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AN ALTERNATIVE EQUATION FOR GENERALIZED POLYNOMIALS OF DEGREE TWO

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Abstract. In this paper we consider a generalized polynomial $f\colon \mathbb{R} \to \mathbb{R}$ of degree two that satisfies the additional equation $f(x)f(y) = 0$ for the pairs $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is given by some algebraic condition. In the particular cases when there exists a positive rational m fulfilling

$$D = \{\, (x, y) \in \mathbb{R}^2 \,|\, x^2 - my^2 = 1 \,\},$$

we prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

1. Introduction

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the set of all real numbers, rationals, and positive integers, respectively.

We call a function $f\colon \mathbb{R} \to \mathbb{R}$ *additive* if

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. The function f is called $\mathbb{Q}$-*homogeneous* if the equation $f(qx) = qf(x)$ is fulfilled by every $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. As it is well-known

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(see M. Kuczma [6, Theorem 5.2.1]), if $f\colon \mathbb{R} \to \mathbb{R}$ is additive, then f is $\mathbb{Q}$-homogeneous as well. For more information concerning these notions the reader is referred to the monograph [6].

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *quadratic* if it satisfies the Jordan-von Neumann functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x,y \in \mathbb{R}$. In what follows we will apply the fact that f is quadratic if and only if there exists a bi-additive and symmetric functional $B\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $f(x)=B(x,x)$ for every $x \in \mathbb{R}$ (see e.g. J. Aczél, J. Dhombres [1, Chapter 11, Proposition 1]). Quadratic functions are also called generalized monomials of degree 2. Further, additive functions are generalized monomials of degree 1 and real constants are generalized monomials of degree 0. Generalized polynomials are defined as sums of generalized monomials of respective degrees. For more facts on generalized polynomials the reader is referred to [6, Chapter 15.9] and L. Székelyhidi [8]. In particular, we call a function $f\colon \mathbb{R} \to \mathbb{R}$ *generalized polynomial of degree two* if we can write it with the following decomposition $f(x) = g(x) + a(x) + b$ for every $x \in \mathbb{R}$, where $g\colon \mathbb{R} \to \mathbb{R}$ is a quadratic function, $a\colon \mathbb{R} \to \mathbb{R}$ is additive and $b \in \mathbb{R}$. Here we do not exclude the particular cases when $g(x) = 0$ or $a(x) = 0$ identically, or $b = 0$. Therefore, we consider constant functions and additive mappings (as well as their sums) as particular generalized polynomials of degree two.

For any positive rational m we define the following sets:

$$S_0 = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\},$$

$$S_{1,m} = \{(x,y) \in \mathbb{R}^2 \mid x^2 - my^2 = 1\},$$

$$S_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Z. Kominek, L. Reich and J. Schwaiger [5] investigated additive real functions that satisfy the additional equation

$$f(x)f(y) = 0 \tag{1}$$

for every $(x,y) \in D$, considering various subsets D of $\mathbb{R}^2$. In several cases they obtained $f(x) = 0$ for every $x \in \mathbb{R}$. Their result for $D = S_2$ was extended by Z. Boros and W. Fechner [2] to the situation when f is a generalized polynomial. On the other hand, P. Kutas [7] has recently established the existence of a non-zero additive function $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling (1) for all $(x,y) \in S_0$. The case of bounded $f(x)f(y)$ on S_2 was investigated by these authors [3] for particular generalized polynomials of degree two. In a recent paper [4] the present authors obtained analogous results, including one for $D = S_{1,m}$, assuming that f is a generalized monomial. However, it remained an open problem whether one can extend the latter result to the more general case when f is a generalized polynomial of an arbitrary degree. The purpose of the present paper is to prove such a theorem when f is a generalized polynomial of degree two.

2. Main results

Now we can establish our main theorems.

Theorem 2.1. *Let m denote a positive rational. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial of degree two and $f(x)f(y) = 0$ for all solutions of the equation $x^2 - my^2 = 1$. Then f is identically equal to zero.*

Proof. Given a generalized polynomial f of degree two, we can associate a quadratic function $g\colon \mathbb{R} \to \mathbb{R}$, an additive function $a\colon \mathbb{R} \to \mathbb{R}$ and a constant $b \in \mathbb{R}$ with f such that

$$f(x) = g(x) + a(x) + b \tag{2}$$

for every $x \in \mathbb{R}$. Moreover, there exists a symmetric and bi-additive functional $G\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g(x) = G(x, x)$ for every $x \in \mathbb{R}$.

Now, let $x \in \mathbb{R}$ be such that $x \geq 1$. Then there exists $0 \leq y \in \mathbb{R}$ such that $x^2 - my^2 = 1$. If α, β are rational numbers such that $\alpha^2 - m\beta^2 = 1$, it is easy to verify the equality

$$(\alpha x + \beta m y)^2 - m(\beta x + \alpha y)^2 = 1\,,$$

hence our assumptions on f imply

$$f(\alpha x + \beta m y) f(\beta x + \alpha y) = 0\,. \tag{3}$$

Considering the decomposition (2) of f we can calculate that

$$\begin{aligned} f(\alpha x + \beta m y) &= g(\alpha x + \beta m y) + a(\alpha x + \beta m y) + b \\ &= \alpha^2 g(x) + (\beta m)^2 g(y) + 2\alpha\beta m G(x, y) + \alpha a(x) + \beta m a(y) + b \end{aligned}$$

and

$$\begin{aligned} f(\beta x + \alpha y) &= g(\beta x + \alpha y) + a(\beta x + \alpha y) + b \\ &= \beta^2 g(x) + \alpha^2 g(y) + 2\alpha\beta G(x, y) + \beta a(x) + \alpha a(y) + b\,. \end{aligned}$$

Due to equation (3), for every pair of rationals (α, β) fulfilling $\alpha^2 - m\beta^2 = 1$, at least one of the foregoing expressions is equal to zero.

What is more, we can find infinitely many distinct pairs (α_j, β_j) such that $\alpha_j^2 - m\beta_j^2 = 1$ and both α_j and β_j are rationals. Namely, let

$$\alpha_j = \frac{mj^2+1}{mj^2-1} \quad \text{and} \quad \beta_j = \frac{2j}{mj^2-1}$$

for $j \in \mathbb{N}$ such that $mj^2 \neq 1$.

Thus, for every $j \in N_m \doteq \mathbb{N} \setminus \{1/\sqrt{m}\}$, we have either

$$0 = \left(\frac{mj^2+1}{mj^2-1}\right)^2 g(x) + \left(\frac{2jm}{mj^2-1}\right)^2 g(y) + \frac{4jm(mj^2+1)}{(mj^2-1)^2} G(x,y)$$
$$+ \frac{mj^2+1}{mj^2-1} a(x) + \frac{2jm}{mj^2-1} a(y) + b$$

or

$$0 = \left(\frac{2j}{mj^2-1}\right)^2 g(x) + \left(\frac{mj^2+1}{mj^2-1}\right)^2 g(y) + \frac{4j(mj^2+1)}{(mj^2-1)^2} G(x,y)$$
$$+ \frac{mj^2+1}{mj^2-1} a(y) + \frac{2j}{mj^2-1} a(x) + b.$$

Multiplying both equations by $(mj^2-1)^2$ and introducing the functions

$$P(j) = j^4 m^2 (g(x) + a(x) + b) + j^3 m^2 (4G(x,y) + 2a(y))$$
$$+ j^2 (2mg(x) + 4m^2 g(y) - 2mb)$$
$$+ j(4mG(x,y) - 2ma(y)) + g(x) - a(x) + b$$

and

$$\tilde{P}(j) = j^4 m^2 (g(y) + a(y) + b) + j^3 m (4G(x,y) + 2a(x))$$
$$+ j^2 (4g(x) + 2mg(y) - 2mb)$$
$$+ j(4G(x,y) - 2a(x)) + g(y) - a(y) + b,$$

we have $P(j) = 0$ or $\tilde{P}(j) = 0$ for each integer $j \in N_m$. Hence either P or $\tilde{P}$ has infinitely many zeros. On the other hand, both P and $\tilde{P}$ are polynomials of degree not greater than 4. Therefore, one of them has to be identically equal to 0. So either each coefficient of P equals zero, or each coefficient of $\tilde{P}$ equals zero. In the first case, considering the coefficient of j^4, we obtain

$$m^2(g(x) + a(x) + b) = 0,$$

which obviously implies

$$f(x) = g(x) + a(x) + b = 0.$$

Now let us consider the second case, when we obtain the following system of equations:

$$g(y) + a(y) + b = 0, \tag{4}$$

$$4G(x, y) + 2a(x) = 0, \tag{5}$$

$$4g(x) + 2mg(y) - 2mb = 0, \tag{6}$$

$$4G(x, y) - 2a(x) = 0, \tag{7}$$

$$g(y) - a(y) + b = 0. \tag{8}$$

By summing (4) and (8) we get $g(y) = -b$. Then substituting the value of $g(y)$ in (6) we get $g(x) = mb$. Finally, by summing (5) and (7) we get $G(x, y) = 0$ and $a(x) = 0$.

We have thus proved, for an arbitrary real number $x \geq 1$, that either $f(x) = 0$ or we have $a(x) = 0$ and $g(x) = mb$.

Suppose that $f(x_0) \neq 0$ for some $x_0 \geq 1$. Then we must have $a(x_0) = 0$, $g(x_0) = mb$ and

$$0 \neq f(x_0) = g(x_0) + a(x_0) + b = mb + 0 + b = (m + 1)b,$$

which implies $b \neq 0$. Then we also have $2x_0 \geq 2$ (and thus $2x_0 \geq 1$),

$$g(2x_0) = 4g(x_0) = 4mb \neq mb$$

and

$$\begin{aligned} f(2x_0) &= g(2x_0) + a(2x_0) + b = 4g(x_0) + 2a(x_0) + b \\ &= 4mb + 0 + b = (4m + 1)b \neq 0 \end{aligned}$$

since $b \neq 0$ and $m \neq -\frac{1}{4}$ (as m is positive). So our conclusions for $2x_0$ do not satisfy our previous results for all $x \geq 1$. This contradiction shows that such an element x_0 does not exist.

We have thus proved $f(x) = 0$ for every real number $x \geq 1$. Applying [4, Lemma 2.1], we obtain that $f(x) = 0$ for all $x \in \mathbb{R}$. $\square$

Corollary 2.2. *Let a and b denote positive real numbers such that $\frac{a^2}{b^2}$ is rational. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial of degree two and $f(x)f(y) = 0$ for all solutions of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then f is identically equal to zero.*

Proof. Let u and w be real numbers fulfilling the condition $u^2 - \frac{a^2}{b^2}w^2 = 1$. Moreover, let $f_a(t) = f(at)$ for all $t \in \mathbb{R}$. Clearly, then f_a is a generalized polynomial of degree two as well. For $x = au$ and $y = aw$ we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = u^2 - \frac{a^2}{b^2}w^2 = 1,$$

hence our assumption yields

$$f_a(u)f_a(w) = f(au)f(aw) = f(x)f(y) = 0.$$

Therefore f_a satisfies the assumptions in Theorem 2.1 with $m = \frac{a^2}{b^2}$, hence f_a is identically equal to zero, which yields $f(x) = f_a(x/a) = 0$ for every $x \in \mathbb{R}$ as well. $\square$

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