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STRONG m-CONVEXITY OF SET-VALUED FUNCTIONS

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Abstract. In this research we introduce the concept of strong m-convexity for set-valued functions defined on m-convex subsets of real linear normed spaces, a variety of properties and examples of these functions are shown, an inclusion of Jensen type is also exhibited.

1. Introduction

In this research we introduce the notion of a strongly m-convex set-valued function, which represents a generalization of the usual concept of m-convexity for the real case that can be found in [3] and references therein. The idea of this new approach involves the concepts of strong convexity and m-convexity of set-valued functions. This is the main reason for which we start off by recalling both definitions. Along this paper X, Y will denote any real normed linear spaces, D an m-convex subset of X ([1]), B the closed unit ball in Y and $n(Y)$ the family of all nonempty subsets of Y.

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Definition 1.1 ([4]). Let $c > 0$. A set-valued function $F\colon D \to n(Y)$ is called *strongly convex with modulus* c if it satisfies the inclusion

$$tF(x) + (1-t)F(y) + ct(1-t)\|x-y\|^2 B \subseteq F(tx + (1-t)y),$$

for all $x, y \in D$ and $t \in [0,1]$.

Definition 1.2 ([3]). Let $m \in [0,1]$. A set-valued function $F\colon D \to n(Y)$ is called *m-convex* if the inclusion

$$tF(x) + m(1-t)F(y) \subseteq F(tx + m(1-t)y),$$

holds for all $x, y \in D$ and $t \in [0,1]$.

Our first definition runs as follows:

Definition 1.3. Let $c > 0$ and $m \in [0,1]$. A set-valued function $F\colon D \to n(Y)$ is called *strongly m-convex with modulus* c if

$$tF(x) + m(1-t)F(y) + cmt(1-t)\|x-y\|^2 B \subseteq F(tx + m(1-t)y), \tag{1.1}$$

for any $x, y \in D$, $t \in [0,1]$.

Remark 1.4. Notice that (1.1) is equivalent to

$$mtF(x) + (1-t)F(y) + cmt(1-t)\|x-y\|^2 B \subseteq F(mtx + (1-t)y),$$

with x, y, t as before.

Remark 1.5. If a set-valued function F is strongly m-convex with modulus c, then it is also m-convex. It follows immediately from the fact that $0 \in B$.

The converse in the foregoing remark is not true. Namely, we have the following.

Example 1.1. The set-valued function $F\colon [0,1] \subseteq \mathbb{R} \to n(\mathbb{R})$, given by $F(x) = [0, x]$, is m-convex ([3, Example 2.17]). But for all $x, y, t \in [0,1]$

$$\begin{aligned} tF(x)+&m(1-t)F(y) + cmt(1-t)\|x-y\|^2 B \\ &= \left[-cmt(1-t)\|x-y\|^2, tx + m(1-t)y + cmt(1-t)\|x-y\|^2\right], \end{aligned}$$

while that

$$F(tx + m(1-t)y) = [0, tx + m(1-t)y],$$

so F can not be a strongly m-convex function.

EXAMPLE 1.2. If $b > 0$ and $f, g\colon [0,b] \to \mathbb{R}$ are two real functions, f and $-g$ being strongly m-convex with the same modulus ([2]) and $f \leq g$ on $[0,b]$, it is not difficult to verify (by reasoning as in Example 2.2 from [3]) that the set-valued functions $F_1, F_2, F_3\colon [0,b] \subseteq \mathbb{R} \to n(\mathbb{R})$ given by

$$F_1(x) = [f(x), g(x)], \quad F_2(x) = [f(x), +\infty), \quad F_3(x) = (-\infty, g(x)]$$

are strongly m-convex (with the same modulus). So, for example, functions $f_1, g_1\colon [0,1] \to \mathbb{R}$ defined as $f_1(x) = 0$ and $g_1(x) = -1$ are clearly m-convex ([5, 6]), while functions $f(x) = \frac{1}{2}x^2$, $g(x) = 1 - \frac{1}{2}x^2$ are such that f and $-g$ are strongly m-convex with modulus $c = \frac{1}{2}$; moreover $f \leq g$ on $[0,1]$. Consequently the set-valued function $F\colon [0,1] \to n(\mathbb{R})$ defined by $F(x) = \left[\frac{1}{2}x^2, 1 - \frac{1}{2}x^2\right]$ is strongly m-convex with modulus $\frac{1}{2}$, and so is $G(x) = \left[\frac{1}{2}x^2 - 1, -\frac{1}{2}x^2\right]$. The graphs of F and G are shown in Figures 1 and 2, respectively.

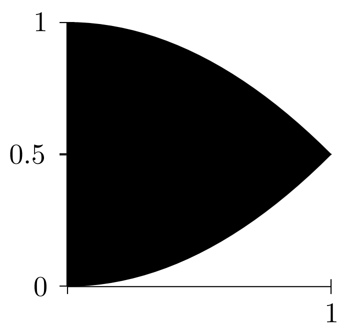

Figure 1. Graph of F

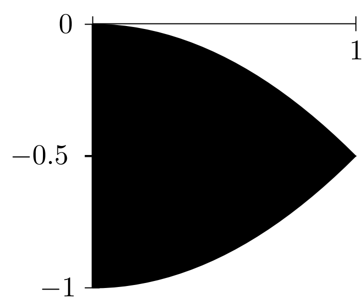

Figure 2. Graph of G

2. Results

In this section we present some set-properties of the unit ball B. At the same time, a characterization of the family of all the strongly m-convex functions is given and illustrate with an interesting example. We begin with a lemma related to two well-known properties of convexity whose proofs are omitted.

LEMMA 2.1. (1) *If* $0 \le \alpha_1 \le \alpha_2$, *then* $\alpha_1 B \subseteq \alpha_2 B$.
(2) *If* $\alpha_1\alpha_2 \ge 0$, *then* $(\alpha_1 + \alpha_2)B = \alpha_1 B + \alpha_2 B$.

PROPOSITION 2.2. *A set-valued function* $F\colon D \to n(Y)$ *is strongly* m-*convex with modulus* c *if and only if*

$$tF(A_1)+m(1-t)F(A_2)+cmt(1-t)\|A_1-A_2\|^2B \subseteq F(A_1+m(1-t)A_2) \tag{2.1}$$

for all $A_1, A_2 \subseteq D$ *and* $t \in [0,1]$, *where* $F(A_i) = \{F(x) : x \in A_i\}$ $(i = 1,2)$ *and* $\|A_1 - A_2\| = \inf\{\|x - y\| : x \in A_1,\ y \in A_2\}$.

PROOF. ($\Rightarrow$) Let A_1, A_2 be two fixed but arbitrary subsets of D and $z \in tF(A_1) + m(1-t)F(A_2) + cmt(1-t)\|A_1 - A_2\|^2B$. Then

$$z \in tF(a) + m(1-t)F(b) + cmt(1-t)\|A_1 - A_2\|^2B \tag{2.2}$$

for some $a \in A_1$ and $b \in A_2$. Since $0 \le \|A_1 - A_2\| \le \|a - b\|$, $0 \le cmt(1-t)\|A_1 - A_2\|^2 \le cmt(1-t)\|a-b\|^2$ and from Lemma 2.1(1), the inclusion $cmt(1-t)\|A_1 - A_2\|^2B \subseteq cmt(1-t)\|a-b\|^2B$ takes place. Hence,

$$\begin{aligned} &tF(a) + m(1-t)F(b) + cmt(1-t)\|A_1 - A_2\|^2B \\ &\qquad\subseteq tF(a) + m(1-t)F(b) + cmt(1-t)\|a-b\|^2B. \end{aligned} \tag{2.3}$$

Furthermore, since $ta + m(1-t)b \in tA_1 + m(1-t)A_2$, it is clear that

$$F(ta + m(1-t)b) \subseteq F(tA_1 + m(1-t)A_2). \tag{2.4}$$

So, (2.1) follows from (2.2), (2.3), the strong m-convexity of F and (2.4).

($\Leftarrow$) Let $x, y \in D$ and $t \in [0,1]$. The strong m-convexity with modulus c of F is obtained by considering in (2.1) the singletons $A_1 = \{x\}$ and $A_2 = \{y\}$. □

PROPOSITION 2.3. *Let* $b \in \mathbb{R}\backslash\{0\}$ *and* $D = [\min\{0,b\}, \max\{0,b\}] \subseteq \mathbb{R}$. *If* $F\colon D \to n(Y)$ *is strongly* m-*convex with modulus* c, *and* $0 < n \le m < 1$, *then* F *is strongly* n-*convex with modulus* c.

PROOF. If $b < 0$, then $D = [b, 0]$. Let $t \in [0, 1]$ and $x, y \in D$ with $x \leq y$. So, $x - \frac{n}{m}y \leq x - y \leq 0$ and therefore, $\|x - y\|^2 \leq \left\|x - \frac{n}{m}y\right\|^2$. Since F is strongly m-convex with modulus c, F is m-convex (Remark 1.5). Thus, from [3, Proposition 2.11], Lemma 2.1(1), and the strong m-convexity of F,

$$\begin{aligned}
tF(x)+&n(1-t)F(y) + cnt(1-t)\|x-y\|^2 B \\
&= tF(x) + m(1-t)\left(\frac{n}{m}\right)F(y) + cmt(1-t)\left(\frac{n}{m}\right)\|x-y\|^2 B \\
&\subseteq tF(x) + m(1-t)F\left(\frac{n}{m}y\right) + cmt(1-t)\left\|x - \frac{n}{m}y\right\|^2 B \\
&\subseteq F(tx + n(1-t)y).
\end{aligned}$$

And for $y < x$, $\|x-y\|^2 \leq \left\|\frac{n}{m}x - y\right\|^2$, hence

$$\begin{aligned}
ntF(x)+&(1-t)F(y) + cnt(1-t)\|x-y\|^2 B \\
&= mt\left(\frac{n}{m}\right)F(x) + (1-t)F(y) + cmt(1-t)\left(\frac{n}{m}\right)\|x-y\|^2 B \\
&\subseteq mtF\left(\frac{n}{m}x\right) + (1-t)F(y) + cmt(1-t)\left\|\frac{n}{m}x - y\right\|^2 B \\
&\subseteq F(ntx + (1-t)y),
\end{aligned}$$

where the last inclusion arises from the strong m-convexity of F and Remark 1.4.

If $b > 0$, $D = [0, b]$ and the proof runs in a similar way, this time for $x \leq y$, we obtain $\|x-y\|^2 \leq \left\|\frac{n}{m}x - y\right\|^2$, and the result follows from Remark 1.4; while for $y < x$, $\|x-y\|^2 \leq \left\|x - \frac{n}{m}y\right\|^2$ and the conclusion follows from (1.1). □

For the next proposition, X is a real inner product space, $cc(Y)$ denotes the subfamily of $n(Y)$ of all convex closed sets. We also recall the cancellation law of Rådström ([4]):

LEMMA 2.4. *Let A, B, C be subsets of X such that $A + C \subseteq B + C$. If B is convex closed and C is nonempty bounded, then $A \subseteq B$.*

PROPOSITION 2.5. *If $F\colon D \subseteq X \to n(Y)$ is m-convex, $c > 0$, and there exists a function $G\colon D \to cc(Y)$ such $F(x) = G(x) + c\|x\|^2 B$ for all $x \in D$, then G is strongly m-convex with modulus c.*

PROOF. Let $x, y \in D$ and $t \in [0,1]$. By the m-convexity of F,

$$t[G(x) + c\|x\|^2 B] + m(1-t)[G(y) + c\|y\|^2 B] \subseteq G(tx + m(1-t)y) + c\|tx + m(1-t)y\|^2 B,$$

which in turn implies, multiplying by $t+m(1-t)$ and applying Lemma 2.1(1),

$$\begin{aligned}[t + m(1-t)](tG(x) + m(1-t)G(y)) &+ [t + m(1-t)](ct\|x\|^2 B + cm(1-t)\|y\|^2 B) \\ &\subseteq [t + m(1-t)]G(tx + m(1-t)y) + c\|tx + m(1-t)y\|^2 B;\end{aligned} \tag{2.5}$$

or

$$[t + m(1-t)](t\|x\|^2 + m(1-t)\|y\|^2) = mt(1-t)\|x - y\|^2 + \|tx + m(1-t)y\|^2.$$

So, by this equality, (2.5), and Lemma 2.1(2), we obtain

$$\begin{aligned}&[t+m(1-t)](tG(x)+m(1-t)G(y))+cmt(1-t)\|x-y\|^2 B+c\|tx+m(1-t)y\|^2 B \\ &\subseteq [t + m(1-t)]G(tx + m(1-t)y) + c\|tx + m(1-t)y\|^2 B.\end{aligned}$$

On the other hand, Lemma 2.1(1) implies

$$[t + m(1-t)]cmt(1-t)\|x - y\|^2 B \subseteq cmt(1-t)\|x - y\|^2 B. \tag{2.6}$$

Then, by Lemma 2.4 and (2.6),

$$[t + m(1-t)](tG(x) + m(1-t)G(y) + cmt(1-t)\|x - y\|^2 B) \subseteq [t + m(1-t)]G(tx + m(1-t)y);$$

or better,

$$tG(x) + m(1-t)G(y) + cmt(1-t)\|x - y\|^2 B \subseteq G(tx + m(1-t)y). \qquad \square$$

Example 2.1. The set-valued function $F\colon [0,1] \subseteq \mathbb{R} \to n(\mathbb{R})$, defined by $F(x) = [0,1]$ is m-convex ([3, Example 2.2]). Moreover, the function $G\colon [0,1] \subseteq \mathbb{R} \to cc(\mathbb{R})$ given by $G(x) = \left[\frac{1}{2}x^2, 1 - \frac{1}{2}x^2\right]$, is such that

$$F(x) = [0,1] = G(x) + \frac{1}{2}x^2\,[-1,1]\,.$$

Hence, from Proposition 2.5, G is a strongly m-convex function with modulus 1/2. Note that this fact agrees with Example 1.2.

3. More results

We finish the paper with this section, in which some properties of the union, intersection and sum of strongly m-convex set-valued functions are shown same as a Jensen type inclusion for this class of functions.

Proposition 3.1. *Let $F_1, F_2\colon D \to n(Y)$ be two strongly m-convex functions with modulus c, such that*

$$F_1(x) \subseteq F_2(x) \quad (\text{or } F_2(x) \subseteq F_1(x)) \tag{3.1}$$

for each $x \in D$. Then the union function ([3, Definition 2.18]*) of F_1 and F_2 is also strongly m-convex function with modulus c.*

Proof. It is straightforward from assumption (3.1). □

The following example shows that the condition (3.1) can not be omitted.

Example 3.1. In Example 1.2 was shown that the functions $F, G\colon [0,1] \to n(\mathbb{R})$ defined by $F(x) = [\frac{1}{2}x^2, 1-\frac{1}{2}x^2]$ and $G(x) = [\frac{1}{2}x^2-1, -\frac{1}{2}x^2]$, are strongly m-convex with modulus $\frac{1}{2}$. Nevertheless, the function $F \cup G$ is not, since it is not m-convex (Remark 1.5). We may notice that its graph (Figure 3) clearly is not an m-convex set ([3, Theorem 2.10]).

For any nonempty subsets A, B, C, D of a linear space and α any scalar, the following properties hold:
- $\alpha(A \cap B) = (\alpha A) \cap (\alpha B)$,
- $A \cap B + C \cap D \subseteq (A + C) \cap (B + D)$,
- If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$,

with these in mind, proof of following result comes out.

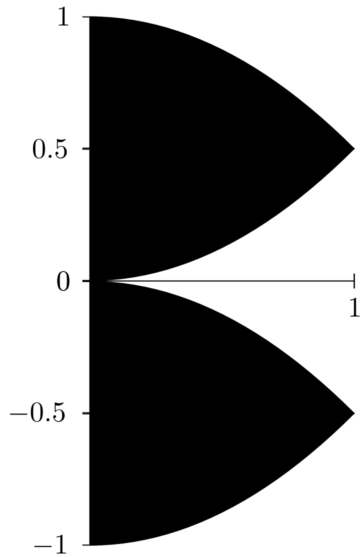

Figure 3. Graph of $F \cup G$

Proposition 3.2. *Let $F_1, F_2 \colon D \to n(Y)$ be two set-valued functions, such that F_1 is strongly m-convex with modulus c_1 and F_2 is strongly m-convex with modulus c_2. Then the intersection function (*[3, Definition 2.18]*) $F_1 \cap F_2$ is strongly m-convex with modulus c, where $c = \min\{c_1, c_2\}$.*

Proof. Let $x, y \in D$ and $t \in [0,1]$. From Lemma 2.1(1) it follows that if $c = \min\{c_1, c_2\}$, then $cmt(1-t)\|x-y\|^2 B \subseteq c_1 mt(1-t)\|x-y\|^2 B \cap c_2 mt(1-t)\|x-y\|^2 B$. Hence,

$$\begin{aligned}
t(F_1 \cap F_2)(x) + m(1-t)(F_1 \cap F_2)(y) + cmt(1-t)\|x-y\|^2 B& \\
\subseteq t[F_1(x) \cap F_2(x)] + m(1-t)[F_1(y) \cap F_2(y)]& \\
+ c_1 mt(1-t)\|x-y\|^2 B \cap c_2 mt(1-t)\|x-y\|^2 B& \\
= tF_1(x) \cap tF_2(x) + m(1-t)F_1(y) \cap m(1-t)F_2(y)& \\
+ c_1 mt(1-t)\|x-y\|^2 B \cap c_2 mt(1-t)\|x-y\|^2 B& \\
\subseteq [tF_1(x) + m(1-t)F_1(y) + c_1 mt(1-t)\|x-y\|^2 B]& \\
\cap [tF_2(x) + m(1-t)F_2(y) + c_2 mt(1-t)\|x-y\|^2 B]& \\
\subseteq F_1(tx + m(1-t)y) \cap F_2(tx + m(1-t)y)& \\
= (F_1 \cap F_2)(tx + m(1-t)y).&
\end{aligned}$$

□

Proposition 3.3. *Let $F_1, F_2 \colon D \to n(Y)$ be two strongly m-convex functions with modulus c_1 and c_2, respectively. Then the sum function (*[3, Definition 2.18]*) $F_1 + F_2$ is strongly m-convex with modulus $c_1 + c_2$.*

Proof. If $x, y \in D$ and $t \in [0,1]$, then

$$\begin{aligned}
t(F_1 + F_2)(x) + m(1-t)(F_1 + F_2)(y) + (c_1 + c_2)mt(1-t)\|x-y\|^2 B \\
&= [tF_1(x) + m(1-t)F_1(y) + c_1 mt(1-t)\|x-y\|^2 B] \\
&\quad + [tF_2(x) + m(1-t)F_2(y) + c_2 mt(1-t)\|x-y\|^2 B] \\
&\subseteq F_1(tx + m(1-t)y) + F_2(tx + m(1-t)y) \\
&= (F_1 + F_2)(tx + m(1-t)y). \qquad \square
\end{aligned}$$

Proposition 3.4. *Let $F_1 \colon D \to n(Y)$ and $F_2 \colon D \to n(Z)$ be two strongly m-convex functions with modulus c_1 and c_2, respectively. Then the Cartesian product function (*[3, Definition 2.19]*) $F_1 \times F_2$ is strongly m-convex with modulus c, where $c = \min\{c_1, c_2\}$, B_Y, B_Z are the closed unit balls in Y and Z, and $B = \{(y,z) \in Y \times Z : \max\{\|y\|, \|z\|\} \le 1\} \subseteq B_Y \times B_Z$.*

Proof. Let $x, y \in D$ and $t \in [0,1]$. Because $c \le c_1, c_2$, Lemma 2.1(1) implies

$$\left.\begin{aligned} cmt(1-t)\|x-y\|^2 B_Y \subseteq c_1 mt(1-t)\|x-y\|^2 B_Y \\ cmt(1-t)\|x-y\|^2 B_Z \subseteq c_2 mt(1-t)\|x-y\|^2 B_Z \end{aligned}\right\}. \tag{3.2}$$

Taking into account (3.2) and properties of Cartesian product ([3]),

$$\begin{aligned}
[cmt(1-t)\|x-y\|^2 B_Y] \times [cmt(1-t)\|x-y\|^2 B_Z] \\
\subseteq [c_1 mt(1-t)\|x-y\|^2 B_Y] \times [c_2 mt(1-t)\|x-y\|^2 B_Z].
\end{aligned}$$

Then,

$$\begin{aligned}
t(F_1 \times F_2)(x) + m(1-t)(F_1 \times F_2)(y) + cmt(1-t)\|x-y\|^2 B \\
&\subseteq t[F_1(x) \times F_2(x)] + m(1-t)[F_1(y) \times F_2(y)] \\
&\quad + cmt(1-t)\|x-y\|^2 (B_Y \times B_Z) \\
&= tF_1(x) \times tF_2(x) + m(1-t)F_1(y) \times m(1-t)F_2(y) \\
&\quad + cmt(1-t)\|x-y\|^2 B_Y \times cmt(1-t)\|x-y\|^2 B_Z
\end{aligned}$$

$$\begin{aligned}
&\subseteq tF_1(x)\times tF_2(x)+m(1-t)F_1(y)\times m(1-t)F_2(y)\\
&\quad + c_1mt(1-t)\|x-y\|^2B_Y \times c_2mt(1-t)\|x-y\|^2B_Z\\
&= [tF_1(x)+m(1-t)F_1(y)+c_1mt(1-t)\|x-y\|^2B_Y]\\
&\quad \times [tF_2(x)+m(1-t)F_2(y)+c_2mt(1-t)\|x-y\|^2B_Z]\\
&\subseteq F_1(tx+m(1-t)y)\times F_2(tx+m(1-t)y)\\
&= (F_1\times F_2)(tx+m(1-t)y). \qquad \square
\end{aligned}$$

We finish the work by presenting a Jensen type inclusion for strongly m-convex set-valued functions, for the discrete case. Thereon, we simplify the notation by employing the well-known Delta of Kronecker $\delta_{ij} = \begin{cases} 0, & \text{if } i\neq j,\\ 1, & \text{if } i=j.\end{cases}$

Theorem 3.5. *Let $t_1,\dots,t_n$ be positive real numbers ($n\geq 2$) such that $T_n=\sum_{i=1}^{n} t_i\in(0,1]$. If $F\colon D\subseteq X\to n(Y)$ is a strongly m-convex function with modulus c, then*

$$\sum_{i=1}^{n} m^{1-\delta_{i1}}t_iF(x_i)+cm\sum_{i=2}^{n}\frac{t_i}{T_{i-1}T_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}t_kx_k-T_{i-1}x_i\Big\|^2B \subseteq F\Big(\sum_{i=1}^{n}m^{1-\delta_{i1}}t_ix_i\Big),$$

for all $x_1,\dots,x_n\in D$.

Proof. The proof runs by induction on n. For $n=2$,

$$\begin{aligned}
&\sum_{i=1}^{2} m^{1-\delta_{i1}}t_iF(x_i)+cm\sum_{i=2}^{2}\frac{t_i}{T_{i-1}T_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}t_kx_k-T_{i-1}x_i\Big\|^2B\\
&= t_1F(x_1)+mt_2F(x_2)+cm\frac{t_2}{T_1T_2}\|t_1x_1-T_1x_2\|^2B\\
&= t_1F(x_1)+mt_2F(x_2)+cm\frac{t_2}{t_1(t_1+t_2)}\|t_1x_1-t_1x_2\|^2B\\
&= (t_1+t_2)\Big[\frac{t_1}{t_1+t_2}F(x_1)+m\frac{t_2}{t_1+t_2}F(x_2)+cm\frac{t_1t_2}{(t_1+t_2)^2}\|x_1-x_2\|^2B\Big]\\
&\subseteq (t_1+t_2)F\Big(\frac{t_1}{t_1+t_2}x_1+m\frac{t_2}{t_1+t_2}x_2\Big),
\end{aligned}$$

where the last inclusion results from the strong m-convexity of F. From Remark 1.5 and [3, Proposition 2.11] we obtain the following inclusion

$$(t_1+t_2)F\left(\frac{t_1}{t_1+t_2}x_1+m\frac{t_2}{t_1+t_2}x_2\right)\subseteq F(t_1x_1+mt_2x_2) = F\Big(\sum_{i=1}^{2}m^{1-\delta_{i1}}t_ix_i\Big).$$

We assume now the result is true for n. So for $n+1$, let $t_1,\ldots,t_{n+1}$ be positive real numbers with $T_{n+1}=\sum_{i=1}^{n+1}t_i\in(0,1]$, and $x_1,\ldots,x_{n+1}\in D$. Then,

$$\begin{aligned}
&\sum_{i=1}^{n+1}m^{1-\delta_{i1}}t_iF(x_i)+cm\sum_{i=2}^{n+1}\frac{t_i}{T_{i-1}T_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}t_kx_k-T_{i-1}x_i\Big\|^2B\\
&\quad=t_1F(x_1)+mt_2F(x_2)+cm\frac{t_2}{T_1T_2}\|t_1x_1-t_1x_2\|^2B\\
&\qquad+\sum_{i=3}^{n+1}m^{1-\delta_{i1}}t_iF(x_i)+cm\sum_{i=3}^{n+1}\frac{t_i}{T_{i-1}T_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}t_kx_k-T_{i-1}x_i\Big\|^2B\\
&\quad=(t_1+t_2)\Big[\frac{t_1}{t_1+t_2}F(x_1)+m\frac{t_2}{t_1+t_2}F(x_2)+cm\frac{t_1t_2}{(t_1+t_2)^2}\|x_1-x_2\|^2B\Big]\\
&\qquad+\sum_{i=3}^{n+1}m^{1-\delta_{i1}}t_iF(x_i)+cm\sum_{i=3}^{n+1}\frac{t_i}{T_{i-1}T_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}t_kx_k-T_{i-1}x_i\Big\|^2B\\
&\quad\subseteq(t_1+t_2)F\Big(\frac{t_1}{t_1+t_2}x_1+m\frac{t_2}{t_1+t_2}x_2\Big)+\sum_{i=3}^{n+1}m^{1-\delta_{i1}}t_iF(x_i)\\
&\qquad+cm\sum_{i=3}^{n+1}\frac{t_i}{T_{i-1}T_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}t_kx_k-T_{i-1}x_i\Big\|^2B\\
&\quad=(t_1+t_2)F\Big(\frac{t_1}{t_1+t_2}x_1+m\frac{t_2}{t_1+t_2}x_2\Big)+m\sum_{i=2}^{n}t_{i+1}F(x_{i+1})\\
&\qquad+cm\sum_{i=2}^{n}\frac{t_{i+1}}{T_iT_{i+1}}\Big\|\sum_{k=1}^{i}m^{1-\delta_{k1}}t_kx_k-T_ix_{i+1}\Big\|^2B\\
&\quad=(t_1+t_2)F\Big(\frac{t_1}{t_1+t_2}x_1+m\frac{t_2}{t_1+t_2}x_2\Big)+m\sum_{i=2}^{n}t_{i+1}F(x_{i+1})\\
&\qquad+cm\sum_{i=2}^{n}\frac{t_{i+1}}{T_iT_{i+1}}\Big\|t_1x_1+mt_2x_2+\sum_{k=3}^{i}m^{1-\delta_{k1}}t_kx_k-T_ix_{i+1}\Big\|^2B
\end{aligned}$$

$$= (t_1 + t_2)F\Big(\frac{t_1}{t_1+t_2}x_1 + m\frac{t_2}{t_1+t_2}x_2\Big) + m\sum_{i=2}^{n} t_{i+1}F(x_{i+1})$$

$$+ cm\sum_{i=2}^{n}\frac{t_{i+1}}{T_iT_{i+1}}\Big\|(t_1+t_2)\Big(\frac{t_1}{t_1+t_2}x_1 + m\frac{t_2}{t_1+t_2}x_2\Big)$$

$$+ \sum_{k=2}^{i-1} m^{1-\delta_{(k+1)1}}t_{k+1}x_{k+1} - T_ix_{i+1}\Big\|^2 B.$$

Now we set

$$\bar{t}_i = \begin{cases} t_1 + t_2, & \text{if } i = 1,\\ t_{i+1}, & \text{if } i \in \{2,\dots,n\},\end{cases}$$

and

$$\overline{x}_i = \begin{cases} \dfrac{t_1}{t_1+t_2}x_1 + m\dfrac{t_2}{t_1+t_2}x_2, & \text{if } i = 1,\\ x_{i+1}, & \text{if } i \in \{2,\dots,n\},\end{cases}$$

then $T_{n+1} = t_1 + t_2 + \cdots + t_{n+1} = \bar{t}_1 + \bar{t}_2 + \cdots + \bar{t}_n := \overline{T}_n$. With this in mind the latter expression can be rewritten as

$$\bar{t}_1F(\overline{x}_1) + m\sum_{i=2}^{n}\bar{t}_iF(\overline{x}_i) + cm\sum_{i=2}^{n}\frac{\bar{t}_i}{\overline{T}_{i-1}\overline{T}_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}\bar{t}_k\overline{x}_k - \overline{T}_{i-1}\overline{x}_i\Big\|^2 B$$

or better,

$$\sum_{i=1}^{n} m^{1-\delta_{i1}}\bar{t}_iF(\overline{x}_i) + cm\sum_{i=2}^{n}\frac{\bar{t}_i}{\overline{T}_{i-1}\overline{T}_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}\bar{t}_k\overline{x}_k - \overline{T}_{i-1}\overline{x}_i\Big\|^2 B, \tag{3.3}$$

where $\bar{t}_1,\dots,\bar{t}_n > 0$ with $\overline{T}_n = \sum_{i=1}^{n}\bar{t}_i \in (0,1]$ and $\overline{x}_1,\dots,\overline{x}_n \in D$. Therefore, by using the inductive hypothesis, (3.3) is a subset of $F\big(\sum_{i=1}^{n} m^{1-\delta_{i1}}\bar{t}_i\overline{x}_i\big)$. In conclusion,

$$\sum_{i=1}^{n+1} m^{1-\delta_{i1}}t_iF(x_i) + cm\sum_{i=2}^{n+1}\frac{t_i}{T_{i-1}T_i}\Big\|\sum_{k=1}^{i-1}m^{1-\delta_{k1}}t_kx_k - T_{i-1}x_i\Big\|^2 B$$

$$\subseteq F\Big(\sum_{i=1}^{n} m^{1-\delta_{i1}}\bar{t}_i\overline{x}_i\Big) = F\Big(\sum_{i=1}^{n+1} m^{1-\delta_{i1}}t_ix_i\Big)$$

and the result is true for $n+1$ as well. □

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