Annales Mathematicae Silesianae **37** (2023), no. 1, 67–81
DOI: 10.2478/amsil-2022-0018

SOME OBSERVATIONS ON THE GREATEST PRIME FACTOR OF AN INTEGER

Rafael Jakimczuk

Abstract. We examine the multiplicity of the greatest prime factor in k-full numbers and k-free numbers. We generalize a well-known result on greatest prime factors and obtain formulas related with the Riemann zeta function.

1. Introduction

We examine the multiplicity of the greatest prime factor in k-full numbers and k-free numbers (Theorem 2.1 and Theorem 2.2). We define some new arithmetical functions related with the greatest prime factor and obtain in Theorem 2.3 asymptotic formulas related with the Riemann zeta function. Finally in Theorem 2.4 we obtain an asymptotic formula for the sum of m-th powers of greatest prime factors obtaining a better result than the previous one of the author.

Let us consider the prime factorization of a positive integral number n, namely $n = q_1^{s_1} \cdots q_r^{s_r}$, where the q_i $(i = 1, \ldots, r)$ are its distinct prime factors and the s_i $(i = 1, \ldots, r)$ are their respective multiplicities. We shall need the following well-known arithmetical functions, $\sigma(n)$ is the sum of the positive divisors of n, $k(n) = q_1 \cdots q_r$ is the kernel of n and hence $\sigma(k(n)) = (q_1 + 1) \cdots (q_r + 1)$.

Received: 03.03.2022. Accepted: 02.11.2022. Published online: 23.11.2022.
(2020) Mathematics Subject Classification: 11A99, 11B99.
Key words and phrases: greatest prime factor, zeta function.

A positive integer n is square-free if and only if its prime factorization has not factors with an exponent larger than one, that is, $n = q_1 \cdots q_r$, where the q_i $(i = 1, \ldots, r)$ are distinct. Let $Q_2(x)$ denote the cardinality of the set of square-free numbers not exceeding x. It is well-known that these numbers have density $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$, where $\zeta(s)$ denotes the Riemann zeta function. More exactly,

$$Q_2(x) = \frac{1}{\zeta(2)} x + O(\sqrt{x}) = \frac{6}{\pi^2} x + O(\sqrt{x}), \tag{1.1}$$

where the error term can be improved.

Lemma 1.1. *Let $Q_{q_1 \cdots q_r}(x)$ denote the number of square-free not exceeding x relatively prime to the square-free $q_1 \cdots q_r$. The following formula holds*

$$Q_{q_1 \cdots q_r}(x) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} x + O(2^r \sqrt{x}).$$

Proof. See [4]. □

Let $s \geq 2$ be an arbitrary fixed integer. A positive integer is said to be s-full if all the factors in its prime factorization have exponent greater than or equal to s. That is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is s-full if $s_i \geq s$ $(i = 1, \ldots, r)$. Let n_s denote a general s-full number. If $s = 1$ then we obtain the positive integers. If $s = 2$ these numbers are called square-full or powerful.

Lemma 1.2. *Let $s \geq 1$ be an arbitrary fixed integer. Let $A_s(x)$ denote the cardinality of the set of s-full numbers not exceeding x. We have*

$$A_s(x) = \frac{6}{\pi^2} C_s x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right),$$

where

$$C_s = \sum_{n=1}^{\infty} \frac{1}{\sigma(k(n))} \frac{1}{n^{\frac{1}{s}}} = \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{s}} - 1)}\right).$$

Proof. See [5]. □

Lemma 1.3. *Let $s \geq 1$ be an arbitrary fixed integer. The following series converges*

$$\sum_Q \frac{1}{Q^{\frac{1}{s}}},$$

where Q runs over the $(s+1)$-full numbers $Q = q_1^{s_1} \cdots q_r^{s_r}$.

Proof. Let a_n denote the n-th $(s+1)$-full number. We have (see Lemma 1.2) $A_{s+1}(x) \sim c_1 \sqrt[s+1]{x}$, where c_1 is a constant. Therefore if $x = a_n$ then we obtain $n = A_{s+1}(a_n) \sim c_1 \sqrt[s+1]{a_n}$, that is, $a_n \sim \frac{n^{s+1}}{c_1^{s+1}}$. The lemma follows by the comparison criterion, since the series $\sum \frac{1}{n^{\frac{s+1}{s}}}$ converges. The lemma is proved. □

Let us consider the prime factorization of a positive integral number n, namely $n = q_1^{s_1} \cdots q_r^{s_r}$, where the q_i $(i = 1, \ldots, r)$ are the distinct prime factors and the s_i $(i = 1, \ldots, r)$ are their respective multiplicities. Let $k \geq 2$ be an arbitrary fixed positive integer. We shall say that n is k-free if $s_i \leq k-1$ $(i = 1, \ldots, r)$. In particular, if $k = 2$ we obtain the square-free numbers. Let s_k denote a general k-free number. Let $Q_k(x)$ be the cardinality of the set of k-free numbers not exceeding x. It is well-known that these numbers have density $\frac{1}{\zeta(k)}$, that is,

$$Q_k(x) = \frac{1}{\zeta(k)} x + O\left(x^{1/k}\right), \qquad (k \geq 2). \tag{1.2}$$

In particular, if $k = 2$ then equation (1.2) becomes equation (1.1).

We shall need the following well-known lemmas.

Lemma 1.4. *We have*

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + M + O\left(\delta_c(x)\right),$$

where $M \approx 0,26149\ldots$ is the Mertens's constant and $\delta_c(x)$ is the usual number theoretic function $\delta_c(x) = e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}$ for some $c > 0$. Note that for all positive integer N we have $\delta_c(x) = O\left(\frac{1}{\log^N x}\right)$.

Lemma 1.5. *Let $\pi(x)$ denote the prime counting function. We have*

$$\pi(x) = \sum_{p \leq x} 1 = Li(x) + O\left(x\delta_c(x)\right),$$

where $Li(x) = \int_2^x \frac{dt}{\log t}$ is the logarithmic integral.

We also shall need the following fundamental lemma.

Lemma 1.6. *Let j, ℓ nonnegative integers and α a positive integer. Then for all positive integer N*

$$\sum_{p\le x^{\frac{1}{\alpha}}} p^{\ell}\left\{\frac{x}{p^{\alpha}}\right\}^{j} = \frac{x^{\frac{\ell+1}{\alpha}}}{\log x}\sum_{h=0}^{N-1}\frac{a_{j,h,\ell,\alpha}}{(\log x)^{h}} + O\left(\frac{x^{\frac{\ell+1}{\alpha}}}{(\log x)^{N+1}}\right),$$

where

$$a_{j,h,\ell,\alpha} = \int_{1}^{\infty}\frac{\{u\}^{j}(\log u)^{h}}{u^{1+\frac{\ell+1}{\alpha}}}\,du$$

and (as usual) $\{x\} = x - \lfloor x\rfloor$ is the fractional part of x.

The error term depends at most on j, ℓ, α, N.

Proof. See [3]. $\square$

Let us consider the prime factorization of a positive integer n, namely $n = q_1^{s_1}\cdots q_r^{s_r}$, where $q_1 > q_2 > \cdots > q_r$ are the distinct prime factors and the s_i $(i = 1, \ldots, r)$ are the multiplicities. Note that q_1 is the greatest prime factor of n and note that n can be considered as a product of prime powers $q_i^{s_i}$ $(i = 1, \ldots, r)$.

Let $G(n) = q_1$ denote the greatest prime factor in the prime factorization of n and let $A(n)$ denote the sum of all prime factors in the prime factorization of n, that is, $A(n) = s_1q_1 + s_2q_2 + \cdots + s_rq_r$. Alladi and Erdős ([1]) proved the following formulas

$$\sum_{n\le x} G(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right),$$

$$\sum_{n\le x} A(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right).$$

Therefore

$$\sum_{n\le x} A(n) \sim \sum_{n\le x} G(n) \sim \frac{\pi^2}{12}\frac{x^2}{\log x}. \tag{1.3}$$

Below, in Remark 3.1, we shall prove (1.3) and more precise formulas, as a consequence of our results in Theorem 2.3.

We shall establish the following definitions. Let $a_k(n)$ be the sum of prime powers $q_i^{s_i}$ such that $s_i = k$. If all prime powers $q_i^{s_i}$ are such that $s_i \neq k$ we put $a_k(n) = 0$. Let $A_k(n)$ be the sum of prime powers $q_i^{s_i}$ such that $s_i \ge k$.

If all prime powers $q_i^{s_i}$ are such that $s_i < k$ we put $A_k(n) = 0$. We put $g_k(n) = q_1^{s_1}$ if $s_1 = k$ and $g_k(n) = 0$ if $s_1 \neq k$. We put $G_k(n) = q_1^{s_1}$ if $s_1 \geq k$ and $G_k(n) = 0$ if $s_1 < k$. We recall that q_1 is the greatest prime factor of n. For example, $a_4(19^4 11^5 5^5) = 19^4$, $a_5(19^4 11^5 5^5) = 11^5 + 5^5$, $a_3(19^4 11^5 5^5) = 0$, $A_4(19^4 11^5 5^5) = 19^4 + 11^5 + 5^5$, $A_3(19^4 11^5 5^5) = 19^4 + 11^5 + 5^5$, $A_8(19^4 11^5 5^5) = 0$, $g_4(19^4 11^5 5^5) = 19^4$, $g_h(19^4 11^5 5^5) = 0$ $(h \neq 4)$, $G_2(19^4 11^5 5^5) = 19^4$, $G_7(19^4 11^5 5^5) = 0$.

Alladi and Erdős ([1]) proved that

$$\sum_{n \leq x} G(n) = \frac{\zeta(2)}{2} \frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right). \tag{1.4}$$

Jakimczuk ([6]) proved the following generalization

$$\sum_{n \leq x} G(n)^m \sim \frac{\zeta(m+1)}{m+1} \frac{x^{m+1}}{\log x},$$

where m is an arbitrary fixed positive integer.

In Theorem 2.4 we obtain a more precise result.

2. Main Results

Theorem 2.1. *Let $s \geq 1$ be an arbitrary fixed integer. Let $G_s(x)$ denote the cardinality of the set of s-full numbers n_s not exceeding x such that the greatest prime factor of n_s has multiplicity s in the prime factorization of n_s. Then $G_s(x) \sim A_s(x)$. That is, in almost all s-full numbers the greatest prime factor has multiplicity s. If $s = 1$ then in almost all integers the greatest prime factor has multiplicity* 1.

Theorem 2.2. *Let $k \geq 3$ be an arbitrary fixed integer. Let $H_k(x)$ denote the cardinality of the set of k-free numbers s_k not exceeding x such that the greatest prime factor of s_k has multiplicity* 1 *in the prime factorization of s_k. Then $H_k(x) \sim Q_k(x)$. That is, in almost all k-free numbers s_k the greatest prime factor has multiplicity* 1.

THEOREM 2.3. *Let k and N be arbitrary fixed positive integers. We have the following asymptotic formulas*

$$\sum_{n\leq x} a_k(n) = \frac{x^{1+\frac{1}{k}}}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right), \tag{2.1}$$

$$\sum_{n\leq x} A_k(n) = \frac{x^{1+\frac{1}{k}}}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right), \tag{2.2}$$

$$\sum_{n\leq x} g_k(n) = \frac{x^{1+\frac{1}{k}}}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right), \tag{2.3}$$

$$\sum_{n\leq x} G_k(n) = \frac{x^{1+\frac{1}{k}}}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right), \tag{2.4}$$

where the error terms depend at most on k and N and where $b_{h,k}$ depends on the zeta function $\zeta(s) = \zeta^{(0)}(s)$ and its successive derivatives $\zeta^{(i)}(s)$ in the point $s = 1 + \frac{1}{k}$, as it is showed by the following formula

$$b_{h,k} = h!\sum_{i=0}^{h}\frac{(-1)^i}{i!}\frac{\zeta^{(i)}\left(1+\frac{1}{k}\right)}{\left(1+\frac{1}{k}\right)^{h+1-i}}. \tag{2.5}$$

The first coefficient is

$$b_{0,k} = \frac{1}{1+\frac{1}{k}}\zeta\left(1+\frac{1}{k}\right).$$

Therefore we have

$$\sum_{n\leq x} A_k(n) \sim \sum_{n\leq x} a_k(n) \sim \sum_{n\leq x} g_k(n) \sim \sum_{n\leq x} G_k(n) \sim \frac{1}{1+\frac{1}{k}}\zeta\left(1+\frac{1}{k}\right)\frac{x^{1+\frac{1}{k}}}{\log x}.$$

THEOREM 2.4. *Let m be an arbitrary fixed positive integer. Then*

$$\sum_{n\leq x} G(n)^m = \frac{\zeta(m+1)}{m+1}\frac{x^{m+1}}{\log x} + O\left(\frac{x^{m+1}}{\log^2 x}\right). \tag{2.6}$$

If $m = 1$ then equation (2.6) becomes equation (1.4).

3. Proofs

Proof of Theorem 2.1. If the s-full number n_s is of the form q^s, where q is a square-free, then clearly the greatest prime factor of n_s has multiplicity s. The cardinality of the set of the numbers $n_s = q^s$ not exceeding x, that is, $q^s \leq x$, is (see (1.1))

$$\frac{6}{\pi^2} x^{1/s} + o\left(x^{1/s}\right). \tag{3.1}$$

Let us consider the prime factorization of a $(s+1)$-full number Q, that is, $Q = q_1^{r_1} \cdots q_t^{r_t}$. Let us consider the s-full numbers n_s of the form $q^s Q$, where q is a square-free, $\gcd(q, Q) = 1$ and Q is fixed. Except by a finite number of cases, the greatest prime factor in $q^s Q$ is in the prime factorization of q and consequently it has multiplicity s. Therefore the number of these numbers $q^s Q$ not exceeding x ($q^s Q \leq x$) such that the greatest prime factor has multiplicity s will be (see Lemma 1.1)

$$\frac{6}{\pi^2} \frac{q_1 \cdots q_t}{(q_1+1) \cdots (q_t+1)} \frac{x^{1/s}}{Q^{1/s}} + o\left(x^{1/s}\right). \tag{3.2}$$

Let $\epsilon > 0$. We choose B such that (see Lemma 1.3)

$$\sum_{Q > B} \frac{1}{Q^{1/s}} < \epsilon. \tag{3.3}$$

Therefore we have (see (3.1), (3.2), (3.3) and Lemma 1.2)

$$G_s(x) = \frac{6}{\pi^2} \left(1 + \sum_{Q \leq B} \frac{q_1 \cdots q_t}{(q_1+1) \cdots (q_t+1)} \frac{1}{Q^{1/s}}\right) x^{1/s} + o\left(x^{1/s}\right) + F(x)$$

$$= \frac{6}{\pi^2} C_s x^{1/s} - \left(\sum_{Q > B} \frac{q_1 \cdots q_t}{(q_1+1) \cdots (q_t+1)} \frac{1}{Q^{1/s}}\right) + o\left(x^{1/s}\right) + F(x), \tag{3.4}$$

where (see (3.3))

$$0 \leq F(x) \leq \sum_{x \geq Q > B} \left\lfloor \frac{x^{1/s}}{Q^{1/s}} \right\rfloor \leq \sum_{Q > B} \frac{x^{1/s}}{Q^{1/s}} \leq \epsilon x^{1/s}. \tag{3.5}$$

Note that

$$1+\sum_{Q}\frac{q_1\cdots q_t}{(q_1+1)\cdots(q_t+1)}\frac{1}{Q^{\frac{1}{s}}}=\prod_{p}\left(1+\frac{p}{p+1}\left(\frac{1}{p^{\frac{s+1}{s}}}+\frac{1}{p^{\frac{s+2}{s}}}+\cdots\right)\right)$$

$$=\prod_{p}\left(1+\frac{1}{p+1}\frac{1}{p^{\frac{1}{s}}}\frac{1}{1-\frac{1}{p^{\frac{1}{s}}}}\right)$$

$$=\prod_{p}\left(1+\frac{1}{p+1}\frac{1}{p^{\frac{1}{s}}-1}\right)=C_s.$$

Consequently (see (3.4) and (3.5))

$$\left|\frac{G_s(x)}{x^{1/s}}-\frac{6}{\pi^2}C_s\right|\leq\epsilon+\epsilon+\epsilon=3\epsilon \qquad (x\geq x_\epsilon).$$

That is, since $\epsilon>0$ can be arbitrarily small,

$$G_s(x)\sim\frac{6}{\pi^2}C_s x^{1/s}\sim A_s(x).$$

The theorem is proved. □

Proof of Theorem 2.2. The proof is identical to the proof of Theorem 2.1. If the k-free number is a square-free q then clearly the greatest prime factor of q has multiplicity 1. The number of these numbers not exceeding x, that is, $q\leq x$, is (see (1.1))

$$\frac{6}{\pi^2}x+o\left(x\right).$$

Let us consider the prime factorization of a square-full number Q, that is, $Q=q_1^{r_1}\cdots q_t^{r_t}$, where $2\leq r_i\leq k-1$ $(i=1,\ldots,t)$. Let us consider the k-free numbers s_k of the form qQ, where q is a square-free, $\gcd(q,Q)=1$ and Q is fixed. Except for a finite number of cases the greatest prime factor in $s_k=qQ$ is in the prime factorization of q and consequently it has multiplicity 1. Therefore the number of these numbers $s_k=qQ$ not exceeding x $(qQ\leq x)$ such that the greatest prime factor has multiplicity 1 will be (see Lemma 1.1)

$$\frac{6}{\pi^2}\frac{q_1\cdots q_t}{(q_1+1)\cdots(q_t+1)}\frac{x}{Q}+o\left(x\right).$$

Now, the proof follows as the proof of Theorem 2.1. Note that

$$\begin{aligned}
&\frac{6}{\pi^2}\left(1+\sum_{Q}\frac{q_1\cdots q_t}{(q_1+1)\cdots(q_t+1)}\frac{1}{Q}\right)\\
&=\prod_p\left(1-\frac{1}{p^2}\right)\prod_p\left(1+\frac{p}{p+1}\left(\frac{1}{p^2}+\cdots+\frac{1}{p^{k-1}}\right)\right)\\
&=\prod_p\left(\frac{p^2-1}{p^2}\right)\prod_p\left(1+\frac{1}{p^2-1}\left(1-\frac{1}{p^{k-2}}\right)\right)=\prod_p\left(1-\frac{1}{p^k}\right)\\
&=\frac{1}{\zeta(k)}.
\end{aligned}$$

The theorem is proved. □

Proof of Theorem 2.3. It is well-known the asymptotic formula

$$Li(x)=\frac{x}{\log x}\sum_{h=0}^{N-1}\frac{h!}{(\log x)^h}+O_N\left(\frac{x}{(\log x)^{N+1}}\right). \tag{3.6}$$

We have (Lemma 1.4, Lemma 1.5, Lemma 1.6 and equality (3.6))

$$\begin{aligned}
\sum_{n\le x}a_k(n)&=\sum_{p\le x}p^k\left(\left\lfloor\frac{x}{p^k}\right\rfloor-\left\lfloor\frac{x}{p^{k+1}}\right\rfloor\right)\\
&=x\pi\left(x^{\frac{1}{k}}\right)-x\sum_{p\le x^{\frac{1}{k+1}}}\frac{1}{p}-\sum_{p\le x^{\frac{1}{k}}}p^k\left\{\frac{x}{p^k}\right\}+\sum_{p\le x^{\frac{1}{k+1}}}p^k\left\{\frac{x}{p^{k+1}}\right\}\\
&=x\left(Li\left(x^{\frac{1}{k}}\right)+O\left(x^{\frac{1}{k}}\delta_c\left(x^{\frac{1}{k}}\right)\right)\right)\\
&\quad-x\left(\log\log x^{\frac{1}{k}}+M+O\left(\delta_c\left(x^{\frac{1}{k+1}}\right)\right)\right)\\
&\quad-\left(\frac{x^{1+\frac{1}{k}}}{\log x}\sum_{h=0}^{N-1}\frac{a_{1,h,k,k}}{(\log x)^h}+O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right)\right)\\
&\quad+\left(\frac{x}{\log x}\sum_{h=0}^{N-1}\frac{a_{1,h,k,k+1}}{(\log x)^h}+O\left(\frac{x}{(\log x)^{N+1}}\right)\right)
\end{aligned}$$

$$= xLi\left(x^{\frac{1}{k}}\right) - \frac{x^{1+\frac{1}{k}}}{\log x}\sum_{h=0}^{N-1}\frac{a_{1,h,k,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right)$$

$$= \frac{x^{1+\frac{1}{k}}}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right), \tag{3.7}$$

where

$$b_{h,k} = k^{h+1}h! - a_{1,h,k,k} = k^{h+1}h! - \int_1^\infty \frac{\{u\}(\log u)^h}{u^{2+\frac{1}{k}}}du. \tag{3.8}$$

Now, by application of Euler-Maclaurin summation formula ([2]), we obtain

$$\zeta^{(h)}(s) = \frac{(-1)^h h!}{(s-1)^{h+1}} + (-1)^h h\int_1^\infty \frac{\{u\}\log^{h-1}u}{u^{s+1}}du$$
$$+ (-1)^{h+1}s\int_1^\infty \frac{\{u\}\log^h u}{u^{s+1}}du.$$

Therefore by mathematical induction we find that

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}du,$$

$$-\zeta'(s) = -\frac{\zeta(s)}{s} + \frac{s}{(s-1)^2} - s\int_1^\infty \frac{\{u\}\log u}{u^{s+1}}du,$$

and in general

$$\int_1^\infty \frac{\{u\}\log^h u}{u^{s+1}}du = \frac{h!}{(s-1)^{h+1}} + h!\sum_{i=0}^{h}\frac{(-1)^{i+1}}{i!}\frac{\zeta^{(i)}(s)}{s^{h+1-i}}. \tag{3.9}$$

Substituting $s = 1 + \frac{1}{k}$ into (3.9) and by application of (3.8) we obtain formula (2.5). Therefore equality (2.1) is proved.

We have

$$\sum_{n\le x} A_k(n) = \sum_{p\le x} p^k\left(\left\lfloor\frac{x}{p^k}\right\rfloor - \left\lfloor\frac{x}{p^{k+1}}\right\rfloor\right) + F(x)$$

$$= \sum_{n\le x} a_k(n) + F(x). \tag{3.10}$$

Note that the inequality $p^h > x$ holds for $h = \left\lfloor \frac{\log x}{\log 2} \right\rfloor + 1 > \frac{\log x}{\log 2} \geq \frac{\log x}{\log p}$ and therefore we have

$$0 \leq F(x) \leq \sum_{p \leq x^{\frac{1}{k+1}}} \left(p^{k+1} \left\lfloor \frac{x}{p^{k+1}} \right\rfloor + p^{k+2} \left\lfloor \frac{x}{p^{k+2}} \right\rfloor + \cdots + p^h \left\lfloor \frac{x}{p^h} \right\rfloor \right)$$

$$\leq hx \sum_{p \leq x^{\frac{1}{k+1}}} 1 \leq \left(\frac{\log x}{\log 2} + 1 \right) xc \frac{x^{\frac{1}{k+1}}}{\log x^{\frac{1}{k+1}}} = O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}} \right). \tag{3.11}$$

Properties (3.10), (3.11) and (2.1) give equality (2.2).

Note that $g_k(n) \leq a_k(n)$ and hence formula (2.1) gives

$$\sum_{n \leq x} g_k(n) \leq \frac{x^{1+\frac{1}{k}}}{\log x} \sum_{h=0}^{N-1} \frac{b_{h,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}} \right). \tag{3.12}$$

Note that if $x^{\frac{1}{k+1}} < p \leq x^{\frac{1}{k}}$ then $p^k \leq x$ and $p^{k+1} > x$. Now, the multiples of p^k not exceeding x are $p^k, p^k 2, \ldots, p^k \left\lfloor \frac{x}{p^k} \right\rfloor$, where $\left\lfloor \frac{x}{p^k} \right\rfloor < p$. Therefore p is the greatest prime factor of these numbers. Consequently, we have (see (3.7))

$$\sum_{n \leq x} g_k(n) \geq \sum_{x^{\frac{1}{k+1}} < p \leq x^{\frac{1}{k}}} p^k \left\lfloor \frac{x}{p^k} \right\rfloor = x\pi\left(x^{\frac{1}{k}}\right) - \sum_{p^k \leq x} p^k \left\{ \frac{x}{p^k} \right\} - x\pi\left(x^{\frac{1}{k+1}}\right)$$

$$+ \sum_{p \leq x^{\frac{1}{k+1}}} p^k \left\{ \frac{x}{p^k} \right\} = \frac{x^{1+\frac{1}{k}}}{\log x} \sum_{h=0}^{N-1} \frac{b_{h,k}}{(\log x)^h} + O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}} \right), \tag{3.13}$$

since

$$0 \leq \sum_{p \leq x^{\frac{1}{k+1}}} p^k \left\{ \frac{x}{p^k} \right\} \leq \sum_{p \leq x^{\frac{1}{k+1}}} p^k \leq \left(x^{\frac{1}{k+1}} \right)^k \sum_{p \leq x^{\frac{1}{k+1}}} 1$$

$$\leq c \frac{x}{\log x} = O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}} \right)$$

and

$$x\pi\left(x^{\frac{1}{k+1}}\right) \leq x^{1+\frac{1}{k+1}} = O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}} \right).$$

Inequalities (3.12) and (3.13) give formula (2.3).

Finally, we have

$$\sum_{n\le x} G_k(n) = \sum_{n\le x} g_k(n) + F_1(x), \tag{3.14}$$

where (see (3.11))

$$0 \le F_1(x) \le F(x) = O\left(\frac{x^{1+\frac{1}{k}}}{(\log x)^{N+1}}\right). \tag{3.15}$$

Properties (2.3), (3.14) and (3.15) give equality (2.4), which completes the proof. □

Remark 3.1. If $k=1$ then equalities (2.3) and (2.2) give

$$\sum_{n\le x} g_1(n) = \frac{x^2}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,1}}{(\log x)^h} + O\left(\frac{x^2}{(\log x)^{N+1}}\right),$$

$$\sum_{n\le x} A_1(n) = \frac{x^2}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,1}}{(\log x)^h} + O\left(\frac{x^2}{(\log x)^{N+1}}\right),$$

where $A_1(n)$ is the sum of the prime powers in the prime factorization of n, that is, $A_1(n) = q_1^{s_1} + q_2^{s_2} + \cdots + q_r^{s_r}$. Therefore, since $s_i q_i \le q_i^{s_i}$ $(i = 1, \ldots, r)$, we have

$$\sum_{n\le x} g_1(n) \le \sum_{n\le x} G(n) \le \sum_{n\le x} A(n) \le \sum_{n\le x} A_1(n).$$

Therefore we obtain

$$\sum_{n\le x} G(n) = \frac{x^2}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,1}}{(\log x)^h} + O\left(\frac{x^2}{(\log x)^{N+1}}\right) \sim \frac{\pi^2}{12}\frac{x^2}{\log x},$$

$$\sum_{n\le x} A(n) = \frac{x^2}{\log x}\sum_{h=0}^{N-1}\frac{b_{h,1}}{(\log x)^h} + O\left(\frac{x^2}{(\log x)^{N+1}}\right) \sim \frac{\pi^2}{12}\frac{x^2}{\log x}.$$

Consequently $\sum_{n\le x} G(n)$ and $\sum_{n\le x} A(n)$ have the same asymptotic expansion.

PROOF OF THEOREM 2.4. By the prime number theorem we have

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Abel's summation gives

$$S_m(x) = \sum_{p \le x} p^m = \frac{1}{m+1}\frac{x^{m+1}}{\log x} + O\left(\frac{x^{m+1}}{\log^2 x}\right).$$

That is

$$S_m(x) = \sum_{p \le x} p^m = \frac{1}{m+1}\frac{x^{m+1}}{\log x} + f(x)\left(\frac{x^{m+1}}{\log^2 x}\right) \qquad (x \ge 2), \tag{3.16}$$

where $|f(x)| < M$ for $x \ge 2$. Note that M depends on m.

Let us consider the positive integer $s = \left\lfloor (\log x)^{\frac{1}{m}} \right\rfloor + 1$ and a positive integer k such that $1 \le k \le s-1$. Now, consider the primes p such that $\frac{x}{s} \le \frac{x}{k+1} < p \le \frac{x}{k}$. The numbers multiples of p not exceeding x are $p, 2p, 3p, \dots, kp$ and since $p > k$, if x is sufficiently large, we obtain that p is the greatest prime factor of these k numbers. Hence if $F(x)$ is the contribution to the sum $\sum_{n \le x} G(n)^m$ of the numbers not exceeding x such that their greatest prime factor is in the interval $\left[2, \frac{x}{s}\right]$ then we have

$$\begin{aligned}
\sum_{n \le x} G(n)^m &= \sum_{k=1}^{s-1} \sum_{\frac{x}{k+1} < p \le \frac{x}{k}} kp^m + F(x) \\
&= \sum_{i=1}^{s-1}\left(\sum_{\frac{x}{s} < p \le \frac{x}{i}} p^m\right) + F(x) = \sum_{i=1}^{s-1}\left(S_m\left(\frac{x}{i}\right) - S_m\left(\frac{x}{s}\right)\right) + F(x) \\
&= \sum_{i=1}^{s-1} S_m\left(\frac{x}{i}\right) - (s-1)S_m\left(\frac{x}{s}\right) + F(x), \qquad (3.17)
\end{aligned}$$

where

$$(s-1)S_m\left(\frac{x}{s}\right) = O\left(\frac{x^{m+1}}{\log^2 x}\right), \tag{3.18}$$

$$0 \le F(x) \le \sum_{p \le \frac{x}{s}} p^m \left\lfloor \frac{x}{p} \right\rfloor = x\sum_{p \le \frac{x}{s}} p^{m-1} = O\left(\frac{x^{m+1}}{\log^2 x}\right), \tag{3.19}$$

and (see (3.16))

$$\sum_{i=1}^{s-1} S_m\left(\frac{x}{i}\right) = \sum_{i=1}^{s-1}\left(\frac{1}{m+1}\left(\frac{x}{i}\right)^{m+1}\frac{1}{\log\left(\frac{x}{i}\right)} + f\left(\frac{x}{i}\right)\left(\frac{x}{i}\right)^{m+1}\frac{1}{\log^2\left(\frac{x}{i}\right)}\right). \tag{3.20}$$

Now, we have the formula $\frac{1}{1-t} = 1 + g(t)t$, where $g(t) \to 1$ as $t \to 0$. Therefore we have (see (3.20))

$$\begin{aligned}
\sum_{i=1}^{s-1} \frac{1}{m+1}\left(\frac{x}{i}\right)^{m+1}\frac{1}{\log\left(\frac{x}{i}\right)} &= \frac{x^{m+1}}{m+1}\frac{1}{\log x}\sum_{i=1}^{s-1}\frac{1}{i^{m+1}}\left(1 + g\left(\frac{\log i}{\log x}\right)\frac{\log i}{\log x}\right) \\
&= \frac{\zeta(m+1)}{m+1}\frac{x^{m+1}}{\log x} - \frac{x^{m+1}}{(m+1)\log x}\sum_{i\geq s}\frac{1}{i^{m+1}} \\
&\quad + \frac{x^{m+1}}{(m+1)\log^2 x}\sum_{i=1}^{s-1}\frac{\log i}{i^{m+1}} g\left(\frac{\log i}{\log x}\right) \\
&= \frac{\zeta(m+1)}{m+1}\frac{x^{m+1}}{\log x} + O\left(\frac{x^{m+1}}{\log^2 x}\right), \qquad (3.21)
\end{aligned}$$

and (see (3.16))

$$\begin{aligned}
\left|\sum_{i=1}^{s-1} f\left(\frac{x}{i}\right)\left(\frac{x}{i}\right)^{m+1}\frac{1}{\log^2\left(\frac{x}{i}\right)}\right| &= \left|\sum_{i=1}^{s-1} f\left(\frac{x}{i}\right)\frac{x^{m+1}}{i^{m+1}}\frac{1}{\log^2 x}\frac{1}{\left(1-\frac{\log i}{\log x}\right)^2}\right| \\
&\leq 2M\zeta(m+1)\frac{x^{m+1}}{\log^2 x}. \qquad (3.22)
\end{aligned}$$

Substituting (3.21) and (3.22) into (3.20) and then substituting (3.20), (3.19) and (3.18) into (3.17) we obtain (2.6). The theorem is proved. □

Acknowledgement. The author would like to thank the editor and the anonymous referee who provided extensive arguments, constructive remarks and helpful suggestions on the earlier version of this article. The author is also very grateful to Universidad Nacional de Luján.

References

[1] K. Alladi and P. Erdős, *On an additive arithmetic function*, Pacific J. Math. **71** (1977), no. 2, 275–294.

[2] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York-Heidelberg, 1976.

[3] O. Bordellés and L. Tóth, *Additive arithmetic functions meet the inclusion-exclusion principle, II*, to appear in Res. Number Theory. Avaliable at arXiv: 2112.13409.

[4] R. Jakimczuk and M. Lalín, *Sums of* $\omega(n)$ *and* $\Omega(n)$ *on the* k*-free parts and* k*-full parts of some particular sequences*, preprint 2022.

[5] R. Jakimczuk and M. Lalín, *The number of prime factors on average in certain integer sequences*, J. Integer Seq. **25** (2022), no. 2, Art. 22.2.3, 15 pp.

[6] R. Jakimczuk, *A note on sums of greatest (least) prime factors*, Int. J. Contemp. Math. Sci. **8** (2013), no. 9–12, 423–432.

División Matemática
Universidad Nacional de Luján
Buenos Aires
Argentina
e-mail: jakimczu@mail.unlu.edu.ar