots & & \vdots \\ 0 & -1 & 3 & & & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & -1 & 3 & 0 \\ 0 & \dots & \dots & \dots & 0 & -1 \end{bmatrix}.$$

Once more, the use of the Laplace expansion along the last column of the second determinant yields

$$\det(A_j) = 3\det(A_{j-1}) + 1\,(-1)^{2j-1}\,(-1)\,(-1)^{2j-2}\det(A_{j-2}),$$

and so $\det(A_j) = 3\det(A_{j-1}) + \det(A_{j-2})$ and by the induction hypothesis the result follows. $\square$

The next result also uses other matrices and with it we can determine the components of $GBFQ_n$. We have

PROPOSITION 4.2. *Let n be integer and $n \geq 1$. Then, it holds that*

$$\begin{bmatrix} GBFQ_n & GBFQ_{n-1} \\ GBFQ_{n+1} & GBFQ_n \end{bmatrix} = \begin{bmatrix} GBFQ_1 & GBFQ_0 \\ GBFQ_2 & GBFQ_1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^{n-1}.$$

PROOF. We use induction on n. For $n = 1$, we get

$$\begin{bmatrix} GBFQ_1 & GBFQ_0 \\ GBFQ_2 & GBFQ_1 \end{bmatrix} = \begin{bmatrix} GBFQ_1 & GBFQ_0 \\ GBFQ_2 & GBFQ_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is true. Let us assume that the above equality is true for n. Then, for $n+1$, we write

$$\begin{aligned} \begin{bmatrix} GBFQ_1 & GBFQ_0 \\ GBFQ_2 & GBFQ_1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^{n} &= \begin{bmatrix} GBFQ_1 & GBFQ_0 \\ GBFQ_2 & GBFQ_1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} GBFQ_n & GBFQ_{n-1} \\ GBFQ_{n+1} & GBFQ_n \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3GBFQ_n + GBFQ_{n-1} & GBFQ_n \\ 3GBFQ_{n+1} + GBFQ_n & GBFQ_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} GBFQ_{n+1} & GBFQ_n \\ GBFQ_{n+2} & GBFQ_{n+1} \end{bmatrix}, \end{aligned}$$

as required. $\square$

Taking advantage of the previous proposition, we state an identity related with this type of quaternion sequence designated by Cassini's identity.

THEOREM 4.3. *Let n be integer and $n \geq 1$. Then the following identity follows*

$$\begin{aligned} GBFQ_n^2 - GBFQ_{n+1}GBFQ_{n-1} &= (-1)^{n-1}\left(GBFQ_1^2 - GBFQ_2GBFQ_0\right) \\ &= (-1)^{n-1}(26 - 117j). \end{aligned}$$

PROOF. From the equality of the previous proposition, we get

$$\det\begin{bmatrix} GBFQ_n & GBFQ_{n-1} \\ GBFQ_{n+1} & GBFQ_n \end{bmatrix} = \det\begin{bmatrix} GBFQ_1 & GBFQ_0 \\ GBFQ_2 & GBFQ_1 \end{bmatrix}\left(\det\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}\right)^{n-1}$$

and then the result follows from equalities (2.1), (2.2) and (2.3). □

Let us consider now the following $n \times n$ tridiagonal matrices:

$$M_n = \begin{bmatrix} a & b & 0 & \dots & \dots & \dots & 0 \\ c & d & e & \ddots & & & \vdots \\ 0 & c & d & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & d & e & 0 \\ \vdots & & & \ddots & c & d & e \\ 0 & \dots & \dots & \dots & 0 & c & d \end{bmatrix}.$$

We get, see [13],

$$\begin{aligned} |M_1| &= a, \\ |M_2| &= d|M_1| - bc, \\ |M_3| &= d|M_2| - ce|M_1|, \\ |M_4| &= d|M_3| - ce|M_2|, \\ &\vdots \\ |M_{n+1}| &= d|M_n| - ce|M_{n-1}|. \end{aligned}$$

If $a = d = 3, b = e = 1$ and $c = -1$ the above matrix M_n becomes, in this case,

$$M_n = \begin{bmatrix} 3 & 1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 3 & 1 & \ddots & & & \vdots \\ 0 & -1 & 3 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 3 & 1 & 0 \\ \vdots & & & \ddots & -1 & 3 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 3 \end{bmatrix}, \tag{4.2}$$

and, accordingly, we also have

$$\begin{aligned} |M_1| &= 3 = BF_2, \\ |M_2| &= 10 = BF_3, \\ |M_3| &= 33 = BF_4, \\ |M_4| &= 109 = BF_5, \\ &\vdots \\ |M_{n+1}| &= 3|M_n| + |M_{n-1}|. \end{aligned}$$

We thus get the following result:

Proposition 4.4. *If M_n is the $n \times n$ tridiagonal matrix defined in (4.2), then the n^{th} Bronze Fibonacci number is given by*

$$BF_n = |M_{n-1}|, \quad n \geq 2.$$

Accordingly with the recurrence relation (1.7), we immediately obtain the next result, which offers an alternative way to calculate the n^{th} Gaussian Bronze Fibonacci numbers using determinants.

Corollary 4.5. $GBF_n = |M_{n-1}| + i|M_{n-2}|, \quad n \geq 3.$

Also, by Definition 2.1, we can express the quaternion-Gaussian Bronze Fibonacci numbers $GBFQ_n$ by the use of some determinants.

Corollary 4.6. *For $n \geq 3$,*

$$\begin{aligned} GBFQ_n = (&|M_{n-1}| + i|M_{n-2}|, \\ &|M_n| + i|M_{n-1}|, |M_{n+1}| + i|M_n|, |M_{n+2}| + i|M_{n+1}|). \end{aligned}$$

In Proposition 4.1 we proved that the determinant of the matrix A_j defined in (4.1) gives the scalar part of $GBFQ_j$, for $j \geq 0$. Our purpose now is to obtain an alternative way, using determinants, to compute BF_n, GBF_n and $GBFQ_n$. We start by recalling that, accordingly with [24] (see also [23]), for any sequence $\{x_n\}_{n\geq 0}$ satisfying the second order linear recurrence

$$x_{n+1} = Ax_n + Bx_{n-1}, \quad n \geq 1,$$

with $x_0 = C, x_1 = D$, and A, B, C, D real numbers, we have, for $n \geq 0$

$$x_n = \begin{vmatrix} C & D & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & B & 0 & \dots & 0 & 0 \\ 0 & -1 & A & B & \dots & 0 & 0 \\ 0 & 0 & -1 & A & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A & B \\ 0 & 0 & 0 & 0 & \dots & -1 & A \end{vmatrix}_{(n+1)\times(n+1)}.$$

In particular, considering $A = 3, B = 1, C = 0$ and $D = 1$ (compare with the recurrence relation (1.2)) we obtain

$$BF_n = \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 3 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 3 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 3 \end{vmatrix}_{(n+1)\times(n+1)}, \quad n \geq 0.$$

Similarly, from the recurrence relation (1.7) with initial conditions $GBF_0 = i$ and $GBF_1 = 1$ we conclude that

$$GBF_n = \begin{vmatrix} i & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 3 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 3 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 3 \end{vmatrix}_{(n+1)\times(n+1)}, \quad n \geq 0,$$

and, finally, considering the recurrence relation (2.1) with initial conditions $GBFQ_0 = 2i + 11k$ and $GBFQ_1 = 3i + 36k$ we get

$$GBFQ_n = \begin{vmatrix} 2i+11k & 3i+36k & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 3 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 3 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 3 \end{vmatrix}_{(n+1)\times(n+1)}, \quad n \geq 0.$$

5. Future work

The subject of this paper has the potential to motivate future researches on the applications of these sequences in matrix theory, combinatorial number theory, and other areas involving matrix algebras. Also, it is our goal to study more algebraic properties and the hybrid, the polynomial, and the hybrinomial versions of these sequences.

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