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GRADIENT INEQUALITIES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. Then

$$0 \leq -m\mathcal{D}'(w,\mu)(\delta) \leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \leq -M\mathcal{D}'(w,\mu)(\alpha),$$

where $\mathcal{D}'(w,\mu)(t)$ is the derivative of $\mathcal{D}(w,\mu)(t)$ as a function of $t > 0$.

If $f\colon [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$ with $f(0) = 0$, then

$$\begin{aligned} 0 &\leq \frac{m}{\delta^2}\left[f(\delta) - f'(\delta)\delta\right] \leq f(A)A^{-1} - f(B)B^{-1} \\ &\leq \frac{M}{\alpha^2}\left[f(\alpha) - f'(\alpha)\alpha\right]. \end{aligned}$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

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1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be *positive* (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions ([7], [6]), see for instance [1, p. 144-145]:

Theorem 1. *A function $f\colon [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda),$$

where $b \geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.1}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)\, (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions ([1, p. 147]):

Theorem 2. *A function $f\colon [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0)\, t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda),$$

where $c \geq 0$ and a positive measure μ on $[0,\infty)$ such that (1.1) *holds.*

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda.$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$. By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\quad t>0, \tag{1.2}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.2) exists for all $t>0$. For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\lambda,\quad t>0.$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\quad t>0. \tag{1.3}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\quad t>0. \tag{1.4}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda,$$

for $T > 0$.

From (1.3) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(T)$$

where $T > 0$ and from (1.4)

$$(T - 1)^{-1} \ln T = \mathcal{D}(w_{\ln})(T)$$

provided $T > 0$ and $T - 1$ is invertible.

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. In this paper we show among others that

$$0 \leq -m\mathcal{D}'(w, \mu)(\delta) \leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \leq -M\mathcal{D}'(w, \mu)(\alpha),$$

where $\mathcal{D}'(w, \mu)(t)$ is the derivative of $\mathcal{D}(w, \mu)(t)$ as a function of $t > 0$. Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

Let f be an operator convex function on I. For $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}\colon [0,1] \to \mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t) := f((1-t)A + tB).$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}\colon [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \left\langle \varphi_{(A,B)}(t)\, x, x \right\rangle = \langle f((1-t)A + tB)\, x, x\rangle.$$

We have the following basic fact ([2]):

Lemma 1. *Let f be an operator convex function on I. For any A, $B \in \mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $A, B \in \mathcal{SA}_I(H)$ and $x \in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on* $[0,1]$.

A continuous function $g\colon \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s \to 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H). \tag{2.1}$$

If the limit (2.1) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilizing the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A, B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that A, $B \in [A, B]$ and $[A, B] \subset \mathcal{SA}_I(H)$.

We also have ([2]):

LEMMA 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A).$$

In particular,

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A)$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A).$$

and, see [2],

LEMMA 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$*

$$\nabla f_{(1-t_1)A+t_1B}(B-A) \leq \nabla f_{(1-t_2)A+t_2B}(B-A)$$

in the operator order.

In particular,

$$\nabla f_A(B-A) \leq \nabla f_{(1-t_1)A+t_1B}(B-A)$$

and

$$\nabla f_{(1-t_2)A+t_2B}(B-A) \leq \nabla f_B(B-A).$$

Also, we have

$$\nabla f_A(B-A) \leq \nabla f_{(1-t)A+tB}(B-A) \leq \nabla f_B(B-A) \tag{2.2}$$

for all $t \in (0,1)$.

We have the following gradient inequalities:

LEMMA 4. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\nabla f_B(B-A) \geq f(B) - f(A) \geq \nabla f_A(B-A). \tag{2.3}$$

Proof. By the properties of Bochner integral, we have

$$f(B)-f(A)=\varphi_{(A,B)}(1)-\varphi_{(A,B)}(0)=\int_0^1\varphi'_{(A,B)}(t)\,dt$$
$$=\int_0^1\nabla f_{(1-t)A+tB}(B-A)\,dt.$$

From (2.2) we have, by integration, that

$$\nabla f_A(B-A)\le\int_0^1\nabla f_{(1-t)A+tB}(B-A)\,dt\le\nabla f_B(B-A),$$

and the inequality (2.3) is proved. □

Let T, $S>0$. The function $f(t)=t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=-T^{-1}ST^{-1}\tag{2.4}$$

for T, $S>0$.

Using (2.4) for the operator convex function $f(t)=t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1}\ge D^{-1}-C^{-1}\ge -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1}\le C^{-1}-D^{-1}\le C^{-1}(D-C)C^{-1}\tag{2.5}$$

for all C, $D>0$. If

$$m\le D-C\le M$$

for some constants m,M, then

$$mD^{-2}\le D^{-1}(D-C)D^{-1}$$

and

$$C^{-1}(D-C)C^{-1}\le MC^{-2}$$

and by (2.5) we derive

$$mD^{-2}\le C^{-1}-D^{-1}\le MC^{-2}.$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2}.$$

We have the following lower and upper bounds for $\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)$ which is a nonnegative operator in the general case when $B - A \geq 0$.

THEOREM 3. *Assume that* $A \geq \alpha > 0$, $\delta \geq B > 0$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, δ, m, M. *Then*

$$0 \leq -m\mathcal{D}'(w,\mu)(\delta) \leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \leq -M\mathcal{D}'(w,\mu)(\alpha), \tag{2.6}$$

where $\mathcal{D}'(w,\mu)(t)$ *is the derivative of* $\mathcal{D}(w,\mu)(t)$ *as a function of* $t > 0$.

PROOF. We have

$$\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) = \int_0^\infty w(\lambda)\left[(\lambda + A)^{-1} - (\lambda + B)^{-1}\right] d\mu(\lambda).$$

From (2.5) we get for $C = \lambda + A$ and $D = \lambda + B$ that

$$\begin{aligned}(\lambda + B)^{-1}(B - A)(\lambda + B)^{-1} &\leq (\lambda + A)^{-1} - (\lambda + B)^{-1} \\ &\leq (\lambda + A)^{-1}(B - A)(\lambda + A)^{-1}\end{aligned} \tag{2.7}$$

for all $\lambda \geq 0$.

If we multiply (2.7) by $w(\lambda) \geq 0$ and integrate over $d\mu(\lambda)$ we get

$$\begin{aligned}&\int_0^\infty w(\lambda)(\lambda + B)^{-1}(B - A)(\lambda + B)^{-1} d\mu(\lambda) \\ &\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\ &\leq \int_0^\infty w(\lambda)(\lambda + A)^{-1}(B - A)(\lambda + A)^{-1} d\mu(\lambda).\end{aligned} \tag{2.8}$$

Since $m \leq B - A \leq M$ hence

$$m(\lambda + B)^{-2} \leq (\lambda + B)^{-1}(B - A)(\lambda + B)^{-1},$$

which implies, by integration, that

$$m\int_0^\infty w(\lambda)(\lambda+B)^{-2}\,d\mu(\lambda) \le \int_0^\infty w(\lambda)(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1}\,d\mu(\lambda). \tag{2.9}$$

Also

$$(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \le M(\lambda+A)^{-2},$$

which implies, by integration, that

$$\int_0^\infty w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}\,d\mu(\lambda) \le M\int_0^\infty w(\lambda)(\lambda+A)^{-2}\,d\mu(\lambda). \tag{2.10}$$

Since $B \le \delta$, then $\lambda+B \le \lambda+\delta$ for all $\lambda \ge 0$ which implies that $(\lambda+B)^{-1} \ge (\lambda+\delta)^{-1}$ and therefore $(\lambda+B)^{-2} \ge (\lambda+\delta)^{-2}$. Consequently

$$m\int_0^\infty w(\lambda)(\lambda+B)^{-2}\,d\mu(\lambda) \ge m\int_0^\infty w(\lambda)(\lambda+\delta)^{-2}\,d\mu(\lambda). \tag{2.11}$$

Also, since $A \ge \alpha > 0$, then $\lambda+A \ge \lambda+\alpha > 0$, which implies that $(\lambda+A)^{-1} \le (\lambda+\alpha)^{-1}$, therefore $(\lambda+A)^{-2} \le (\lambda+\alpha)^{-2}$ and

$$M\int_0^\infty w(\lambda)(\lambda+A)^{-2}\,d\mu(\lambda) \le M\int_0^\infty w(\lambda)(\lambda+\alpha)^{-2}\,d\mu(\lambda). \tag{2.12}$$

From (2.8)–(2.12) we get

$$m\int_0^\infty w(\lambda)(\lambda+\delta)^{-2}\,d\mu(\lambda) \le \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \le M\int_0^\infty w(\lambda)(\lambda+\alpha)^{-2}\,d\mu(\lambda). \tag{2.13}$$

For $h \ne 0$ small,

$$\frac{\mathcal{D}(w,\mu)(t+h) - \mathcal{D}(w,\mu)(t)}{h} = \frac{1}{h}\int_0^\infty \left(\frac{w(\lambda)}{t+h+\lambda} - \frac{w(\lambda)}{t+\lambda}\right) d\mu(\lambda) = -\int_0^\infty \frac{w(\lambda)}{(t+h+\lambda)(t+\lambda)}\,d\mu(\lambda).$$

By taking the limit over $h \to 0$ and using the properties of limits and integrals, we get the derivative of $\mathcal{D}(w,\mu)$ as

$$\mathcal{D}'(w,\mu)(t) = -\int_0^\infty \frac{w(\lambda)}{(t+\lambda)^2} d\mu(\lambda) \le 0, \quad t > 0. \tag{2.14}$$

From (2.13) and (2.14) we derive (2.6). □

We know that for $T > 0$, we have the operator inequalities

$$0 < \left\|T^{-1}\right\|^{-1} \le T \le \|T\|. \tag{2.15}$$

Indeed, it is well known that, if $P \ge 0$, then

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$. Therefore, if $T > 0$, then

$$\begin{aligned} 0 &\le \langle x, x\rangle^2 = \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2 \\ &\le \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \end{aligned}$$

for all $x \in H$. If $x \in H$, $\|x\| = 1$, then

$$1 \le \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \le \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} 1_H \le T.$$

The second inequality in (2.15) is obvious.

Corollary 1. *If A, $B > 0$ and $B - A > 0$, then*

$$\begin{aligned} 0 &\le -\left\|(B-A)^{-1}\right\|^{-1} \mathcal{D}'(w,\mu)(\|B\|) \le \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\ &\le -\|B-A\| \mathcal{D}'(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right). \end{aligned} \tag{2.16}$$

PROOF. Since $A \geq \left\|A^{-1}\right\|^{-1} = \alpha > 0$, $\delta = \|B\| \geq B > 0$ and

$$0 < m = \left\|(B-A)^{-1}\right\|^{-1} \leq B - A \leq \|B-A\| = M,$$

then by (2.6) we get (2.16). □

The case of operator monotone functions is as follows:

COROLLARY 2. *Assume that* $A \geq \alpha > 0$, $\delta \geq B > 0$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, δ, m, M. *If* $f\colon [0,\infty) \to \mathbb{R}$ *is operator monotone on* $[0,\infty)$, *then*

$$\begin{aligned} 0 &\leq \frac{m}{\delta^2}\left[f(\delta) - f(0) - f'(\delta)\delta\right] \\ &\leq f(A)A^{-1} - f(B)B^{-1} - f(0)\left(A^{-1} - B^{-1}\right) \\ &\leq \frac{M}{\alpha^2}\left[f(\alpha) - f(0) - f'(\alpha)\alpha\right]. \end{aligned} \tag{2.17}$$

If $f(0) = 0$, *then*

$$\begin{aligned} 0 &\leq \frac{m}{\delta^2}\left[f(\delta) - f'(\delta)\delta\right] \leq f(A)A^{-1} - f(B)B^{-1} \\ &\leq \frac{M}{\alpha^2}\left[f(\alpha) - f'(\alpha)\alpha\right]. \end{aligned} \tag{2.18}$$

PROOF. We have that

$$\frac{f(t) - f(0)}{t} - b = \int_0^\infty \frac{\lambda}{\lambda + t} d\mu(\lambda) = \mathcal{D}(\ell,\mu)(t), \quad t > 0$$

with $\ell(\lambda) = \lambda$, for some positive measure $\mu(\lambda)$ and nonnegative b. From this,

$$\mathcal{D}'(\ell,\mu)(t) = \frac{f'(t)t - f(t) + f(0)}{t^2}, \quad t > 0.$$

Then by (2.6) we get

$$\begin{aligned} 0 &\leq \frac{m}{\delta^2}\left[f(\delta) - f(0) - f'(\delta)\delta\right] \\ &\leq \left[f(A) - f(0)\right]A^{-1} - \left[f(B) - f(0)\right]B^{-1} \leq \frac{M}{\alpha^2}\left[f(\alpha) - f(0) - f'(\alpha)\alpha\right], \end{aligned}$$

which is equivalent to (2.17). □

Remark 1. If we write the inequality (2.18) for the operator monotone function $f(t) = t^r$, $r \in (0, 1]$, then we get the power inequalities

$$0 < (1-r)\,\delta^{r-2} m \le A^{r-1} - B^{r-1} \le (1-r)\,\alpha^{r-2} M,$$

provided that A, B satisfy the assumptions in Corollary 2.

We also have the logarithmic inequalities

$$\begin{aligned}0 &\le \frac{m}{\delta^2}\left[\ln(\delta+1) - (\delta+1)^{-1}\delta\right] \le A^{-1}\ln(A+1) - B^{-1}\ln(B+1)\\ &\le \frac{M}{\alpha^2}\left[\ln(\alpha+1) - (\alpha+1)^{-1}\alpha\right].\end{aligned}$$

We also have:

Corollary 3. *Let A, $B > 0$ and $B - A > 0$. If $f\colon [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$, then*

$$\begin{aligned}0 &\le \frac{1}{\|B\|^2 \left\|(B-A)^{-1}\right\|}\left[f(\|B\|) - f(0) - f'(\|B\|)\,\|B\|\right]\\ &\le f(A)\,A^{-1} - f(B)\,B^{-1} - f(0)\left(A^{-1} - B^{-1}\right)\\ &\le \|B-A\|\,\|A^{-1}\|^2\left[f\left(\|A^{-1}\|^{-1}\right) - f(0) - \frac{f'\left(\|A^{-1}\|^{-1}\right)}{\|A^{-1}\|}\right].\end{aligned}$$

If $f(0) = 0$, then

$$\begin{aligned}0 &\le \frac{1}{\|B\|^2 \left\|(B-A)^{-1}\right\|}\left[f(\|B\|) - f'(\|B\|)\,\|B\|\right]\\ &\le f(A)\,A^{-1} - f(B)\,B^{-1}\\ &\le \|B-A\|\,\|A^{-1}\|^2\left[f\left(\|A^{-1}\|^{-1}\right) - \frac{f'\left(\|A^{-1}\|^{-1}\right)}{\|A^{-1}\|}\right].\end{aligned}\tag{2.19}$$

If we take $f(t) = t^r$, $r \in (0, 1]$ in (2.19), then we get the power inequalities

$$0 < \frac{(1-r)\,\|B\|^{r-2}}{\left\|(B-A)^{-1}\right\|} \le A^{r-1} - B^{r-1} \le (1-r)\,\|B-A\|\left\|A^{-1}\right\|^{2-r},$$

for A, $B > 0$ and $B - A > 0$.

We also have the logarithmic inequalities

$$0 \leq \frac{1}{\|B\|^2 \left\|(B-A)^{-1}\right\|} \left[\ln\left(\|B\|+1\right) - \left(\|B\|+1\right)^{-1}\|B\|\right]$$
$$\leq A^{-1}\ln(A+1) - B^{-1}\ln(B+1)$$
$$\leq \|B-A\| \left\|A^{-1}\right\|^2 \left[\ln\left(\left\|A^{-1}\right\|^{-1}+1\right) - \left(\left\|A^{-1}\right\|^{-1}+1\right)^{-1}\left\|A^{-1}\right\|^{-1}\right].$$

The case of operator convex functions is as follows:

COROLLARY 4. *Assume that A, B are as in Corollary 2. If $f\colon[0,\infty)\to\mathbb{R}$ is operator convex on $[0,\infty)$, then*

$$0 \leq \frac{2m}{\delta^2}\left(\frac{f(\delta)-f(0)}{\delta} - \frac{f'(\delta)+f'_+(0)}{2}\right) \tag{2.20}$$
$$\leq f(A)A^{-2} - f(B)B^{-2} - f(0)\left(A^{-2}-B^{-2}\right)$$
$$- f'_+(0)\left(A^{-1}-B^{-1}\right)$$
$$\leq \frac{2M}{\alpha^2}\left(\frac{f(\alpha)-f(0)}{\alpha} - \frac{f'(\alpha)+f'_+(0)}{2}\right).$$

If $f(0)=0$, then

$$0 \leq \frac{2m}{\delta^2}\left(\frac{f(\delta)}{\delta} - \frac{f'(\delta)+f'_+(0)}{2}\right) \tag{2.21}$$
$$\leq f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1}-B^{-1}\right)$$
$$\leq \frac{2M}{\alpha^2}\left(\frac{f(\alpha)}{\alpha} - \frac{f'(\alpha)+f'_+(0)}{2}\right).$$

PROOF. We have that

$$\frac{f(t)-f(0)-f'_+(0)\,t}{t^2} - c = \int_0^\infty \frac{\lambda}{\lambda+t}d\mu(\lambda) = \mathcal{D}(\ell,\mu)(t), \quad t\geq 0$$

with $\ell(\lambda)=\lambda$ for some positive measure $\mu(\lambda)$ and nonnegative c.

We have that

$$\mathcal{D}'(\ell,\mu)(t) = \frac{\left(f'(t) - f'_+(0)\right)t^2 - 2t\left(f(t) - f(0) - f'_+(0)\,t\right)}{t^4}$$
$$= \frac{2}{t^2}\left(\frac{f'(t) + f'_+(0)}{2} - \frac{f(t) - f(0)}{t}\right).$$

Since

$$\mathcal{D}(\ell,\mu)(A) - \mathcal{D}(\ell,\mu)(B)$$
$$= \left[f(A) - f(0) - f'_+(0)\,A\right]A^{-2} - \left[f(B) - f(0) - f'_+(0)\,B\right]B^{-2}$$
$$= f(A)\,A^{-2} - f(B)\,B^{-2} - f(0)\left(A^{-2} - B^{-2}\right) - f'_+(0)\left(A^{-1} - B^{-1}\right),$$

$$-m\mathcal{D}'(\ell,\mu)(\delta) = \frac{2m}{\delta^2}\left(\frac{f(\delta) - f(0)}{\delta} - \frac{f'(\delta) + f'_+(0)}{2}\right)$$

and

$$-M\mathcal{D}'(\ell,\mu)(\alpha) = \frac{2M}{\alpha^2}\left(\frac{f(\alpha) - f(0)}{\alpha} - \frac{f'(\alpha) + f'_+(0)}{2}\right),$$

hence by (2.6) we derive (2.20). □

Corollary 5. *Let A, $B > 0$ and $B - A > 0$. If $f\colon[0,\infty) \to \mathbb{R}$ is operator convex on $[0,\infty)$, then*

$$0 \le \frac{2}{\|B\|^2\left\|(B-A)^{-1}\right\|}\left(\frac{f(\|B\|) - f(0)}{\|B\|} - \frac{f'(\|B\|) + f'_+(0)}{2}\right)$$
$$\le f(A)\,A^{-2} - f(B)\,B^{-2} - f(0)\left(A^{-2} - B^{-2}\right) - f'_+(0)\left(A^{-1} - B^{-1}\right)$$
$$\le 2\,\|B - A\|\,\left\|A^{-1}\right\|^2$$
$$\times\left(\left\|A^{-1}\right\|\left[f\left(\left\|A^{-1}\right\|^{-1}\right) - f(0)\right] - \frac{f'\left(\left\|A^{-1}\right\|^{-1}\right) + f'_+(0)}{2}\right).$$

If $f(0)=0$, *then*

$$0\le\frac{2}{\|B\|^{2}\left\|(B-A)^{-1}\right\|}\left(\frac{f(\|B\|)}{\|B\|}-\frac{f'(\|B\|)+f'_{+}(0)}{2}\right) \tag{2.22}$$
$$\le f(A)A^{-2}-f(B)B^{-2}-f'_{+}(0)\left(A^{-1}-B^{-1}\right)$$
$$\le 2\|B-A\|\left\|A^{-1}\right\|^{2}$$
$$\times\left(\left\|A^{-1}\right\|f\left(\left\|A^{-1}\right\|^{-1}\right)-\frac{f'\left(\left\|A^{-1}\right\|^{-1}\right)+f'_{+}(0)}{2}\right).$$

Remark 2. Consider the operator convex function $f(t)=-\ln(t+1)$, $t\ge0$. Assume that $A\ge\alpha>0$, $\delta\ge B>0$ and $0<m\le B-A\le M$ for some constants α, δ, m, M. Then by (2.21) we derive

$$0\le\frac{2m}{\delta^{2}}\left(\frac{\delta+2}{2(\delta+1)}-\frac{\ln(\delta+1)}{\delta}\right)$$
$$\le B^{-2}\ln(B+1)-A^{-2}\ln(A+1)+A^{-1}-B^{-1}$$
$$\le\frac{2M}{\alpha^{2}}\left(\frac{\alpha+2}{2(\alpha+1)}-\frac{\ln(\alpha+1)}{\alpha}\right).$$

If A, $B>0$ and $B-A>0$, then by (2.22)

$$0\le\frac{2}{\|B\|^{2}\left\|(B-A)^{-1}\right\|}\left(\frac{\|B\|+2}{2(\|B\|+1)}-\frac{\ln(\|B\|+1)}{\|B\|}\right)$$
$$\le B^{-2}\ln(B+1)-A^{-2}\ln(A+1)+A^{-1}-B^{-1}$$
$$\le 2\|B-A\|\left\|A^{-1}\right\|^{2}\left(\frac{1+2\left\|A^{-1}\right\|}{2(\|A^{-1}\|+1)}-\left\|A^{-1}\right\|\ln\left(\left\|A^{-1}\right\|^{-1}+1\right)\right).$$

3. More Examples

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then

$$\mathcal{D}(e_{-a})(t) := \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at), \quad t \geq 0,$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du, \quad t \geq 0.$$

For $a = 1$ we have

$$\mathcal{D}(e_{-1})(t) := \int_0^\infty \frac{\exp(-\lambda)}{t+\lambda} d\lambda = E_1(t)\exp(t), \quad t \geq 0.$$

Since $E_1'(t) = -\frac{e^{-t}}{t}$, $t > 0$, then

$$\mathcal{D}'(e_{-a})(t) = E_1'(at)\exp(at) + E_1(at)(\exp(at))' = aE_1(at)\exp(at) - \frac{1}{t}.$$

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. Then by (2.6) we get

$$\begin{aligned} 0 &\leq m\left[\frac{1}{\delta} - aE_1(a\delta)\exp(a\delta)\right] \\ &\leq E_1(aA)\exp(aA) - E_1(aB)\exp(aB) \\ &\leq M\left[\frac{1}{\alpha} - aE_1(a\alpha)\exp(a\alpha)\right], \end{aligned}$$

for $a > 1$, and in particular

$$\begin{aligned} 0 &\leq m\left[\frac{1}{\delta} - E_1(\delta)\exp(\delta)\right] \\ &\leq E_1(A)\exp(A) - E_1(B)\exp(B) \\ &\leq M\left[\frac{1}{\alpha} - E_1(\alpha)\exp(\alpha)\right]. \end{aligned}$$

If A, $B > 0$ and $B - A > 0$, then by (2.16),

$$
\begin{aligned}
0 &\le \left\| (B-A)^{-1} \right\|^{-1} \left[\|B\|^{-1} - aE_1 (a\|B\|) \exp (a\|B\|) \right] \\
&\le E_1 (aA) \exp (aA) - E_1 (aB) \exp (aB) \\
&\le \|B - A\| \left[\left\|A^{-1}\right\| - aE_1 \left(a \left\|A^{-1}\right\|^{-1} \right) \exp \left(a \left\|A^{-1}\right\|^{-1} \right) \right],
\end{aligned}
$$

for $a > 1$, and in particular

$$
\begin{aligned}
0 &\le \left\| (B-A)^{-1} \right\|^{-1} \left[\|B\|^{-1} - E_1 (\|B\|) \exp (\|B\|) \right] \\
&\le E_1 (A) \exp (A) - E_1 (B) \exp (B) \\
&\le \|B - A\| \left[\left\|A^{-1}\right\| - E_1 \left(\left\|A^{-1}\right\|^{-1} \right) \exp \left(\left\|A^{-1}\right\|^{-1} \right) \right].
\end{aligned}
$$

More examples of such transforms are

$$
\mathcal{D}(w_{1/(\ell^2+a^2)})(t) := \int_0^\infty \frac{1}{(t+\lambda)(\lambda^2+a^2)} d\lambda = \frac{\pi t - 2a \ln(t/a)}{2a(t^2+a^2)}, \quad t \ge 0
$$

and

$$
\mathcal{D}(w_{\ell/(\ell^2+a^2)})(t) := \int_0^\infty \frac{\lambda}{(t+\lambda)(\lambda^2+a^2)} d\lambda = \frac{\pi a + 2t \ln(t/a)}{2a(t^2+a^2)}, \quad t \ge 0
$$

for $a > 0$.

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [3], [4], [5], [8] and [9].

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