Annales Mathematicae Silesianae **36** (2022), no. 2, 176–183
DOI: 10.2478/amsil-2022-0014

GENERAL LIMIT FORMULAE INVOLVING PRIME NUMBERS

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Abstract. Let p_n be the nth prime number. In this note, we study strictly increasing sequences of positive integers A_n such that the limit $\lim_{n\to\infty}(A_1A_2\cdots A_n)^{1/p_n} = e$ holds. This limit formula is in fact a generalization of some previously known results. Furthermore, some other generalizations are established.

1. Introduction

Euler's number e, is a mathematical constant approximately equal to 2.718281828459045..., and can be characterized in many ways. The most well-known of them is the following fundamental and primordial limit formula (see, e.g., [2], [8])

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

A better approximation than the above common limit is obtained by the limit (see, e.g., [8])

$$\lim_{n\to\infty}\left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}} - \frac{(n+1)^{n+1}}{n^n}\right] = e.$$

Received: 28.03.2022. Accepted: 16.08.2022. Published online: 08.09.2022.
(2020) Mathematics Subject Classification: 11A41, 11A99, 40A20, 11Y60, 11B99.
Key words and phrases: limit formula, generalization, the number e, prime numbers.

It can also be expressed exactly by the following infinite series and limit in which $n!$ appears (see, e.g., [6] and [8])

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots,$$

and

$$\lim_{n \to \infty} \frac{n}{\sqrt[n]{n!}} = e.$$

There are also some limit formulae involving recursive sequences that tend to the number e. For example, let F_n be the nth Fibonacci number. Following [3] and [5], we have

$$\lim_{n \to \infty} \left(\frac{\ln F_{n+1}}{\ln F_n} \right)^n = e,$$

$$\lim_{n \to \infty} \frac{\sqrt[n]{\ln F_3 \ln F_4 \cdots \ln F_n}}{\ln F_n} = \frac{1}{e}.$$

This amazing mathematical constant can also be expanded by prime numbers. Let p_n be the nth prime number, we have (see [4], [9])

$$\lim_{n \to \infty} \frac{p_n}{\sqrt[n]{p_1 p_2 \cdots p_n}} = e.$$

Furthermore, the well-known prime number theorem (PNT) in the form $\vartheta(x) = \sum_{p \le x} \ln p \sim x$ implies that $\sum_{i=1}^{n} \ln p_i \sim p_n$, which gives (see [1] and [7])

$$\lim_{n \to \infty} (p_1 p_2 \cdots p_n)^{\frac{1}{p_n}} = e. \tag{1.1}$$

On the other hand, from the well-known Stirling's approximation $n! \sim \sqrt{2\pi} \frac{n^n \sqrt{n}}{e^n}$ and by use of the PNT in the form $p_n \sim n \ln n$, we obtain

$$\ln 1 + \ln 2 + \cdots + \ln n = n \ln n - n + o(n) \sim n \ln n, \tag{1.2}$$

which gives

$$\lim_{n \to \infty} (1 \cdot 2 \cdots n)^{\frac{1}{p_n}} = \lim_{n \to \infty} (n!)^{\frac{1}{p_n}} = e. \tag{1.3}$$

In this paper, we wish to establish sufficient conditions for a strictly increasing sequence A_n of positive integers satisfying the following limit formula

$$\lim_{n\to\infty} (A_1 A_2 \cdots A_n)^{\frac{1}{p_n}} = e, \tag{1.4}$$

that is, a generalization of limit formulae in (1.1) and (1.3). Furthermore, some other generalizations are established.

2. Main results

In this section we aim to present our main results. We first prove the following theorem.

THEOREM 2.1. *Let's consider a strictly increasing sequence A_n of positive integers such that $A_n = n^{1+o(1)}$. Then, limit formula* (1.4) *holds.*

PROOF. We give two proofs.

1) For sake of clarity we put $o(1) = g(n)$. We shall prove that A_n satisfies the asymptotic formula

$$(\ln A_1 + \ln A_2 + \cdots + \ln A_n) \sim n \ln n. \tag{2.1}$$

We have (see (1.2))

$$\begin{aligned}\sum_{i=1}^{n} \ln A_i = \sum_{i=1}^{n} \ln\left(i^{1+g(i)}\right) &= \sum_{i=1}^{n} \ln i + \sum_{i=1}^{n} g(i) \ln i \\ &= n \ln n + o(n \ln n) + \sum_{i=1}^{n} g(i) \ln i. \end{aligned} \tag{2.2}$$

Since $o(1) = g(i)$, we have $\lim_{i\to\infty} g(i) = 0$. Therefore given $\epsilon > 0$ there exists N (depending of ϵ) such that if $i > N$ then $|g(i)| < \epsilon$. Therefore we have

$$\begin{aligned}\left|\sum_{i=1}^{n} g(i) \ln i\right| &\le \sum_{i=1}^{n} |g(i)| \ln i \le \sum_{i=1}^{N} |g(i)| \ln i + \epsilon \sum_{i=N+1}^{n} \ln i \\ &\le \sum_{i=1}^{N} |g(i)| \ln i + \epsilon \sum_{i=1}^{n} \ln i \\ &= \sum_{i=1}^{N} |g(i)| \ln i + \epsilon\left(n \ln n + o(n \ln n)\right) \le 2\epsilon n \ln n. \end{aligned} \tag{2.3}$$

Hence, since $\epsilon > 0$ can be arbitrarily small, equation (2.3) gives

$$\sum_{i=1}^{n} g(i)\ln i = o\left(n\ln n\right). \tag{2.4}$$

Substituting equation (2.4) into equation (2.2) we obtain (2.1). The theorem is proved.

2) The following proposition is well-known ([11]): Let $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} a_i$ be two series of positive terms such that $b_i \sim a_i$. Then if $\sum_{i=1}^{\infty} a_i$ diverges we have $\sum_{i=1}^{n} b_i \sim \sum_{i=1}^{n} a_i$.

Now, equation $A_n = n^{1+o(1)}$ is equivalent to the limit $\frac{\ln A_n}{\ln n} \to 1$. Therefore the mentioned proposition and equality (1.2) gives (2.1). □

A family of sequences A_n that satisfy $A_n = n^{1+o(1)}$ is given in the next theorem. Before, we need the following definition.

Definition 2.2. Let $f(x)$ be a function defined on the interval $[a, \infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The *function* $f(x)$ is *of slow increase* if the following condition holds

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \tag{2.5}$$

Typical functions of slow increase are $f(x) = \ln x$, $f(x) = \ln\ln x$, $f(x) = \ln^2 x$, $f(x) = \frac{\ln x}{\ln\ln x}$. Functions of slow increase are studied in [10].

Theorem 2.3. *Let A_n be a strictly increasing sequence of positive integers such that $A_n \sim nf(n)$, where $f(x)$ is a function of slow increase. Then the sequence A_n satisfies limit in* (1.4).

Proof. We shall prove (see Theorem 2.1) that $A_n = n^{1+o(1)}$. We have

$$A_n = h(n)nf(n) = nn^{\frac{\ln h(n)}{\ln n} + \frac{\ln f(n)}{\ln n}},$$

where $h(n) \to 1$. Now, we have the trivial limit

$$\lim_{x\to\infty} \frac{\ln h(n)}{\ln n} = 0$$

and the limit (use L'Hospital's rule and (2.5))

$$\lim_{x\to\infty} \frac{\ln f(x)}{\ln x} = \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0.$$ □

Since (prime number theorem) $p_n \sim n \ln n$ and $\ln x$ is a function of slow increase, Theorem 2.3 is applicable and we obtain again limit in (1.1). That is,

$$\lim_{n\to\infty} (p_1 p_2 \cdots p_n)^{\frac{1}{p_n}} = e.$$

Let us consider the sequence $A_n = c_{n,k}$, where $c_{n,k}$ is the nth number with exactly $k(\geq 2)$ prime factors in their prime factorization. It is well-known ([10]) that these numbers satisfy the property $c_{n,k} \sim n f(n)$, where $f(x)$ is a function of slow increase. Therefore limit in (1.4) holds for these numbers. That is, we have

$$\lim_{n\to\infty} (c_{1,k} c_{2,k} \cdots c_{n,k})^{\frac{1}{p_n}} = e.$$

Now, we prove two curious theorems that relate an arbitrary sequence A_n, such that $\frac{A_{n+1}}{A_n} \to 1$, the prime numbers and the e number. These theorems generalize limit formula in (1.4).

THEOREM 2.4. *Let k be an arbitrary but fixed positive integer. Let us consider a strictly increasing sequence A_n $(n \geq 1)$ of positive integers such that*

$$A_{n+1} \sim A_n. \tag{2.6}$$

Let p_{A_n} be the A_nth prime number. The following asymptotic formulae hold:

$$\sum_{i=1}^{n} \left(A_{i+1}^k - A_i^k\right) \ln^k A_i \sim \left(p_{A_n}\right)^k, \tag{2.7}$$

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} A_i^{\left(A_{i+1}^k - A_i^k\right) \ln^{k-1} A_i}\right)^{\frac{1}{\left(p_{A_n}\right)^k}} = e. \tag{2.8}$$

In particular if $k = 1$, we obtain

$$\sum_{i=1}^{n} d_i \ln A_i \sim p_{A_n},$$

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} A_i^{d_i}\right)^{\frac{1}{p_{A_n}}} = e,$$

where $d_i = A_{i+1} - A_i$.

PROOF. Note that the function $\ln^k x$ is strictly increasing and continuous in the interval $[1,\infty)$. Therefore the integral mean value theorem applied in the interval $[A_n^k, A_{n+1}^k]$ gives

$$\int_{A_n^k}^{A_{n+1}^k} \ln^k x\, dx = \left(A_{n+1}^k - A_n^k\right) \ln^k c, \tag{2.9}$$

where c is such that

$$A_n^k < c < A_{n+1}^k. \tag{2.10}$$

Note that (see (2.6)) $A_{n+1} \sim A_n$ implies

$$\ln A_{n+1} \sim \ln A_n. \tag{2.11}$$

Properties (2.9) and (2.10) give

$$\left(A_{n+1}^k - A_n^k\right) \ln^k A_n^k < \int_{A_n^k}^{A_{n+1}^k} \ln^k x\, dx < \left(A_{n+1}^k - A_n^k\right) \ln^k A_{n+1}^k. \tag{2.12}$$

Properties (2.12) and (2.11) give

$$1 < \frac{\int_{A_n^k}^{A_{n+1}^k} \ln^k x\, dx}{\left(A_{n+1}^k - A_n^k\right) \ln^k A_n^k} < \frac{\ln^k A_{n+1}^k}{\ln^k A_n^k} = \left(\frac{\ln A_{n+1}}{\ln A_n}\right)^k \to 1,$$

that is, by the compression theorem,

$$\int_{A_n^k}^{A_{n+1}^k} \ln^k x\, dx \sim \left(A_{n+1}^k - A_n^k\right) \ln^k A_n^k. \tag{2.13}$$

Note that by L'Hospital's rule we have

$$\lim_{x\to\infty} \frac{\int_{A_1^k}^{x} \ln^k t\, dt}{x \ln^k x} = 1. \tag{2.14}$$

Now, we use the same well-known proposition that we use before in the second proof of Theorem 2.1. This proposition, equalities (2.13), (2.14), (2.6), (2.11) and the prime number theorem ($p_n \sim n \ln n$) give

$$k^k \sum_{i=1}^{n} \left(A_{i+1}^k - A_i^k\right) \ln^k A_i \sim \sum_{i=1}^{n} \int_{A_i^k}^{A_{i+1}^k} \ln^k x\, dx = \int_{A_1^k}^{A_{n+1}^k} \ln^k x\, dx$$
$$\sim A_{n+1}^k \ln^k A_{n+1}^k \sim A_n^k \ln^k A_n^k = k^k \left(A_n \ln A_n\right)^k \sim k^k \left(p_{A_n}\right)^k,$$

that is, property (2.7). Equality (2.8) is an immediate consequence of (2.7). The theorem is proved. □

It can be seen that Theorem 2.4 gives limit formula (1.3) when $A_i = i$ and $k = 1$.

THEOREM 2.5. *Let k be an arbitrary but fixed positive integer. Let us consider a strictly increasing sequence A_n $(n \geq 1)$ of positive integers such that*

$$A_{n+1} \sim A_n.$$

Let p_{A_i} be the A_ith prime number. The following asymptotic formulae hold:

$$\sum_{i=1}^{n} (A_{i+1} - A_i)\, p_{A_i}^{k-1} \ln A_i \sim \frac{1}{k} \left(p_{A_n}\right)^k,$$

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} A_i^{(A_{i+1}-A_i)p_{A_i}^{k-1}} \right)^{\frac{k}{\left(p_{A_n}\right)^k}} = e.$$

PROOF. The proof is the same as the proof of Theorem 2.4. In this case we use the function $x^{k-1} \ln^k x$. Note that (L'Hospital's rule) we have

$$\lim_{x\to\infty} \frac{\int_{A_1}^{x} t^{k-1} \ln^k t \, dt}{\frac{x^k}{k} \ln^k x} = 1. \qquad \square$$

Note that taking $A_i = i$ and $k = 1$ in Theorem 2.5 gives limit formula (1.3).

Acknowledgements. The authors would like to thank the editor and the anonymous referee for their valuable suggestions.

References

[1] R. Farhadian, *A remark on* $\lim_{n\to\infty} \sqrt[p_n]{p_1 p_2 \cdots p_n} = e$, Math. Gaz. **105** (2021), 311–312.

[2] R. Farhadian, *A generalization of Euler's limit*, Amer. Math. Monthly **129** (2022), 384.

[3] R. Farhadian and R. Jakimczuk, *Notes on a general sequence*, Ann. Math. Sil. **34** (2020), 193–202.

[4] R. Farhadian and R. Jakimczuk, *A note on the geometric mean of prime numbers and generalizations*, J. Discrete Math. Sci. Cryptogr. (2020). DOI: 10.1080/09720529.2020.1723920

[5] R. Farhadian and R. Jakimczuk, *A note on two fundamental recursive sequences*, Ann. Math. Sil. **35** (2021), 172–183.
[6] R. Farhadian and R. Jakimczuk, *A further generalization of* $\lim_{n\to\infty} \sqrt[n]{n!}/n = 1/e$, Ann. Math. Sil. (2022). DOI: 10.2478/amsil-2022-0006
[7] R. Farhadian and R. Jakimczuk, *On a sequence involving products of primes*, J. Interdiscip. Math. (2022). DOI: 10.1080/09720502.2021.1961981
[8] S.R. Finch, *Mathematical Constants*, Cambridge University Press, Cambridge, 2003.
[9] R. Jakimczuk, *The ratio between the average factor in a product and the last factor*, Math. Sci. Q. J. **1** (2007), 53–62.
[10] R. Jakimczuk, *Functions of slow increase and integer sequences*, J. Integer Seq. **13** (2010), Article 10.1.1, 14 pp.
[11] J. Rey Pastor, P. Pi Calleja, and C.A. Trejo, *Análisis Matemático*, Vol. 1, Editorial Kapelusz, Buenos Aires, 1969.

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