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NEW PEXIDERIZATIONS OF DRYGAS' FUNCTIONAL EQUATION ON ABELIAN SEMIGROUPS

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Abstract. Let $(S,+)$ be an abelian semigroup, let $(H,+)$ be an abelian group which is uniquely 2-divisible, and let φ be an endomorphism of S. We find the solutions $f,h\colon S\to H$ of each of the functional equations

$$f(x+y)+f(x+\varphi(y))=h(x)+f(y)+f\circ\varphi(y),\quad x,y\in S,$$
$$f(x+y)+f(x+\varphi(y))=h(x)+2f(y),\quad x,y\in S,$$

in terms of additive and bi-additive maps. Moreover, as applications, we determine the solutions of some related functional equations.

1. Introduction

Throughout this paper, let $(S,+)$ be an abelian semigroup (a set equipped with an associative composition rule $(x,y)\mapsto x+y$), $\varphi\colon S\to S$ be an endomorphism of S. Let $(H,+)$ denote an abelian group which is uniquely 2-divisible, i.e., for any $h\in H$ the equation $2x=h$ has exactly one solution $x\in H$.

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This article concerns primarily the following functional equations

$$f(x+y)+f(x+\varphi(y))=h(x)+f(y)+f\circ\varphi(y),\quad x,y\in S, \tag{1.1}$$

and

$$f(x+y)+f(x+\varphi(y))=h(x)+2f(y),\quad x,y\in S, \tag{1.2}$$

where $f,h\colon S\to H$ are the unknown functions.

Equations (1.1) and (1.2) contain as special cases the Drygas functional equation

$$f(x+y)+f(x+\varphi(y))=2f(x)+f(y)+f\circ\varphi(y),\quad x,y\in S, \tag{1.3}$$

and the quadratic functional equation

$$f(x+y)+f(x+\varphi(y))=2f(x)+2f(y),\quad x,y\in S. \tag{1.4}$$

Equations (1.3) and (1.4) have been studied by a number of mathematicians. Let us mention Sabour and Kabbaj [12], Sabour [11], and Akkaoui et al. [3]. We also call attention to the paper [4] where Fadli et al. determined the solutions of (1.3) and (1.4) on semigroups under the additional condition that φ is involutive ($\varphi\circ\varphi(x)=x$ for all $x\in S$).

The monographs and the papers [1, 2, 5–7, 10, 13–16] have references and detailed discussions of the classical results on Drygas' and the quadratic equations.

Our main results here are the following:

(1) We determine the structure of all solutions $\{f,h\}$ of (1.1). It turns out that in this structure, additive maps, symmetric bi-additive maps and solutions of the homogeneous equation (2.1) play important role.
(2) We determine the structure of all solutions $\{f,h\}$ of (1.2). Here we involve only additive maps and symmetric bi-additive maps.
(3) These results enable us to find the solutions of some related equations like (1.3), (1.4),

$$f(x+y)+f(x+\varphi(y))=\gamma+f(y)+f\circ\varphi(y),\quad x,y\in S,$$

$$f(x+y)+f(x+\varphi(y))=f(x)+f\circ\varphi(x)+f(y)+f\circ\varphi(y),\quad x,y\in S, \tag{1.5}$$

and

$$f(x+y)+f(x+\varphi(y))=f\circ\phi(x)+f\circ\psi(x)+2f(y),\quad x,y\in S, \tag{1.6}$$

where $\gamma\in H$ is a constant and $\varphi,\phi,\psi\colon S\to S$ are endomorphisms.

2. Set up, notation and terminology

The following notation and terminology will be used throughout the paper unless explicitly stated otherwise.

S is an abelian semigroup, $(H,+)$ denotes an abelian group which is uniquely 2-divisible, and the map $\varphi\colon S \to S$ is an endomorphism of S. By φ^2 we mean $\varphi \circ \varphi$.

A function $a\colon S \to H$ is said to be additive if

$$a(x+y) = a(x) + a(y) \quad \text{for all } x, y \in S.$$

A function $Q\colon S \times S \to H$ is bi-additive if it is additive in each variable.

By $\mathcal{N}(S,H,\varphi)$ we mean the set of the solutions $\theta\colon S \to H$ of the homogeneous equation

$$\theta(x+y) + \theta(x+\varphi(y)) = 0, \quad x, y \in S. \tag{2.1}$$

3. Main results

In this section, we seek the solutions of the functional equations (1.1) and (1.2) in terms of additive and symmetric, bi-additive maps and solutions of the homogeneous equation

$$\theta(x+y) + \theta(x+\varphi(y)) = 0, \quad x, y \in S.$$

To form our main results (Theorems 3.4 and 3.6), we start with the following lemmas.

Lemma 3.1. *Assume that the triple $f, h, k\colon S \to H$ is a solution of*

$$f(x+y) + f(x+\varphi(y)) = h(x) + k(y), \quad x, y \in S. \tag{3.1}$$

Then $h + h\circ\varphi - 2k$ is a constant function.

Proof. Let $f, h, k\colon S \to H$ be a solution of (3.1). Replacing x by $\varphi(x)$ in (3.1), we obtain

$$f(\varphi(x)+y) + f(\varphi(x)+\varphi(y)) = h\circ\varphi(x) + k(y), \quad x, y \in S. \tag{3.2}$$

If we add (3.1) and (3.2) side by side, we find that

$$f(x+y)+f(\varphi(x)+y)+f(x+\varphi(y))+f(\varphi(x)+\varphi(y)) \\ = h(x)+h\circ\varphi(x)+2k(y), \quad x,y\in S. \tag{3.3}$$

The left hand side of (3.3) is invariant under interchange of x and y. Hence so is its right hand side, and this implies that $h+h\circ\varphi-2k$ is a constant function. □

Lemma 3.2. *Assume that the pair $f,h\colon S\to H$ satisfies* (1.1). *Then h is a solution of Drygas' functional equation* (1.3).

Proof. Let $f,h\colon S\to H$ be a solution of (1.1). Making the following substitutions $(x,y+z)$ and $(x,y+\varphi(z))$ in (1.1), we get respectively

$$f(x+y+z)+f(x+\varphi(y)+\varphi(z))=h(x)+f(y+z)+f(\varphi(y)+\varphi(z)), \tag{3.4}$$

and

$$f(x+y+\varphi(z))+f(x+\varphi(y)+\varphi^2(z)) \\ = h(x)+f(y+\varphi(z))+f(\varphi(y)+\varphi^2(z)), \tag{3.5}$$

for all $x,y,z\in S$. Adding (3.4) and (3.5) side by side, we get by using (1.1) that

$$h(x+y)+f(z)+2f\circ\varphi(z)+h(x+\varphi(y))+f\circ\varphi^2(z) \\ = 2h(x)+h(y)+h\circ\varphi(y)+f(z)+2f\circ\varphi(z)+f\circ\varphi^2(z),$$

for all $x,y,z\in S$, and hence h satisfies (1.3). □

Lemma 3.3. *Assume that the pair $f,h\colon S\to H$ satisfies* (1.2). *Then h is a solution of the quadratic functional equation* (1.4).

Proof. Let $f,h\colon S\to H$ be a solution of (1.2). Making the substitutions $(x,y+z)$ and $(x,y+\varphi(z))$ in (1.2), we get respectively

$$f(x+y+z)+f(x+\varphi(y)+\varphi(z))=h(x)+2f(y+z), \quad x,y,z\in S, \tag{3.6}$$

and

$$f(x+y+\varphi(z))+f(x+\varphi(y)+\varphi^2(z))=h(x)+2f(y+\varphi(z)), \quad x,y,z\in S. \tag{3.7}$$

Adding (3.6) and (3.7) side by side, we get by using (1.2) that

$$h(x+y)+2f(z)+h(x+\varphi(y))+2f\circ\varphi(z)=2h(x)+2h(y)+4f(z),$$

for all $x,y,z\in S$. This yields that

$$h(x+y)+h(x+\varphi(y))=2h(x)+2h(y)+2f(z)-2f\circ\varphi(z),\quad x,y,z\in S. \tag{3.8}$$

Fix $z\in S$ and let α be the constant defined by $\alpha:=f(z)-f\circ\varphi(z)$. If we add 2α to the two sides of (3.8), we get

$$(h+\alpha)(x+y)+(h+\alpha)(x+\varphi(y))=2(h+\alpha)(x)+2(h+\alpha)(y),\quad x,y\in S.$$

So, according to [12, Lemma 4.1] we infer that $h\circ\varphi=h$. Applying Lemma 3.1 with $k=2f$, we conclude that $2h-4f$ is a constant, say $2c$. Since H is uniquely 2-divisible, we deduce that

$$h(x)-2f(x)=c\quad\text{for all } x\in S. \tag{3.9}$$

From (3.9), the fact that $h\circ\varphi=h$, and that H is uniquely 2-divisible, we conclude that $f\circ\varphi=f$. Hence (3.8) yields that h is a solution of the quadratic functional equation (1.4). □

Now, we are ready to state our first main result.

Theorem 3.4. *The solutions $f,h\colon S\to H$ of* (1.1) *are the functions of the following form*

$$f(x)=Q(x,x)+A(x)+\theta(x)+\alpha\quad\textit{and}\quad h(x)=2Q(x,x)+2A(x), \tag{3.10}$$

for all $x\in S$, where $\alpha\in H$ is a constant, $A\colon S\to H$ is an additive map, $Q\colon S\times S\to H$ is a symmetric, bi-additive map such that $Q(x,\varphi(y))=-Q(x,y)$ for all $x,y\in S$, and

$$\theta\in\mathcal{N}(S,H,\varphi)\quad\textit{is such that}\quad\theta\circ\varphi=-\theta. \tag{3.11}$$

Proof. It is easy to check that any pair of functions of the form above satisfies (1.1). Conversely, assume that the pair $f,h\colon S\to H$ is a solution of (1.1). By Lemma 3.2, h satisfies (1.3), then we deduce from [11, Theorem 3.2] that

$$h(x)=2Q(x,x)+2A(x),\quad x\in S,$$

where $Q\colon S\times S\to H$ is a symmetric, bi-additive map satisfying $Q(x,\varphi(y))=-Q(x,y)$ for all $x,y\in S$, and where $A\colon S\to H$ is an additive map. By using

Lemma 3.1, with $k = f + f \circ \varphi$, we get that $h + h\circ\varphi - 2f - 2f\circ\varphi$ is a constant, say -4α. Then,

$$2f + 2f \circ \varphi = h + h \circ \varphi + 4\alpha. \tag{3.12}$$

Multiplying (1.1) by 2 and using (3.12), we get

$$2f(x+y) + 2f(x+\varphi(y)) = 2h(x) + h(y) + h\circ\varphi(y) + 4\alpha, \quad x, y \in S. \tag{3.13}$$

According to Lemma 3.2, we know that h satisfies (1.3). So, if we subtract (3.13) from (1.3), we obtain

$$(2f - h)(x+y) + (2f-h)(x+\varphi(y)) = 4\alpha, \quad x, y \in S,$$

which means that

$$(2f - h - 2\alpha)(x+y) + (2f - h - 2\alpha)(x + \varphi(y)) = 0, \quad x, y \in S.$$

Hence, there exists $2\theta \in \mathcal{N}(S, H, \varphi)$ such that

$$\begin{aligned} 2f(x) &= h(x) + 2\theta + 2\alpha \\ &= 2Q(x,x) + 2A(x) + 2\theta(x) + 2\alpha, \quad x \in S. \end{aligned}$$

Since H is uniquely 2-divisible, we obtain

$$f(x) = Q(x,x) + A(x) + \theta(x) + \alpha, \quad x \in S. \tag{3.14}$$

Furthermore, the symmetricity of Q and the fact that $Q(x, \varphi(y)) = -Q(x,y)$ imply that

$$Q(\varphi(x), \varphi(x)) = Q(x,x), \quad x \in S.$$

By using (3.12) and (3.14), we infer that

$$\begin{aligned} &4Q(x,x) + 2A(x) + 2A\circ\varphi(x) + 2\theta(x) + 2\theta\circ\varphi(x) + 4\alpha \\ &\qquad = 4Q(x,x) + 2A(x) + 2A\circ\varphi(x) + 4\alpha. \end{aligned}$$

This yields that $2\theta\circ\varphi + 2\theta = 0$ and so $\theta \circ \varphi = -\theta$. □

If $S = \{x + y,\ x, y \in S\}$, i.e., for all $z \in S$ there exist $x, y \in S$ such that $z = x + y$, then the solutions $f, h\colon S \to H$ of (1.1) read as follows.

PROPOSITION 3.5. *Let S be an abelian semigroup such that $S = \{x + y,\ x, y \in S\}$. The general solution $f, h\colon S \to H$ of* (1.1) *is given by*

$$f(x) = Q(x, x) + A(x) + \alpha \quad \textit{and} \quad h(x) = 2Q(x, x) + 2A(x), \tag{3.15}$$

for all $x \in S$, where $\alpha \in H$ is a constant, $A\colon S \to H$ is an additive map, and where $Q\colon S \times S \to H$ is a symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x, y)$ for all $x, y \in S$.

Moreover, if S and H are topological semigroups and f is continuous then so are the maps A and Q in (3.15).

PROOF. Let $f, h\colon S \to H$ be a solution of (1.1). From Theorem 3.4, the pair f, h has the form (3.10). Based on the condition (3.11), making the substitutions $(x + y, z)$, $(x + \varphi(z), y)$ and $(x, y + z)$ in (2.1), we get respectively that

$$\theta(x + y + z) + \theta(x + y + \varphi(z)) = 0,$$

$$\theta(x + \varphi(z) + y) + \theta(x + \varphi(z) + \varphi(y)) = 0,$$

and

$$\theta(x + y + z) + \theta(x + \varphi(y) + \varphi(z)) = 0,$$

for all $x, y, z \in S$. Subtracting the middle identity from the sum of the two others, we obtain

$$2\theta(x + y + z) = 0 \quad \text{for all } x, y, z \in S.$$

Applying the assumption on S twice, we obtain that $2\theta = 0$. So, the desired result follows from the fact that H is uniquely 2-divisible.

The converse statement is shown elementarily.

For the continuity statement it is enough to see, from (3.15), that $2Q(x, x) = f(2x) - 2f(x) + \alpha$ and $2A(x) = 4f(x) - f(2x) - 3\alpha$ for all $x \in S$. □

Now we characterize the solutions of Eq. (1.2), that is

$$f(x + y) + f(x + \varphi(y)) = h(x) + 2f(y), \quad x, y \in S.$$

THEOREM 3.6. *The solutions $f, h\colon S \to H$ of* (1.2) *are the functions of the form*

$$f(x) = Q(x,x) + A(x) + \alpha \quad \text{and} \quad h(x) = 2Q(x,x) + 2A(x), \tag{3.16}$$

for all $x \in S$, where $\alpha \in H$ is a constant, $A\colon S \to H$ is an additive function such that $A \circ \varphi = A$, and where $Q\colon S \times S \to H$ is a symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x,y)$ for all $x, y \in S$.

Moreover, if S and H are topological semigroups and f is continuous then so are the maps A and Q in (3.16).

PROOF. Let $f, h\colon S \to H$ be a solution of (1.2). By Lemma 3.2, h is a solution of (1.4). Then, from [12, Theorem 4.2] we infer that

$$h(x) = 2Q(x,x) + 2A(x), \quad x \in S,$$

where $A\colon S \to H$ is an additive map such that $A \circ \varphi = A$, and $Q\colon S \times S \to H$ is a symmetric bi-additive map such that $Q(x, \varphi(y)) = -Q(x,y)$ for all $x, y \in S$. Hence h has the desired form. By Lemma 3.1, we get, with $k = 2f$, that $h + h \circ \varphi - 4f$ is a constant, say -4α. Then,

$$\begin{aligned} 4f(x) &= h(x) + h \circ \varphi(x) + 4\alpha \\ &= 2Q(x,x) + 2A(x) + 2Q(\varphi(x), \varphi(x)) + 2A \circ \varphi(x) + 4\alpha \\ &= 4Q(x,x) + 4A(x) + 4\alpha, \quad x \in S. \end{aligned}$$

Since H is uniquely 2-divisible, we get

$$f(x) = Q(x,x) + A(x) + \alpha, \quad x \in S.$$

The proof of the converse implication is a simple calculation that we omit. The continuity statement can be derived in a similar way as in the proof of Proposition 3.5. □

EXAMPLE 3.7. Let $S := (\mathbb{R}^2, +)$, $H := (\mathbb{C}, +)$ and let φ be the endomorphism of S defined by $\varphi(x) = (x_2, -x_1)$ for all $x := (x_1, x_2) \in S$. It is clear that S is an abelian semigroup satisfying $S = \{x + y,\ x, y \in S\}$ and that φ is not involutive. We solve (1.1) on S. From [15, Lemma 2.14], the continuous bi-additive and symmetric maps $Q\colon S \times S \to \mathbb{C}$ are

$$Q(x,y) = \lambda x_1 y_1 + \beta x_2 y_2 + \gamma(x_1 y_2 + x_2 y_1),$$

for all $x := (x_1, x_2), y := (y_1, y_2) \in S$, where $\lambda, \beta, \gamma \in \mathbb{C}$. We compute that $Q(x, \varphi(y)) = Q((x_1, x_2), (y_2, -y_1))$, so $Q(x, \varphi(y)) = -Q(x, y)$ if and only if

$$(\lambda - \gamma)x_1y_1 + (\beta + \gamma)x_2y_2 + (\lambda + \gamma)x_1y_2 + (\gamma - \beta)x_2y_1 = 0,$$

for all $x := (x_1, x_2), y := (y_1, y_2) \in S$, and in that case

$$\lambda = \beta = \gamma = 0, \quad \text{i.e.} \quad Q \equiv 0.$$

According to [15, Corollary 2.4], the continuous additive maps $A\colon S \to \mathbb{C}$ are parameterized by $a, b \in \mathbb{C}$ as follows

$$A(x_1, x_2) = ax_1 + bx_2, \quad (x_1, x_2) \in S.$$

Hence, from Proposition 3.5, the continuous solutions $f, h\colon \mathbb{R}^2 \to \mathbb{C}$ of the functional equation (1.1), namely

$$f(x_1 + y_1, x_2 + y_2) + f(x_1 + y_2, x_2 - y_1) = h(x_1, x_2) + f(y_1, y_2) + f(y_2, -y_1),$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, are the functions of the form

$$f((x_1, x_2)) = ax_1 + bx_2 + \alpha, \quad h((x_1, x_2)) = 2ax_1 + 2bx_2, \quad x_1, x_2 \in \mathbb{R},$$

where $\alpha, a, b \in \mathbb{C}$.

Our second example shows that it is possible to have non-trivial solutions of (1.2) with a non-involutive endomorphism.

Example 3.8. Let $S := (\mathbb{R}^2, +)$, $H := (\mathbb{C}, +)$ and let φ be the endomorphism of S defined by $\varphi(x) = (-x_1, 0)$ for all $x := (x_1, x_2) \in S$. The continuous additive maps $A\colon S \to \mathbb{C}$ are

$$A(x_1, x_2) = ax_1 + bx_2, \quad (x_1, x_2) \in S,$$

where $a, b \in \mathbb{C}$. We compute that $A \circ \varphi(x_1, x_2) = A(-x_1, 0) = -ax_1$, then $A \circ \varphi = A$ if and only if $a = b = 0$, i.e. $A \equiv 0$.

The continuous bi-additive and symmetric maps $Q\colon S \times S \to \mathbb{C}$ are

$$Q(x, y) = \lambda x_1y_1 + \beta x_2y_2 + \gamma(x_1y_2 + x_2y_1),$$

for all $x := (x_1, x_2), y := (y_1, y_2) \in S$, where $\lambda, \beta, \gamma \in \mathbb{C}$. We compute that $Q(x, \varphi(y)) = Q((x_1, x_2), (-y_1, 0))$, so $Q(x, \varphi(y)) = -Q(x, y)$ if and only if

$\beta = \gamma = 0$, i.e. $Q(x,y) = \lambda x_1 y_1$ for all $x, y \in S$. Hence, from Theorem 3.6, the continuous solutions of the functional equation (1.2), namely

$$f(x_1+y_1, x_2+y_2)+f(x_1-y_1, x_2) = h(x_1, x_2)+2f(y_1, y_2), \quad x_1, x_2, y_1, y_2 \in \mathbb{R},$$

are the functions of the form

$$f(x_1, x_2) = \lambda x_1^2 + \alpha, \quad h(x_1, x_2) = 2\lambda x_1^2, \quad x_1, x_2 \in \mathbb{R},$$

where $\lambda, \alpha \in \mathbb{C}$.

4. Applications

Now we are in the position to express the solutions of some related functional equations to (1.1) and (1.2). In what follows, we need the following lemma.

Lemma 4.1 ([3]). *Let $K\colon S \to H$ be such that $K(nx) = n^2 K(x)$ for all $n = 1, 2, \ldots$ and $x \in S$, let $L\colon S \to H$ be additive, and let $C \in H$ be a constant. If*

$$K(x) + L(x) = C \quad \text{for all } x \in S,$$

then $K(x) = L(x) = C = 0$ for all $x \in S$.

We start with the following corollary, which describes the solutions of Eq. (1.5), that is

$$f(x+y) + f(x+\varphi(y)) = f(x) + f\circ\varphi(x) + f(y) + f\circ\varphi(y), \quad x, y \in S.$$

Corollary 4.2. *The general solution $f\colon S \to H$ of the functional equation (1.5) is given by*

$$f(x) = Q(x,x) + A(x) + \theta(x), \quad x \in S,$$

where $A\colon S \to H$ is an additive map such that $A \circ \varphi = A$, $Q\colon S \times S \to H$ is a symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x,y)$ for all $x, y \in S$, and $\theta \in \mathcal{N}(S, H, \varphi)$ is such that $\theta \circ \varphi = -\theta$.

Proof. It is elementary to check that the formula above of f defines a solution of (1.5). So, it remains to show the other direction. Let $f\colon S \to H$ be a solution of (1.5). By applying Theorem 3.4, with $h = f + f \circ \varphi$, we infer that there exist a constant $\alpha \in H$, a symmetric, bi-additive map $Q\colon S \times S \to H$ with $Q(x, \varphi(y)) = -Q(x, y)$ for all $x, y \in S$, an additive map $A\colon S \to H$, and a function $\theta\colon S \to H$ satisfying $\theta \circ \varphi = -\theta$ and $\theta \in \mathcal{N}(S, H, \varphi)$, such that

$$f(x) = Q(x, x) + A(x) + \theta(x) + \alpha, \quad x \in S, \tag{4.1}$$

and

$$f(x) + f \circ \varphi(x) = 2Q(x, x) + 2A(x), \quad x \in S. \tag{4.2}$$

If we use the symmetricity of Q and $Q(x, \varphi(y)) = -Q(x, y)$ for all $x, y \in S$, we get

$$f \circ \varphi(x) = Q(x, x) + A \circ \varphi(x) - \theta(x) + \alpha, \quad x \in S. \tag{4.3}$$

Hence, if we add (4.1) and (4.3) we get

$$f(x) + f \circ \varphi(x) = 2Q(x, x) + A(x) + A \circ \varphi(x) + 2\alpha, \quad x \in S. \tag{4.4}$$

If we subtract (4.2) from (4.4), we find that

$$A(x) - A \circ \varphi(x) = 2\alpha, \quad x \in S.$$

By using the fact that H is a uniquely 2-divisible and Lemma 4.1, with $K \equiv 0$, $L \equiv A - A \circ \varphi$ and $C = 2\alpha$, we deduce that $A \circ \varphi = A$ and $\alpha = 0$. These complete the proof. $\square$

Corollary 4.3 below describes the solutions of (1.3). This result was treated by Sabour in [11].

Corollary 4.3. *The solutions $f\colon S \to H$ of* (1.3) *are the functions of the form*

$$f(x) = Q(x, x) + A(x), \quad x \in S, \tag{4.5}$$

where $A\colon S \to H$ is an additive map and $Q\colon S \times S \to H$ is a symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x, y)$ for all $x, y \in S$.

Proof. Applying Theorem 3.4 with $h = 2f$, we obtain

$$\begin{cases} f(x) = Q(x,x) + A(x) + \theta(x) + \alpha, \\ 2f(x) = 2Q(x,x) + 2A(x), \quad x \in S, \end{cases} \tag{4.6}$$

where $\alpha \in H$ is a constant, $Q\colon S \times S \to H$ is a symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x,y)$ for all $x, y \in S$, $A\colon S \to H$ is an additive map, and where $\theta \in \mathcal{N}(S, H, \varphi)$ is such that $\theta \circ \varphi = -\theta$. Since H is uniquely 2-divisible, we deduce from (4.6) that

$$\theta(x) + \alpha = 0 \quad \text{for all } x \in S. \tag{4.7}$$

Replacing x by $\varphi(x)$ and subtracting (4.7) from the obtained result, we get that $2\theta \equiv 0$ and hence $\theta = \alpha = 0$. So, f has the form (4.5).

Conversely, it is elementary to show that any function of the form (4.5) satisfies (1.3). □

Corollary 4.4. *The solutions $f\colon S \to H$ of the functional equation*

$$f(x+y) + f(x + \varphi(y)) = f(y) + f \circ \varphi(y), \quad x, y \in S,$$

are the functions of the form

$$f(x) = \theta(x) + \alpha, \quad x \in S,$$

where $\theta \in \mathcal{N}(S, H, \varphi)$ is such that $\theta \circ \varphi = -\theta$, and $\alpha \in H$ is a constant.

Proof. The proof follows easily from Theorem 3.4. □

Remark 4.5. The functional equation

$$f(x+y) + f(x + \varphi(y)) = \gamma + f(y) + f \circ \varphi(y), \quad x, y \in H, \tag{4.8}$$

has no solution when $\gamma \neq 0$. Indeed, assume that the functional equation (4.8) with $\gamma \neq 0$ has a solution. Applying Theorem 3.4 with $h = \gamma$, we obtain

$$2Q(x,x) + 2A(x) = \gamma, \quad x \in S.$$

Using Lemma 4.1 with $K(x) := 2Q(x,x)$, $L(x) := 2A(x)$ and $C = \gamma$, we get that

$$2Q(x,x) = 2A(x) = \gamma = 0,$$

which contradicts our assumption on γ.

As a further result of Theorem 3.6, we obtain the following corollary about the solutions of (1.6), where ψ and ϕ are two endomorphisms of S.

Corollary 4.6. *The solutions $f\colon S \to H$ of (1.6) are the functions of the following form*

$$f(x) = Q(x,x) + A(x), \quad x \in S, \tag{4.9}$$

where $Q\colon S \times S \to H$ is a symmetric, bi-additive map such that

$$Q(x,\varphi(y)) = -Q(x,y) \quad \text{and} \quad Q(\psi(x),\psi(x)) + Q(\phi(x),\phi(x)) = 2Q(x,x),$$

for all $x, y \in S$, and where $A\colon S \to H$ is an additive map such that

$$A \circ \phi + A \circ \psi = 2A \circ \varphi = 2A.$$

Proof. Simple computations based on the properties of Q and A show that the indicated functions (4.9) are solutions of (1.6). Conversely, assume that f is a solution of (1.6), then from Theorem 3.6 there exist an additive map $A\colon S \to H$ with $A \circ \varphi = A$, a symmetric, bi-additive map $Q\colon S \times S \to H$ with $Q(x,\varphi(y)) = -Q(x,y)$ for all $x, y \in S$, and a constant $\alpha \in H$ such that

$$f(x) = Q(x,x) + A(x) + \alpha \quad \text{and} \quad f \circ \phi(x) + f \circ \psi(x) = 2Q(x,x) + 2A(x),$$

for all $x \in S$. These imply that

$$[Q(\phi(x),\phi(x)) + Q(\psi(x),\psi(x)) - 2Q(x,x)] \\ + [A \circ \phi(x) + A \circ \psi(x)) - 2A(x)] = -2\alpha,$$

for all $x \in S$. By using Lemma 3.3 and the fact that H is uniquely 2-divisible, we see that $Q(\phi(x),\phi(x)) + Q(\psi(x),\psi(x)) = 2Q(x,x)$ for all $x \in S$, $A \circ \phi + A \circ \psi = 2A$, and $\alpha = 0$. These complete the proof. □

Remark 4.7. Any solution $f\colon S \to H$ of (1.6) satisfies $f \circ \phi + f \circ \psi = 2f$.

Corollary 4.8. *The solutions $f\colon S \to H$ of the functional equation*

$$f(x+y) + f(x+\varphi(y)) = 2f(y), \quad x, y \in S,$$

are the constant functions.

Proof. The proof follows immediately from Theorem 3.6. □

Now we solve the inhomogeneous quadratic functional equation

$$f(x+y)+f(x+\varphi(y))=\Phi(x)+2f(x)+2f(y), \quad x,y\in S, \tag{4.10}$$

Corollary 4.9. *The solutions $f,\Phi\colon S\to H$ of* (4.10) *are the functions of the form*

$$f(x)=Q(x,x)+A(x)+\alpha \quad \text{and} \quad \Phi(x)=-2\alpha, \quad x\in S, \tag{4.11}$$

where $\alpha\in H$ is a constant, $A\colon S\to H$ is an additive function such that $A\circ\varphi=A$, and where $Q\colon S\times S\to H$ is a symmetric, bi-additive map such that $Q(x,\varphi(y))=-Q(x,y)$ for all $x,y\in S$.

Proof. Let $f,\Phi\colon S\to H$ be a solution of (4.10). By applying Theorem 3.6, with $h=\Phi(x)+2f(x)$, we get

$$\begin{cases} f(x)=Q(x,x)+A(x)+\alpha, \\ \Phi(x)+2f(x)=2Q(x,x)+2A(x), \quad x\in S, \end{cases}$$

where $Q\colon S\times S\to H$ is a symmetric, bi-additive map such that $Q(x,\varphi(y))=-Q(x,y)$ for all $x,y\in S$, $A\colon S\to H$ is an additive function such that $A\circ\varphi=A$, and where $\alpha\in H$ is a constant. Simple computations show that the pair $\{f,\Phi\}$ is of the form (4.11).

The other direction is easy to check. □

Assume additionally that H is a ring of characteristic different from 2. Eq. (4.10) contains as a special case the following equation

$$f(x+y)+f(x+\varphi(y))+f(\psi(x))=sf(x)+2f(x)+2f(y), \tag{4.12}$$

for all $x,y\in S$, where φ and ψ are two endomorphisms of S and $s\in H$ is a constant. Equation (4.12) results from the standard alienation procedure starting in this case from adding the quadratic equation (1.4) and the Schröder equation $f(\psi(x))=sf(x)$ (see e.g., Kuczma [8] or Kuczma, Choczewski and Ger [9]) side by side.

Corollary 4.10. *Assume additionally that H is a ring of characteristic different from 2. The solutions $f\colon S\to H$ of* (4.12) *are the functions of the form*

$$f(x)=Q(x,x)+A(x)+\alpha, \quad x\in S, \tag{4.13}$$

where $Q\colon S\times S\to H$ *is a symmetric, bi-additive function such that*

$$Q(x,\varphi(y))=-Q(x,y)\quad \text{and}\quad Q(\psi(x),\quad \psi(x))=sQ(x,x), \tag{4.14}$$

for all $x,y\in S$*,* $A\colon S\to H$ *is an additive function such that*

$$A\circ\varphi=A\quad \text{and}\quad A\circ\psi=sA, \tag{4.15}$$

and where $\alpha\in H$ *is a constant such that*

$$s\alpha+3\alpha=0. \tag{4.16}$$

Proof. It is easy to check that the functions of the form (4.13) are solutions of (4.12). To see that any solution $f\colon S\to H$ of (4.12) has the form (4.13) we apply Corollary 4.9, and we get that

$$f(x)=Q(x,x)+A(x)+\alpha\quad \text{and}\quad sf(x)-f(\psi(x))=-2\alpha, \tag{4.17}$$

for all $x\in S$. With simple computations and by using Lemma 4.1 with $K(x):=Q(\psi(x),\psi(x))-sQ(x,x)$, $L(x)=A\circ\psi(x)-sA(x)$ and $C=s\alpha+3\alpha$, we see that f has the form (4.13) with the conditions (4.14)-(4.16). □

References

[1] J. Aczél and E. Vincze, *Über eine gemeinsame Verallgemeinerung zweier Funktionalgleichungen von Jensen*, Publ. Math. Debrecen **10** (1963), 326–344.

[2] Y. Aissi, D. Zeglami, and A. Mouzoun, *A quadratic functional equation with involutive automorphisms on semigroups*, Bol. Soc. Mat. Mex. (3) **28** (2022), no. 1, Paper No. 19, 13 pp.

[3] A. Akkaoui, B. Fadli, and M. El Fatini, *The Drygas functional equation on abelian semigroups with endomorphisms*, Results Math. **76** (2021), no. 1, Paper No. 42, 13 pp.

[4] B. Fadli, D. Zeglami, and S. Kabbaj, *A variant of the quadratic functional equation on semigroups*, Proyecciones **37** (2018), no. 1, 45–55.

[5] Pl. Kannappan, *On inner product spaces. I*, Math. Japon. **45** (1997), no. 2, 289–296.

[6] Pl. Kannappan, *On quadratic functional equation*, Int. J. Math. Stat. Sci. **9** (2000), no. 1, 35–60.

[7] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monogr. Math., Springer, New York, 2009.

[8] M. Kuczma, *Functional Equations in a Single Variable*, Monografie Mat., **46**, Polish Scientific Publishers, Warsaw, 1968.

[9] M. Kuczma, B. Choczewski, and R. Ger, *Iterative Functional Equations*, Encyclopedia Math. Appl., **32**, Cambridge University Press, Cambridge, 1990.

[10] S. Kurepa, *On the quadratic functional*, Acad. Serbe Sci. Publ. Inst. Math. **13** (1959), 57–72.

[11] Kh. Sabour, *A variant of Drygas' functional equation with an endomorphism*, Asia Math. **4** (2020), no. 1, 63–68.
[12] Kh. Sabour and S. Kabbaj, *Jensen's and the quadratic functional equations with an endomorphism*, Proyecciones **36** (2017), no. 1, 187–194.
[13] P. Sinopoulos, *Functional equations on semigroups*, Aequationes Math. **59** (2000), no. 3, 255–261.
[14] H. Stetkær, *Functional equations on abelian groups with involution*, Aequationes Math. **54** (1997), no. 1–2, 144–172.
[15] H. Stetkær, *Functional Equations on Groups*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2013.
[16] D. Yang, *The quadratic functional equation on groups*, Publ. Math. Debrecen **66** (2005), no. 3–4, 327–348.

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