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STRONGLY $M_\varphi M_\psi$-CONVEX FUNCTIONS, THE HERMITE–HADAMARD–FEJÉR INEQUALITY AND RELATED RESULTS

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Abstract. We present Hermite–Hadamard–Fejér type inequalities for strongly $M_\varphi M_\psi$-convex functions. Some refinements of them and bounds for the integral mean of the product of two functions are also obtained.

1. Preliminaries

In the last few decades, several articles have been published devoted to the research of generalized convex functions that are defined according to a pair of means, so called MN-convex functions, where M and N are suitable means. Roughly speaking, the following inequality holds for these functions:

$$f(M(x,y)) \leq N(f(x), f(y)),$$

where M and N are means (see [10, Chapter 2]). The most significant among MN-convex functions, apart from the class of convex functions in the usual

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sense, are: the class of log-convex (or AG-convex) functions (M is the arithmetic mean and N is the geometric mean), the class of multiplicatively convex (or GG-convex) functions (M and N are geometric means), the class of p-convex (or M_pA-convex) functions where M is the power mean of order p and N is the arithmetic mean. As we see in the previous texts, very often means M and N are quasi-arithmetic means and in this paper we focus on them. Let us define these objects precisely.

Let φ be a continuous, strictly monotone function defined on the interval I. With M_φ we denote the quasi-arithmetic mean:

$$M_\varphi(x, y; t) := \varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)), \quad x, y \in I, t \in [0, 1].$$

Definition 1.1. Let φ and ψ be two continuous, strictly monotone functions defined on intervals I and K respectively. We say that a function $f\colon I \to K$ is *$M_\varphi M_\psi$-convex* if

$$f(M_\varphi(x, y; t)) \leq M_\psi(f(x), f(y); t) \tag{1.1}$$

for all $x, y \in I$ and all $t \in [0, 1]$. If the inequality sign is reversed in (1.1), then f is called *$M_\varphi M_\psi$-concave.*

The concept of $M_\varphi M_\psi$-convexity can be connected with the classical convexity via the following lemma.

Lemma 1.2 ([10, Lemma 2.6.1.], [18]). *Let ψ, φ and f satisfy the assumptions from Definition* 1.1.

If ψ is strictly increasing on K, then f is $M_\varphi M_\psi$-convex on I if and only if $\psi \circ f \circ \varphi^{-1}$ is convex on $\varphi(I)$.

If ψ is strictly decreasing on K, then f is $M_\varphi M_\psi$-convex on I if and only if $\psi \circ f \circ \varphi^{-1}$ is concave on $\varphi(I)$.

More results about $M_\varphi M_\psi$-convexity can be found in [1], [3], [9], [10], [16], [18].

Sixty years ago B.T. Polyak introduced strongly convex functions that proved to be a useful tool in optimization theory and mathematics economics, [12].

Definition 1.3. Let $I \subseteq \mathbb{R}$ be an interval and c be a positive number. A function $f\colon I \to \mathbb{R}$ is called *strongly convex with modulus c* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2,$$

for all $x, y \in I$ and $t \in [0, 1]$.

Strong convexity can be considered as one kind of strengthening of convexity and many properties of convex functions have counterparts involving strongly convex functions. A very useful tool in the study of the class of strongly convex functions is the following lemma which gives the relationship between convexity and strong convexity.

LEMMA 1.4 ([7]). *A function $f\colon I \to \mathbb{R}$ is strongly convex with modulus c if and only if the function $g\colon I \to \mathbb{R}$ defined by $g(x) = f(x) - cx^2$ is convex.*

The class of strongly convex functions has been intensively studied as we can see from the very rich literature, see for example [2], [6], [8], [13] and references therein. From a large number of results, we will single out the following Hermite–Hadamard–Fejér inequality for strongly convex functions.

THEOREM 1.5 ([2, 8]). *Let $w\colon [a,b] \to \mathbb{R}_0^+$ be a symmetric density function on $[a,b]$ (that is, $w(a+b-x) = w(x)$ for all $x \in [a,b]$ and $\int_a^b w(x)dx = 1$). Let $f\colon [a,b] \to \mathbb{R}$ be a strongly convex function with modulus $c > 0$. Then*

$$f\left(\frac{a+b}{2}\right) + c\left[\int_a^b x^2 w(x)dx - \left(\frac{a+b}{2}\right)^2\right] \leq \int_a^b f(x)w(x)dx$$
$$\leq \frac{f(a)+f(b)}{2} - c\left[\frac{a^2+b^2}{2} - \int_a^b x^2 w(x)dx\right].$$

Moreover,

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2) \leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2.$$

In this paper, we connect these two concepts: strong convexity and MN-convexity and introduce the following new class of functions.

DEFINITION 1.6. Let φ and ψ be two continuous, strictly monotone functions defined on intervals I and K, respectively. Let $c \geq 0$ and $f\colon I \to K$ such that

$$t\psi(f(x)) + (1-t)\psi(f(y)) - ct(1-t)(\varphi(y)-\varphi(x))^2 \in \psi(K) \tag{1.2}$$

for all $x, y \in I, t \in [0,1]$. We say that a function f is *strongly $M_\varphi M_\psi$-convex with modulus c* if

$$f(M_\varphi(x,y;t)) \leq \psi^{-1}\Big(t\psi(f(x)) + (1-t)\psi(f(y)) - ct(1-t)(\varphi(y)-\varphi(x))^2\Big)$$

for all $x, y \in I$ and all $t \in [0,1]$. If the sign of inequality is reversed in the above inequality, then f is called strongly $M_\varphi M_\psi$-concave with modulus c.

Recently, some special cases of the class of functions defined above have been studied. For example, the definition and some properties of the reciprocally strongly convex functions (or strongly HA-convex) are given in [4], some properties of strongly p-convex functions (or strongly M_pA-convex) are obtained in [15], the Hermite–Hadamard type inequalities for strongly GA-convex are studied in [14].

In this paper, the counterpart of the Hermite–Hadamard–Fejér inequality for strongly $M_\varphi M_\psi$-convex functions is the focus of study. The second section is devoted to the above-mentioned result. In the same section, we describe the relationship between $M_\varphi M_\psi$-convex functions and strongly $M_\varphi M_\psi$-convex functions. Refinements of the Hermite–Hadamard inequality are given in the third section while in the fourth section we consider estimates for the integral mean of the product of two functions.

2. Characterization and the Hermite–Hadamard–Fejér inequality for strongly $M_\varphi M_\psi$-convex function

The definition of the strongly $M_\varphi M_\psi$-convexity and the connection of $M_\varphi M_\psi$-convexity with convexity lead us to the following statement.

Lemma 2.1. *Let φ, ψ, f satisfy the assumptions from Definition* 1.6.
If ψ is strictly increasing on K, then f is strongly $M_\varphi M_\psi$-convex (concave) on I if and only if $\psi \circ f \circ \varphi^{-1}$ is strongly convex (concave) on $\varphi(I)$.
If ψ is strictly decreasing on K, then f is strongly $M_\varphi M_\psi$-convex (concave) on I if and only if $\psi \circ f \circ \varphi^{-1}$ is strongly concave (convex) on $\varphi(I)$.

Proof. The statements follow directly from the definition of the strongly $M_\varphi M_\psi$-convex function. □

In the following theorem we connect strongly $M_\varphi M_\psi$-convexity and $M_\varphi M_\psi$-convexity.

Theorem 2.2. *Let f, φ and ψ satisfy assumptions of Definition* 1.6 *and let $\psi(f(x)) - c\varphi^2(x) \in \psi(K)$. Function f is a strongly $M_\varphi M_\psi$-convex (concave) function with modulus c if and only if*

$$g(x) := \psi^{-1}\Big(\psi(f(x)) - c\varphi^2(x)\Big)$$

is an $M_\varphi M_\psi$-convex (concave) function.

Proof. In the following text we use symbols: $F := \psi \circ f$ and $G := \psi \circ g$. Let us assume that ψ is strictly increasing. If f is a strongly $M_\varphi M_\psi$-convex function with modulus c, then for any $x, y \in I$ and any $\alpha, \beta \in [0, 1]$, $\alpha + \beta = 1$

$$F(M_\varphi(x, y; \alpha)) \le \alpha F(x) + \beta F(y) - c\alpha\beta(\varphi(y) - \varphi(x))^2. \tag{2.1}$$

Then, for a function g defined as $g(x) := \psi^{-1}\Big(\psi(f(x)) - c\varphi^2(x)\Big)$, we get

$$\begin{aligned}
&G(M_\varphi(x, y; \alpha)) - (\alpha G(x) + \beta G(y)) \\
&\quad = F(M_\varphi(x, y; \alpha)) - c\varphi^2\big(M_\varphi(x, y; \alpha)\big) \\
&\qquad - \big(\alpha(F(x) - c\varphi^2(x)) + \beta(F(y) - c\varphi^2(y))\big) \\
&\quad = F(M_\varphi(x, y; \alpha)) - \Big(\alpha F(x) + \beta F(y) - c\alpha\beta(\varphi(y) - \varphi(x))^2\Big) \\
&\qquad - c\left\{\varphi^2\big(M_\varphi(x, y; \alpha)\big) - \alpha\varphi^2(x) - \beta\varphi^2(y) + \alpha\beta(\varphi(y) - \varphi(x))^2\right\} \\
&\quad \le 0 - c\left\{\varphi^2\big(M_\varphi(x, y; \alpha)\big) - \alpha\varphi^2(x) - \beta\varphi^2(y) + \alpha\beta(\varphi(y) - \varphi(x))^2\right\} \\
&\quad = -c\Big\{\varphi^2(x)[\alpha^2 - \alpha + \alpha\beta] + \varphi^2(y)[\beta^2 - \beta + \alpha\beta]\Big\} \\
&\quad = -c\Big\{\varphi^2(x)\alpha[\alpha - 1 + \beta] + \varphi^2(y)\beta[\beta - 1 + \alpha]\Big\} = 0.
\end{aligned}$$

Hence,

$$G(M_\varphi(x, y; \alpha)) \le \alpha G(x) + \beta G(y), \tag{2.2}$$

which means that g is $M_\varphi M_\psi$-convex. Conversely, if g is $M_\varphi M_\psi$-convex, then using the above calculation, we conclude that f is a strongly $M_\varphi M_\psi$-convex function with modulus c.

If ψ is strictly decreasing and f is strongly $M_\varphi M_\psi$-convex, then in (2.1) an opposite sign is valid. Then inequality (2.2) holds with the sign $\ge$, which means that g is $M_\varphi M_\psi$-convex. □

Corollary 2.3. *Let the assumptions of Theorem* 2.2 *hold.*

If ψ is strictly increasing on K, then f is strongly $M_\varphi M_\psi$-convex (concave) on I if and only if $\psi \circ f \circ \varphi^{-1} - ce_2$ is convex (concave) on $\varphi(I)$, where $e_2(x) = x^2$.

If ψ is strictly decreasing on K, then f is strongly $M_\varphi M_\psi$-convex (concave) on I if and only if $\psi \circ f \circ \varphi^{-1} - ce_2$ is concave (convex) on $\varphi(I)$.

Proof. It is a consequence of Lemma 2.1 and Lemma 1.4. □

The Hermite–Hadamard inequality for an $M_\varphi M_\psi$-convex function is already known (see [9]). Using the above-proved Theorem 2.2, we can easily prove the Hermite–Hadamard inequality for a strongly $M_\varphi M_\psi$-convex function. But, here we will prove its weighted version, the so-called Hermite–Hadamard–Fejér inequality. Since, we do not find result of that type for the $M_\varphi M_\psi$-convex function, we are unable to prove the Hermite–Hadamard–Fejér inequality using Theorem 2.2. So we prove it directly.

THEOREM 2.4 (The Hermite–Hadamard–Fejér inequality for a strongly $M_\varphi M_\psi$-convex function). *Let φ be a differentiable, strictly monotone function defined on $I=[a,b]$ and let ψ be strictly monotone function on an interval K.*

Let $w\colon [a,b]\to[0,\infty)$ be a function such that $w\varphi'\in L([a,b])$ and

$$w\big(M_\varphi(x,y;\alpha)\big)=w\left(M_\varphi(x,y;\beta)\right) \tag{2.3}$$

for all $\alpha,\beta\in[0,1]$, $\alpha+\beta=1$.

(i) *If ψ is strictly increasing, then for a strongly $M_\varphi M_\psi$-convex function f the following holds:*

$$\begin{aligned} &\psi\left(f\Big(M_\varphi(a,b;\frac{1}{2})\Big)\right)\int_a^b w(x)\varphi'(x)\,dx\\ &\quad+c\Big[\int_a^b w(x)\varphi^2(x)\varphi'(x)\,dx-\frac{\big(\varphi(b)+\varphi(a)\big)^2}{4}\int_a^b w(x)\varphi'(x)\,dx\Big]\\ &\le\int_a^b \psi(f(x))w(x)\varphi'(x)\,dx\\ &\le\frac{\psi(f(a))+\psi(f(b))}{2}\int_a^b w(x)\varphi'(x)\,dx\\ &\quad-c\Big[\frac{\varphi^2(a)+\varphi^2(b)}{2}\int_a^b w(x)\varphi'(x)\,dx-\int_a^b w(x)\varphi^2(x)\varphi'(x)\,dx\Big], \end{aligned} \tag{2.4}$$

provided that all integrals exist. Moreover,

$$\begin{aligned} &\psi\left(f\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\right)+c\frac{\big(\varphi(b)-\varphi(a)\big)^2}{12}\\ &\qquad\le\frac{1}{\varphi(b)-\varphi(a)}\int_a^b\psi(f(x))\varphi'(x)\,dx\\ &\qquad\le\frac{\psi(f(a))+\psi(f(b))}{2}-c\frac{\Big(\varphi(b)-\varphi(a)\Big)^2}{6}, \end{aligned} \tag{2.5}$$

provided that all integrals exist.

If f is strongly $M_\varphi M_\psi$-concave, then signs of inequality in (2.4) *and* (2.5) *are reversed.*

(ii) *If ψ is strictly decreasing, then for a strongly $M_\varphi M_\psi$-concave function f inequalities* (2.4) *and* (2.5) *are valid. But, if f is strongly $M_\varphi M_\psi$-convex, then signs of inequality in* (2.4) *and* (2.5) *are reversed.*

Proof. Let us suppose that ψ is strictly increasing and f is strongly $M_\varphi M_\psi$-convex. Let us write some particular, useful identities. We use abbreviations: $F := \psi \circ f$, $\Delta := \varphi(b) - \varphi(a)$. After substitutions $x = M_\varphi(a, b; 1-t)$ and $x = M_\varphi(a, b; t)$, we get:

$$\int_0^{1/2} w(M_\varphi(a, b; 1-t))dt = \frac{1}{\Delta} \int_a^{M_\varphi(a,b;1/2)} w(x)\varphi'(x)dx, \tag{2.6}$$

$$\int_0^{1/2} w(M_\varphi(a, b; t))dt = \frac{1}{\Delta} \int_{M_\varphi(a,b;1/2)}^b w(x)\varphi'(x)dx. \tag{2.7}$$

Since $w\left(M_\varphi(x, y; t)\right) = w\left(M_\varphi(x, y; 1-t)\right)$, summing identities (2.6) and (2.7), we get

$$\int_0^{1/2} w(M_\varphi(a, b; t))dt = \frac{1}{2\Delta} \int_a^b w(x)\varphi'(x)dx. \tag{2.8}$$

Again, from symmetry, we have:

$$\int_0^{1/2} w(M_\varphi(a, b; 1-t))dt = \frac{1}{2\Delta} \int_a^b w(x)\varphi'(x)dx. \tag{2.9}$$

Summing (2.8) and (2.9) and using the assumption (2.3), we get

$$\int_0^1 w(M_\varphi(a, b; t))dt = \frac{1}{\Delta} \int_a^b w(x)\varphi'(x)dx. \tag{2.10}$$

Let us prove the following equality.

$$\int_a^b w(x)\varphi(x)\varphi'(x)dx = \frac{\varphi(b) + \varphi(a)}{2} \int_a^b w(x)\varphi'(x)dx. \tag{2.11}$$

Firstly, we divide the integral on the right-hand side in two integrals:

$$\begin{aligned}
&\int_a^b w(x)\varphi(x)\varphi'(x)dx \\
&\qquad = \int_a^{M_\varphi(a,b;\frac{1}{2})} w(x)\varphi(x)\varphi'(x)dx + \int_{M_\varphi(a,b;\frac{1}{2})}^b w(x)\varphi(x)\varphi'(x)dx \\
&\qquad = \Delta\int_0^{\frac{1}{2}} ((1-t)\varphi(a)+t\varphi(b))w(M_\varphi(a,b;1-t))\,dt \\
&\qquad\quad + \Delta\int_{\frac{1}{2}}^0 (t\varphi(a)+(1-t)\varphi(b))w(M_\varphi(a,b;t))\,(-dt) \\
&\qquad = \Delta\int_0^{\frac{1}{2}} (\varphi(a)+\varphi(b))w(M_\varphi(a,b;1-t))\,dt \\
&\qquad = \frac{\varphi(b)+\varphi(a)}{2}\int_a^b w(x)\varphi'(x)dx,
\end{aligned}$$

where we use the assumption (2.3) and identity (2.9).

Using (2.11) and the assumption (2.3), after a suitable substitution $x = M_\varphi(a,b;t)$, we get

$$\begin{aligned}
\int_0^1 tw(M_\varphi(a,b;t))dt &= \frac{1}{\Delta^2}\int_a^b (\varphi(b)-\varphi(x))w(x)\varphi'(x)dx \\
&= \frac{1}{\Delta^2}\left[\varphi(b)\int_a^b w(x)\varphi'(x)dx - \int_a^b \varphi(x)w(x)\varphi'(x)dx\right] \\
&= \frac{1}{\Delta^2}\left[\varphi(b)\int_a^b w(x)\varphi'(x)dx - \frac{\varphi(b)+\varphi(a)}{2}\int_a^b w(x)\varphi'(x)dx\right] \\
&= \frac{1}{2\Delta}\int_a^b w(x)\varphi'(x)dx \qquad (2.12)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 t^2 w(M_\varphi(a,b;t))dt &= \frac{1}{\Delta^3}\int_a^b (\varphi(b)-\varphi(x))^2 w(x)\varphi'(x)dx \qquad (2.13) \\
&= \frac{1}{\Delta^3}\Big\{ -\varphi(a)\varphi(b)\int_a^b w(x)\varphi'(x)dx + \int_a^b w(x)\varphi^2(x)\varphi'(x)dx\Big\}.
\end{aligned}$$

Let us prove the left-hand side inequality in (2.4). Since f is strongly $M_\varphi M_\psi$-convex and ψ is strictly increasing, then $F = \psi \circ f$ is strongly $M_\varphi A$-convex and we get:

$$\begin{aligned}
F\left(M_\varphi(a,b;\frac{1}{2})\right) &= F\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\\
&= F\left(\varphi^{-1}\left(\frac{1}{2}[t\varphi(a)+(1-t)\varphi(b)]+\frac{1}{2}[(1-t)\varphi(a)+t\varphi(b)]\right)\right)\\
&= F\left(M_\varphi(u,v;\frac{1}{2})\right) \le \frac{1}{2}F(u)+\frac{1}{2}F(v)-\frac{c}{4}(\varphi(u)-\varphi(v))^2\\
&= \frac{1}{2}F(u)+\frac{1}{2}F(v)-\frac{c}{4}(1-2t)^2\Delta^2,
\end{aligned}$$

where we use abbreviations: $u := M_\varphi(a,b;t)$ and $v := M_\varphi(a,b;1-t)$. Multiplying the above inequality with $w\left(M_\varphi(a,b;t)\right)$, integrating over $[0,1]$, and using condition (2.3), we get

$$\begin{aligned}
&F\left(M_\varphi(a,b;\frac{1}{2})\right)\int_0^1 w\left(M_\varphi(a,b;t)\right)dt\\
&\le \frac{1}{2}\int_0^1 F(u)w\left(u\right)dt+\frac{1}{2}\int_0^1 F(v)w(v)dt-\frac{c}{4}\Delta^2\int_0^1(1-2t)^2w(u)dt\\
&= \frac{1}{\Delta}\int_a^b w(x)F(x)\varphi'(x)dx-\frac{c}{4}\Delta^2\int_0^1(1-2t)^2w(u)dt.
\end{aligned}$$

Using (2.10) and multiplying with Δ, we get:

$$\begin{aligned}
F\left(M_\varphi(a,b;\frac{1}{2})\right)&\int_0^1 w(x)\varphi'(x)dx\\
&\le \int_a^b w(x)F(x)\varphi'(x)\,dx-\frac{c}{4}\Delta^3\int_0^1(1-2t)^2w\left(u\right)\,dt.
\end{aligned}$$

Using (2.10), (2.11), (2.12) and (2.13), we get

$$\begin{aligned}
&\frac{c}{4}\Delta^3\int_0^1(1-2t)^2w\left(u\right)\,dt\\
&\qquad= c\Big[\int_a^b w(x)\varphi^2(x)\varphi'(x)\,dx-\frac{\Big(\varphi(b)+\varphi(a)\Big)^2}{4}\int_a^b w(x)\varphi'(x)\,dx\Big]
\end{aligned}$$

and the proof of the left-hand side of (2.4) is complete.

Let us prove the right-hand side of the Hermite–Hadamard inequality. Since f is strongly $M_\varphi M_\psi$-convex, we get

$$F(M_\varphi(a,b;t)) \le tF(a) + (1-t)F(b) - ct(1-t)(\varphi(b)-\varphi(a))^2. \tag{2.14}$$

Multiplying inequality (2.14) with $w\left(M_\varphi(a,b;t)\right)$ and integrating over $[0,1]$, we get

$$\begin{aligned}\int_0^1 &F(M_\varphi(a,b;t))w(M_\varphi(a,b;t))\,dt\\ &\le \int_0^1 [tF(a)+(1-t)F(b)]w(M_\varphi(a,b;t))\,dt\\ &\qquad - c\Delta^2\int_0^1 (t-t^2)w(M_\varphi(a,b;t))\,dt.\end{aligned}$$

After substitution $x = M_\varphi(a,b;t)$, an integral on the left-hand side in the above inequality becomes $\int_a^b F(x)w(x)\varphi'(x)dx$ and using identities (2.12) and (2.13) on the right-hand side of the above inequality, we get the right-hand side of (2.4).

Formula (2.5) follows from (2.4) for the particular weight $w(t) = 1$. Other cases are proven in a similar way. □

Remark 2.5. It is clear that for $\varphi(x) = \psi(x) = x$ Theorem 2.4 becomes Theorem 1.5, [2].

Particular cases of the Hermite–Hadamard inequality (2.5), but not for the weighted version of the Hermite–Hadamard inequality, for various types of generalized convex functions can be found in the literature. Inequality (2.5) for strongly GA-convex functions ($\varphi(x) = \log x$, $\psi(x) = x$) is given in [14] and for strongly p-convex in [15].

The Hermite–Hadamard inequality (2.5) for strongly HA-convex functions is published in [11], where we can also found the right-hand side of (2.5) for strongly HG-convex functions.

3. Refinements of the Hermite–Hadamard inequalities

In this section, we consider the Hermite–Hadamard inequality (2.5) and obtain its refinements. We are following an idea given in [5] by El Farissi. Let us mention his result.

THEOREM 3.1 ([5]). *Assume that f is a convex function on $[a,b]$. Then for all $\lambda \in [0,1]$ we have*

$$f\Big(\frac{a+b}{2}\Big) \le \delta_1 \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \delta_2 \le \frac{f(a)+f(b)}{2},$$

where

$$\delta_1 := \lambda f\Big(\frac{(2-\lambda)a+\lambda b}{2}\Big) + (1-\lambda) f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2}\Big),$$

$$\delta_2 := \frac{1}{2}\Big[f\big((1-\lambda)a+\lambda b\big) + (1-\lambda)f(b) + \lambda f(a)\Big].$$

Using the above result and Corollary 2.3 we obtain the following refinement of the Hermite–Hadamard inequality.

THEOREM 3.2. *Let φ and ψ be strictly monotone continuous functions defined on intervals I and K respectively such that φ is differentiable on $[a,b] \subseteq I$. Let $f\colon I \to K$ be a function such that $(\psi \circ f)\varphi'$ is integrable and f satisfies* (1.2).

(i) *If ψ is strictly increasing, then for a strongly $M_\varphi M_\psi$-convex function f the following holds*

$$\psi(f(M_\varphi(a,b;\frac{1}{2}))) + \frac{c}{12}\Delta^2 \le \tilde{\delta_1} \le \frac{1}{\Delta}\int_a^b \psi(f(x))\,\varphi'(x)\,dx \tag{3.1}$$
$$\le \tilde{\delta_2} \le \frac{\psi(f(a))+\psi(f(b))}{2} - \frac{c}{6}\Delta^2,$$

where

$$\Delta := \varphi(b) - \varphi(a),$$

$$\tilde{\delta_1} := (1-\lambda)(\psi\circ f)\Big(M_\varphi\Big(a,b;\frac{1-\lambda}{2}\Big)\Big)$$
$$+ \lambda(\psi\circ f)\Big(M_\varphi\Big(a,b;\frac{2-\lambda}{2}\Big)\Big) + \frac{c}{12}\Delta^2(3\lambda^2-3\lambda+1),$$

$$\tilde{\delta_2} := \frac{1}{2}\Big[\psi(f(M_\varphi(a,b;1-\lambda))) + (1-\lambda)\psi(f(b)) + \lambda\psi(f(a))\Big]$$
$$- \frac{c}{6}\Delta^2(3\lambda^2-3\lambda+1).$$

If f is strongly $M_\varphi M_\psi$-concave, then the reversed (3.1) *holds.*

(ii) *If ψ is strictly decreasing and f is strongly $M_\varphi M_\psi$-convex, then the reversed* (3.1) *holds. If ψ is strictly decreasing and f is strongly $M_\varphi M_\psi$-concave, then* (3.1) *is valid.*

PROOF. Let us prove the case when ψ is increasing. Other cases are done in the similar manner. Denote $F := \psi \circ f$. Since f is strongly $M_\varphi M_\psi$-convex on I, then $F \circ \varphi^{-1} - ce_2$ is convex on $\mathrm{Im}(\varphi)$ and applying Theorem 3.1 on function $\tilde{f} := F \circ \varphi^{-1} - ce_2$, we get

$$\tilde{f}\left(\frac{\varphi(a)+\varphi(b)}{2}\right) = F(M_\varphi(a,b;\frac{1}{2})) - \frac{c}{4}(\varphi(a)+\varphi(b))^2,$$

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} \tilde{f}(x)\,dx = \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx - \frac{c}{3}(\varphi^2(a)+\varphi(a)\varphi(b)+\varphi^2(b)),$$

$$\frac{\tilde{f}(\varphi(a))+\tilde{f}(\varphi(a))}{2} = \frac{F(a)+F(b)}{2} - \frac{c}{2}(\varphi^2(a)+\varphi^2(b)),$$

$$\begin{aligned}\delta_1 &= (1-\lambda)(F\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)\\ &\quad + \lambda(F\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)\\ &\quad - \frac{c}{4}\Big((1+\lambda-\lambda^2)(\varphi^2(a)+\varphi^2(b)) + 2(1-\lambda+\lambda^2)\varphi(a)\varphi(b)\Big),\end{aligned}$$

$$\begin{aligned}\delta_2 &= \frac{1}{2}\Big[(F\circ\varphi^{-1})((1-\lambda)\varphi(a)+\lambda\varphi(b)) + (1-\lambda)F(b) + \lambda F(a)\Big]\\ &\quad - \frac{c}{2}\Big((1-\lambda+\lambda^2)(\varphi^2(a)+\varphi^2(b)) + 2(\lambda-\lambda^2)\varphi(a)\varphi(b)\Big).\end{aligned}$$

Since

$$(F\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right) = (\psi\circ f)\left(M_\varphi\Big(a,b;\frac{1-\lambda}{2}\Big)\right),$$

$$(F\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right) = (\psi\circ f)\left(M_\varphi\Big(a,b;\frac{2-\lambda}{2}\Big)\right),$$

$$(F\circ\varphi^{-1})\left((1-\lambda)\varphi(a)+\lambda\varphi(b)\right) = (\psi\circ f)(M_\varphi(a,b;1-\lambda)),$$

and adding to all terms $\frac{c}{3}(\varphi^2(a)+\varphi(a)\varphi(b)+\varphi^2(b))$, we get (3.1). □

COROLLARY 3.3. *Let the assumptions of Theorem 3.2 hold.*

(i) *If ψ is strictly increasing, then for a strongly $M_\varphi M_\psi$-convex function f the following holds:*

$$\begin{aligned}
&\psi\Big(f\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\Big)+\frac{c}{12}\Delta^2\\
&\le \frac{1}{2}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{4}\big)\Big)+\frac{1}{2}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{3}{4}\big)\Big)+\frac{c}{48}\Delta^2\\
&\le \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx\\
&\le \frac{1}{2}\Big\{(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)+\frac{\psi(f(a))+\psi(f(b))}{2}\Big\}-\frac{c}{24}\Delta^2\\
&\le \frac{\psi(f(a))+\psi(f(b))}{2}-\frac{c}{6}\Delta^2. \qquad (3.2)
\end{aligned}$$

If f is strongly $M_\varphi M_\psi$-concave, then (3.2) *is reversed.*

(ii) *If ψ is strictly decreasing and f is strongly $M_\varphi M_\psi$-convex, then* (3.2) *is reversed. If f is strongly $M_\varphi M_\psi$-h-concave, then* (3.2) *is valid.*

PROOF. Putting in inequality (3.1) $\lambda=\frac{1}{2}$, we obtain (3.2). □

REMARK 3.4. Some particular results related to inequality (3.2) are already known. For example, if $\psi(x)=\varphi(x)=x$, then it is published in [2].

Inequality (3.2) for strongly GA-convex functions is given in [14] and for strongly p-convex in [15], while for strongly HA-convex, the corresponding inequality is published in [11].

The following corollary contains a refinement of the Hermite–Hadamard inequality in nodes $\sqrt[4]{a^3b}$ and $\sqrt[4]{ab^3}$ for a strongly multiplicatively convex (GG-convex) function. As we know, there are a lot of examples of multiplicatively convex functions, namely all real analytic functions $f(z)=\sum a_n z^n$ with $a_n\ge 0$ are multiplicatively convex functions.

COROLLARY 3.5. *Let f be a strongly GG-convex function on $[a,b]\subset(0,\infty)$ with modulus c such that $\frac{1}{x}\log f(x)$ is integrable on $[a,b]$. Then*

$$\begin{aligned}
&\exp\left(\frac{c}{12}\log^2\frac{b}{a}\right)f(\sqrt{ab})\le \exp\left(\frac{c}{48}\log^2\frac{b}{a}\right)\sqrt{f(\sqrt[4]{a^3b})f(\sqrt[4]{ab^3})}\\
&\le \exp\left(\frac{1}{\log\frac{b}{a}}\int_a^b\log f(x)\frac{dx}{x}\right)\le \frac{\sqrt{f(\sqrt{ab})\sqrt{f(a)f(b)}}}{\exp\left(\frac{c}{24}\log^2\frac{b}{a}\right)}\le\frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{c}{6}\log^2\frac{b}{a}\right)}.
\end{aligned}$$

PROOF. Putting in (3.2): $\psi=\varphi=\log$, we get the above statement. □

From Corollary 3.3 we get that if ψ is strictly increasing, then the difference between the middle term and the left-hand side of Hermite–Hadamard inequality for strongly $M_\varphi M_\psi$-convex function is stronger than the difference between the right-hand side of the Hermite–Hadamard inequality and its middle term. Precisely, we have the following statement.

Corollary 3.6. *Let the assumptions of Theorem* 3.2 *hold.*

(i) *If ψ is strictly increasing, then for a strongly $M_\varphi M_\psi$-convex function $f\colon I \to \mathbb{R}$ the following holds:*

$$\frac{1}{\Delta}\int_a^b \psi(f(x))\,\varphi'(x)\,dx - \psi\Big(f\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\Big) - \frac{c}{12}\Delta^2$$

$$\leq \frac{\psi(f(a))+\psi(f(b))}{2} - \frac{c}{6}\Delta^2 - \frac{1}{\Delta}\int_a^b \psi(f(x))\,\varphi'(x)\,dx. \tag{3.3}$$

If f is strongly $M_\varphi M_\psi$-concave, then (3.3) *is reversed.*

(ii) *If ψ is strictly decreasing and f is strongly $M_\varphi M_\psi$-convex, then* (3.3) *is reversed. If f is strongly $M_\varphi M_\psi$-h-concave, then* (3.3) *is valid.*

Proof. Let us suppose that ψ is strictly increasing and f is a strongly $M_\varphi M_\psi$-convex function. Multiplying by 2 the third and the fourth lines in (3.2), we get

$$2\frac{1}{\Delta}\int_a^b \psi(f(x))\,\varphi'(x)\,dx \leq \psi\Big(f\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\Big) + \frac{\psi(f(a))+\psi(f(b))}{2} - \frac{c}{12}\Delta^2$$

and this is equivalent to (3.3). Other cases are proven similarly. □

Remark 3.7. Corollary 3.6 when f is strongly convex is discussed in [2], while a similar result for other oftenly used strongly $M_\varphi M_\psi$-convex functions we do not find in the literature.

4. The Hermite–Hadamard type inequalities for the product of two functions

The following theorem contains estimates for the integral mean of the product of two strongly $M_\varphi M_\psi$-convex functions that do not necessarily have the same modulus of convexity.

THEOREM 4.1. *Let φ and ψ be strictly monotone functions defined on intervals $I=[a,b]$ and K such that φ is differentiable. Let $f,g\colon [a,b]\to[0,\infty)$.*

If f is strongly $M_\varphi M_\psi$-convex with modulus c_f, g is strongly $M_\varphi M_\psi$-convex with modulus c_g, and ψ is strictly increasing such that $(\psi\circ f)(\psi\circ f)\varphi'$ is integrable on $[a,b]$, then the following hold:

(i)

$$\begin{aligned}&\frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\psi(g(x))\varphi'(x)\,dx\\ &\le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)+\frac{1}{30}c_f c_g\Delta^4\\ &-\frac{\Delta^2}{12}\Big\{c_g[\psi(f(a))+\psi(f(b))]+c_f[\psi(g(a))+\psi(g(b))]\Big\},\end{aligned}\tag{4.1}$$

(ii)

$$\begin{aligned}&2\psi\bigg(f\bigg(M_\varphi(a,b;\frac{1}{2})\bigg)\bigg)\cdot\psi\bigg(g\bigg(M_\varphi(a,b;\frac{1}{2})\bigg)\bigg)-\frac{1}{6}M(a,b)-\frac{1}{3}N(a,b)\\ &+\frac{1}{12}\Delta^2\left\{c_g[\psi(f(a))+\psi(f(b))]+c_f[\psi(g(a))+\psi(g(b))]\right\}\\ &+\frac{1}{6}\Delta^2\left\{c_g\psi\left(f\left(M_\varphi(a,b;\frac{1}{2})\right)\right)+c_f\psi\left(g\left(M_\varphi(a,b;\frac{1}{2})\right)\right)\right\}\\ &-\frac{1}{120}c_f c_g\Delta^4\le\frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\psi(g(x))\varphi'(x)\,dx,\end{aligned}\tag{4.2}$$

where

$$M(a,b):=\psi(f(a))\psi(g(a))+\psi(f(b))\psi(g(b)),$$

$$N(a,b):=\psi(f(a))\psi(g(b))+\psi(f(b))\psi(g(a)),$$

$$\Delta:=\varphi(b)-\varphi(a).$$

If ψ is strictly decreasing, then the opposite inequality sign holds in (4.1) *and* (4.2).

PROOF. (i) Let us consider the case when ψ is strictly increasing. Since f and g are strongly $M_\varphi M_\psi$-convex with modulus c_f and c_g respectively, we get

$$F(M_\varphi(a,b;t))\le tF(a)+(1-t)F(b)-c_f t(1-t)\Delta^2,$$

$$G(M_\varphi(a,b;t))\le tG(a)+(1-t)G(b)-c_g t(1-t)\Delta^2,$$

where $F=\psi\circ f$ and $G=\psi\circ g$.

Multiplying these two inequalities and integrating the product over $[0,1]$, we get

$$\int_0^1 F(M_\varphi(a,b;t))G(M_\varphi(a,b;t))dt \leq F(a)G(a)\int_0^1 t^2 dt$$
$$+F(a)G(b)\int_0^1 t(1-t)dt + F(b)G(a)\int_0^1 t(1-t)dt$$
$$+F(b)G(b)\int_0^1 (1-t)^2 dt - c_g\Delta^2 F(a)\int_0^1 t^2(1-t)dt$$
$$-c_g\Delta^2 F(b)\int_0^1 t(1-t)^2 dt - c_f\Delta^2 G(a)\int_0^1 t^2(1-t)dt$$
$$-c_f\Delta^2 G(b)\int_0^1 t^2(1-t)dt + c_f c_g\Delta^4\int_0^1 t^2(1-t)^2 dt.$$

Since

$$\int_0^1 F(M_\varphi(a,b;t))G(M_\varphi(a,b;t))dt = \frac{1}{\Delta}\int_a^b F(x)G(x)\varphi'(x)\,dx,$$

after simple calculation we get inequality (4.1).

In the proof of (4.2), we begin with equality

$$\frac{\varphi(a)+\varphi(b)}{2} = \frac{1}{2}(t\varphi(a)+(1-t)\varphi(b)) + \frac{1}{2}((1-t)\varphi(a)+t\varphi(b)).$$

Since φ is strictly monotone and continuous, there exist $u,v\in[a,b]$ such that

$$\varphi(u) = t\varphi(a)+(1-t)\varphi(b), \quad \varphi(v) = (1-t)\varphi(a)+t\varphi(b).$$

Obviously, $M_\varphi(a,b;\frac{1}{2}) = M_\varphi(u,v;\frac{1}{2})$. Since F and G are strongly $M_\varphi A$-convex on $[u,v]$, we get

$$F(M_\varphi(a,b;\frac{1}{2})) \leq \frac{F(u)+F(v)}{2} - \frac{1}{4}c_f(1-2t)^2\Delta^2, \tag{4.3}$$

$$G(M_\varphi(a,b;\frac{1}{2})) \leq \frac{G(u)+G(v)}{2} - \frac{1}{4}c_g(1-2t)^2\Delta^2. \tag{4.4}$$

Multiplying inequalities (4.3) and (4.4), using strong convexity for terms in the expression $F(u)G(v)+F(v)G(u)$, and using (4.3) and (4.4), we get the following:

$$\begin{aligned}
F(M_\varphi(a,b;\frac{1}{2}))G(M_\varphi(a,b;\frac{1}{2})) &= F(M_\varphi(u,v;\frac{1}{2}))G(M_\varphi(u,v;\frac{1}{2}))\\
&\le \left(\frac{1}{2}F(u)+\frac{1}{2}F(v)-\frac{1}{4}c_f(\varphi(u)-\varphi(v))^2\right)\times\\
&\quad\times\left(\frac{1}{2}G(u)+\frac{1}{2}G(v)-\frac{1}{4}c_g(\varphi(u)-\varphi(v))^2\right)\\
&= \frac{1}{4}\Big(F(u)G(u)+F(v)G(v)\Big)+\frac{1}{4}\Big(F(u)G(v)+F(v)G(u)\Big)\\
&\quad-\frac{1}{8}c_g(1-2t)^2\Delta^2\Big(F(u)+F(v)\Big)\\
&\quad-\frac{1}{8}c_f(1-2t)^2\Delta^2\Big(G(u)+G(v)\Big)+\frac{1}{16}\Delta^4c_fc_g(1-2t)^4\\
&\le \frac{1}{4}\Big(F(u)G(u)+F(v)G(v)\Big)\\
&\quad+\frac{1}{4}\Big((tF(a)+(1-t)F(b)-c_ft(1-t)\Delta^2)\times\\
&\qquad\times(tG(a)+(1-t)G(b)-c_gt(1-t)\Delta^2)\\
&\quad+((1-t)F(a)+tF(b)-c_ft(1-t)\Delta^2)\times\\
&\qquad\times((1-t)G(a)+tG(b)-c_gt(1-t)\Delta^2)\Big)\\
&\quad-\frac{1}{4}c_g(1-2t)^2\Delta^2\Big(F(M_\varphi(a,b;\frac{1}{2}))+\frac{1}{4}c_f(1-2t)^2\Delta^2\Big)\\
&\quad-\frac{1}{4}c_f(1-2t)^2\Delta^2\Big(G(M_\varphi(a,b;\frac{1}{2}))+\frac{1}{4}c_g(1-2t)^2\Delta^2\Big)\\
&\quad+\frac{1}{16}\Delta^4c_fc_g(1-2t)^4.
\end{aligned}$$

Integrating over $[0,1]$ with respect to t and using facts that

$$\int_0^1 F(u)G(u)dt=\int_0^1 F(v)G(v)dt=\frac{1}{\Delta}\int_a^b F(x)G(x)\varphi'(x)dx,$$

we get inequality (4.2).

The case when ψ is strictly decreasing is done in the similar way and the proof is complete. □

Remark 4.2. Some particular cases of the first part of Theorem 4.1 are already known. So far, functions with the same module have always been considered. If $\psi(x) = x$ and $\varphi(x) = x^p$, then a result which corresponds to (4.1) for two strongly $M_\varphi A$-convex functions with same modulus $c_f = c_g$ is given in [15], while if φ is monotone function, then we get result from [17]. Result for strongly HA-convex function which is a special case of (4.1) is given in [11].

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