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# ON TRANSCENDENTAL ENTIRE SOLUTION OF FERMAT-TYPE TRINOMIAL AND BINOMIAL EQUATIONS UNDER RESTRICTED HYPER-ORDER

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**Abstract.** In this paper we are focusing on finding the transcendental entire solution of Fermat-type trinomial and binomial equations, by restricting the hyper-order to be less than one. As the hyper-order is a crucial parameter that characterizes the growth of entire functions, it will be interesting to investigate this unexplored domain, as far as practible, with certain restriction on hyper order. Our results are the improvements of previous results reported in recent papers [12], [13]. We have provided a series of examples to demonstrate and validate the effectiveness of our proposed solutions.

## 1. Introduction

Fermat's last theorem [16] states that for an integer $n \geq 3$, there does not exist any non-zero rational numbers $x$, $y$ such that $x^n + y^n = 1$. Building upon this, Gross (see, [3, 4, 5]) extended the theorem to the complex functional field by considering entire and meromorphic functions instead of rational numbers and investigated the existence and forms of solutions in this

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context. Since then, numerous authors have further explored this topic, focusing on Fermat-type differential, difference, and difference-differential equations. They have utilized the logarithmic derivative lemma and its difference analogue (see, [2, 7]) to establish the existence and form of transcendental entire and meromorphic solutions.

Before stating the main content of the paper we assume that the readers are familiar with the Nevanlinna theory [8, 9, 18] such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $N(r,\frac{1}{f})$, $S(r,f)$, etc.

For a meromorphic function $f(z)$ in $\mathbb{C}$, we know that order is defined as

$$\rho(f) = \limsup_{r\longrightarrow\infty} \frac{\log\ T(r,f)}{\log\ r},$$

hyper-order of $f$ as

$$\rho_2(f) = \limsup_{r\longrightarrow\infty} \frac{\log\ \log\ T(r,f)}{\log\ r}.$$

By $S(r,f)$ we will mean any quantity satisfying $o(T(r,f))$, $r \to \infty$, outside possibly an exceptional set of finite logarithmic measure.

In 2013 Liu-Yang [12] investigated the finite-order transcendental entire solution of the following eqaution

$$f^{(k)}(z)^2 + f(z+c)^2 = 1, \tag{1.1}$$

and established the following result.

Theorem A ([12]). *The transcendental entire solution with finite order of the differential-difference equation* (1.1) *must satisfy the following two cases:*
(i) *if $k$ is odd then $f(z) = \mp\sin(Aiz+Bi)$ and $c = \frac{k\pi i}{A}$, $A^k = \pm i$,*
(ii) *if $k$ is even then $f(z) = \pm\cos(Aiz+Bi)$ and $c = \frac{k\pi i + \frac{\pi i}{2}}{A}$, $A^k = \pm 1$,*
*where $B$ is a constant.*

In 2013, Saleeby [14] conducted the initial investigation into the entire and meromorphic solutions of Fermat-type quadratic trinomial equations. Actually, in the paper [14], Saleeby first studied Fermat-type trinomial equation of the form

$$f^2(z) + 2\alpha f(z)g(z) + g^2(z) = 1, \tag{1.2}$$

and obtained the next result.

Theorem B ([14]). *Let $\alpha^2 \neq 0, 1$, then the transcendental entire solution of* (1.2) *must be of the form*

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} + \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right), \quad g(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} - \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right),$$

*where $h(z)$ is an entire function in $\mathbb{C}^n$. The meromorphic solution of* (1.2) *must be of the form*

$$f(z) = \frac{\alpha_1 - \alpha_2\beta(z)^2}{(\alpha_1 - \alpha_2)\beta(z)}, \quad g(z) = \frac{1 - \beta(z)^2}{(\alpha_1 - \alpha_2)\beta(z)},$$

*where $\beta(z)$ is a meromorphic function in $\mathbb{C}^n$ and*

$$\alpha_1 = -\alpha + \sqrt{\alpha^2 - 1}, \quad \alpha_2 = -\alpha - \sqrt{\alpha^2 - 1}.$$

Saleeby's initial work provided insights and information regarding the existence and properties of solutions to quadratic trinomial equations. Consequently, it sparked significant interest among researchers and motivated further study into different variations of Fermat-type quadratic trinomial equations. Building upon this foundation, researchers explored various aspects and variants of Fermat-type quadratic trinomial equations. As a result, several papers have been published in the literature to expand and enhance the understanding of this field. Later in 2016, Liu-Yang [13] further studied on Fermat-type trinomial equations involving derivative and shift operator and obtained the following results.

Theorem C ([13]). *If $\alpha^2 \neq \pm 1, 0$, then the equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = 1,$$

*has no transcendental meromorphic solution.*

Theorem D ([13]). *If $\alpha^2 \neq \pm 1, 0$, then the finite order transcendental entire solution of*

$$f^2(z) + 2\alpha f(z)f(z+c) + f^2(z+c) = 1,$$

*must be of order equal to one.*

## 2. Motivation, main results and examples

In the literature on Fermat-type trinomial and binomial equations, researchers have primarily focused on finding transcendental entire solutions of the equations provided in the previous section. We refer the readers to go through the results in [10, 11, 13, 19] and the references therein to be acquainted with various forms of such solutions and to acquire knowledge about their properties.

However, it appears that most of the previous work has primarily considered solutions of finite order in the complex plane $\mathbb{C}$. The case of solutions with infinite order did not receive much attention.

In 2022 Zhang *et al.,* [19] investigated and established the exact form of finite order transcendental entire solutions of the following Fermat-type trinomial equations:

$$f(z)^2 + 2\alpha f(z)\Delta_c f(z) + \Delta_c f(z)^2 = e^{g(z)},$$

$$f(z+c)^2 + 2\alpha f(z+c)\Delta_c f(z) + \Delta_c f(z)^2 = e^{g(z)},$$

$$f'(z)^2 + 2\alpha f'(z)\Delta_c f(z) + \Delta_c f(z)^2 = e^{g(z)},$$

where $\Delta_c f(z) = f(z+c) - f(z)$ and $g(z)$ is a non-constant polynomial in $\mathbb{C}$.

Inspired by the results of [19], in this paper we aim to fill this gap by investigating the forms of transcendental entire solution of hyper-order strictly less than one for different variants of Fermat-type trinomial and binomial equations. In our equations the left hand side will be more generalized than that was considered in [19]. Actually, hyper-order is a concept that extends the notion of order for entire functions and provides a more refined measure of their growth. So by restricting the hyper-order to be strictly less than one, we are literally interested in understanding the behavior of solutions of different variants of Fermat-type trinomial and binomial equations that exhibit higher growth than typical entire functions. This investigation can provide valuable insights into the nature and properties of solutions with infinite order and definitely shed light on the previously unexplored case of solutions with hyper-order less than one to contribute a more comprehensive understanding of Fermat-type trinomial and binomial equations.

We consider the following equations:

$$(2.1)\quad \{a_1 f(z) + a_2 f(z+c)\}^2$$
$$+2\alpha\{a_1 f(z) + a_2 f(z+c)\}\{b_1 f(z) + b_2 f(z+c)\}\{b_1 f(z) + b_2 f(z+c)\}^2 = 1,$$

$$(2.2)\quad \{a_0 f^{(k)}(z) + a_2 f(z+c)\}^2$$
$$+2\alpha\{a_0 f^{(k)}(z)+a_2 f(z+c)\}\{b_0 f^{(k)}(z)+b_2 f(z+c)\}\{b_0 f^{(k)}(z)+b_2 f(z+c)\}^2 = 1,$$

where $\alpha^2 \neq 0, 1$, $a_0$, $b_0$, $a_1$, $a_2$, $b_1$, $b_2$, $c$ are non-zero constants in $\mathbb{C}$.

Henceforth, we will use the following notations

$$D_t = a_t b_2 - b_t a_2,$$

where $t = 0, 1$ and

$$A_1 = \frac{1}{2\sqrt{1+\alpha}} - \frac{i}{2\sqrt{1-\alpha}}, \quad A_2 = \frac{1}{2\sqrt{1+\alpha}} + \frac{i}{2\sqrt{1-\alpha}}.$$

THEOREM 2.1. *Let $D_1 \neq 0$, $\alpha^2 \neq 0, 1$, $c \neq 0$ be constants in $\mathbb{C}$. Also let $f(z)$ be a non-constant transcendental entire solution of (2.1) with $\rho_2(f) < 1$. Then $f(z)$ has the following form*

$$f(z) = \frac{1}{\sqrt{2}D_1}[(b_2 A_1 - a_2 A_2)e^{i(az+b)} + (b_2 A_2 - a_2 A_1)e^{-i(az+b)}],$$

*such that*

$$(a_1 A_1 - b_1 A_2)(a_1 A_2 - b_1 A_1) = (b_2 A_2 - a_2 A_1)(b_2 A_1 - a_2 A_2),$$

*and*

$$e^{iac} = \frac{a_1 A_2 - b_1 A_1}{b_2 A_1 - a_2 A_2}, \quad e^{-iac} = \frac{a_1 A_1 - b_1 A_2}{b_2 A_2 - a_2 A_1}.$$

The next example justifies Theorem 2.1.

EXAMPLE 2.1. Let $a = 1$, $b = 2$, $\alpha = \frac{1}{2}$, $a_1 = 1$, $a_2 = -2$, $b_1 = -2$, $b_2 = 1$. Then

$$f(z) = \frac{1}{6}[-(\sqrt{3}+i)e^{iz+2i} - (\sqrt{3}-i)e^{-iz-2i}],$$

is a solution of (2.1), where $c$ satisfies the condition defined in Theorem 2.1.

The next example shows that the condition $\rho_2(f) < 1$ is sharp for Theorem 2.1.

EXAMPLE 2.2. Let $\alpha = \frac{1}{2}$, $a_1 = 1$, $a_2 = -2$, $b_1 = -2$, $b_2 = 1$, $c$ be a constant such that $e^{ic} = \frac{A_2+2A_1}{A_1+2A_2}$. Here

$$f(z) = -\frac{1}{3\sqrt{2}}[(A_1 + 2A_2)e^{e^{iz}} + (A_2 + 2A_1)e^{e^{-iz}}],$$

is not a solution of Theorem 2.1 and $\rho_2(f) = 1$.

THEOREM 2.2. *Let $D_0 \neq 0$, $\alpha^2 \neq 0, 1$, $c \neq 0$ be constants in $\mathbb{C}$. Also let $f(z)$ be a non-constant transcendental entire solution of* (2.2) *with $\rho_2(f) < 1$. Then $f(z)$ has the following form*

$$f(z) = \frac{1}{\sqrt{2}D_0}\left[\frac{(b_2A_1 - a_2A_2)}{i^k a^k}e^{i(az+b)} + \frac{(b_2A_2 - a_2A_1)}{(-i)^k a^k}e^{-i(az+b)}\right],$$

*such that*

$$a^{2k}(a_0A_2 - b_0A_1)(a_0A_1 - b_0A_2) = (b_2A_1 - a_2A_2)(b_2A_2 - a_2A_1),$$

*and*

$$e^{iac} = \frac{i^k a^k(a_0A_2 - b_0A_1)}{b_2A_1 - a_2A_2}, \quad e^{-iac} = \frac{(-i)^k a^k(a_0A_1 - b_0A_2)}{b_2A_2 - a_2A_1}.$$

The next two examples show that the conclusion of Theorem 2.2 is precise.

EXAMPLE 2.3. Let $\alpha = \frac{1}{2}$, $a_0 = 1$, $a_2 = 2$, $b_0 = 2$, $b_2 = 1$, $a = 1$, $b = 0$, $k = 2$. Then

$$f(z) = \frac{1}{6\sqrt{3}}[-(1 + i3\sqrt{3})e^{iz} - (1 - i3\sqrt{3})e^{-iz}],$$

is a solution of (2.2), where $c$ satisfies the condition defined in Theorem 2.2.

EXAMPLE 2.4. Let $\alpha = \frac{1}{2}$, $a_0 = 1$, $a_2 = -2$, $b_0 = -2$, $b_2 = 1$, $a = 1$, $b = 0$, $k = 1$. Then

$$f(z) = \frac{1}{6i}[-(\sqrt{3} + i)e^{iz} + (\sqrt{3} - i)e^{-iz}],$$

is a solution of equation (2.2), where $c$ satisfies the condition defined in Theorem 2.2.

The next example shows that the condition $\rho_2(f) < 1$ is sharp for Theorem 2.2.

EXAMPLE 2.5. Let $\alpha = \frac{1}{2}$, $a_0 = 1$, $a_2 = 2$, $b_0 = 2$, $b_2 = 1$, $a = 1$, $b = 0$, $k = 2$, $c$ be a constant such that $e^{ic} = \frac{i(A_2-2A_1)}{A_1-2A_2}$. Here

$$f(z) = \frac{1}{6\sqrt{3}}[-(1+i3\sqrt{3})e^{e^{iz}} - (1-i3\sqrt{3})e^{e^{-iz}}],$$

is not a solution of (2.2) and $\rho_2(f) = 1$.

If $\alpha = 0$ in (2.1) and (2.2) simply reduced to the corresponding binomial equations

$$\{a_1 f(z) + a_2 f(z+c)\}^2 + \{b_1 f(z) + b_2 f(z+c)\}^2 = 1, \tag{2.3}$$

and

$$\{a_0 f^{(k)}(z) + a_2 f(z+c)\}^2 + \{b_0 f^{(k)}(z) + b_2 f(z+c)\}^2 = 1, \tag{2.4}$$

respectively. In the context of (2.3) and (2.4), it will be interesting to investigate the analogous results of Theorems 2.1 and 2.2. So we have the following two theorems.

THEOREM 2.3. *Let $D_1 \neq 0$, $b_2 \neq \pm i a_2$, $\alpha^2 \neq 0, 1$, $c \neq 0$ be constants in $\mathbb{C}$. Also let $f(z)$ be a non-constant transcendental entire solution of* (2.3) *with $\rho_2(f) < 1$. Then $f(z)$ has the following form*

$$f(z) = \frac{1}{2D_1}\left[(b_2 + ia_2)e^{az+b} + (b_2 - ia_2)e^{-az-b}\right],$$

*where $a \neq 0$, $b$ are two constants in $\mathbb{C}$ such that*

$$a_1^2 + b_1^2 = a_2^2 + b_2^2$$

*and*

$$e^{ac} = \frac{b_1 + ia_1}{b_2 + ia_2}, \quad e^{-ac} = \frac{b_1 - ia_1}{b_2 - ia_2}.$$

The next example justifies Theorem 2.3.

EXAMPLE 2.6. Let $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = -1$, $a = 1$, $b = 1$, $c = -\frac{\pi i}{2}$. Then

$$f(z) = \frac{1-i}{4}[e^{z+1} + e^{-z-1}],$$

is a solution of (2.3).

Theorem 2.4. *Let $D_0 \neq 0$, $b_2 \neq \pm i a_2$, $\alpha^2 \neq 0, 1$, $c \neq 0$ be constants in $\mathbb{C}$. Also let $f(z)$ be a non-constant transcendental entire solution of* (2.4) *with $\rho_2(f) < 1$. Then $f(z)$ has the following form*

$$f(z) = \frac{1}{2D_0}\left[\frac{b_2 + ia_2}{a^k}e^{az+b} + \frac{b_2 - ia_2}{(-1)^k a^k}e^{-az-b}\right]$$

*such that*

$$(-1)^k a^{2k}(a_0^2 + b_0^2) = a_2^2 + b_2^2$$

*and*

$$e^{iac} = -\frac{a^k(b_0 + ia_0)}{b_2 + ia_2}, \quad e^{-iac} = \frac{(-1)^{k+1}a^k(b_0 - ia_0)}{b_2 - ia_2},$$

*$a \neq 0$, $b$ are two constants in $\mathbb{C}$.*

Corollary 2.1.

(i) *When $k$ is odd, then we have*

$$f(z) = \frac{1}{2D_0 a^k}\left[(b_2 + ia_2)e^{az+b} - (b_2 - ia_2)e^{-az-b}\right],$$

*such that $a_2^2 + b_2^2 = -a^{2k}(a_0^2 + b_0^2)$.*

(ii) *When $k$ is even, then we have*

$$f(z) = \frac{1}{2D_0 a^k}\left[(b_2 + ia_2)e^{az+b} + (b_2 - ia_2)e^{-az-b}\right],$$

*such that $a_2^2 + b_2^2 = a^{2k}(a_0^2 + b_0^2)$.*

The next example justifies Theorem 2.4.

Example 2.7. Let $k = 2$, $a_0 = 1$, $b_0 = -1$, $a_2 = 1$, $b_2 = 1$, $a = 1$, $b = 0$, $c = -\frac{\pi i}{2}$. Then

$$f(z) = \frac{1}{4}\left[(1 + i)e^{z} + (1 - i)e^{-z}\right],$$

is a solution of the equation (2.4).

The next example justifies the sharpness of the condition of $\rho_2(f) < 1$ in Theorem 2.4.

EXAMPLE 2.8. Let $k = 1$, $a_0 = 1$, $b_0 = 1$, $a_2 = -1$, $b_2 = 1$, $a = i$, $b = 0$, $c = 2\pi i$ but

$$f(z) = \frac{1}{4i}[(1-i)e^{e^{iz}} - (1+i)e^{e^{-iz}}],$$

is not a solution of (2.4). Clearly $\rho_2(f) = 1$.

## 3. Lemmas

LEMMA 3.1 ([15]). *Consider an entire function $F$ in $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then there exists a canonical function $f_F$ and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_{F(z)}e^{g_{F(z)}}$. For special case $n = 1$, $f_F$ is the canonical product of Weierstrass. Here $\rho(n_f)$ denotes the order of the counting function of zeros of $F$.*

LEMMA 3.2 ([1]). *Let $g$ be a transcendental meromorphic function of order less than one and $h_1$ be a positive constant. Then there exists an $\epsilon$-set $E$ such that $\mathbb{C}\backslash\{E\} \ni z \to \infty$, one has*

$$\frac{g'(z+\zeta)}{g(z+\zeta)} \to 0, \qquad \frac{g(z+\zeta)}{g(z)} \to 1,$$

*uniformly in $\zeta$ for $|\zeta| \leq h_1$. Further, the $\epsilon$-set $E$ may be chosen such that for large $z$ not in $E$, the function $g$ has no zeros or poles in $|\zeta - z| \leq h_1$.*

LEMMA 3.3 ([17]). *If $f$ is a non-constant periodic meromorphic function, then $\rho(f) \geq 1$.*

LEMMA 3.4 ([17]). *If $h$ is non-constant entire function, then $\rho_2(e^h) = \rho(h)$.*

LEMMA 3.5 ([6]). *Let $a_j(z)$ be entire functions of finite order $\rho$ and $g_j(z)$ be entire functions such that $g_k(z) - g_j(z)$, $j \neq k$ are transcendental entire functions or polynomials of degree greater than $\rho$. Then*

$$\sum_{j=1}^{n} a_j(z)e^{g_j(z)} = a_0(z),$$

*holds only when $a_0(z) = a_1(z) = a_2(z) = \cdots = a_n(z) \equiv 0$.*

## 4. Proofs of the theorems

Proof of Theorem 2.1. Let $f(z)$ be a non-constant transcendental entire solution of (2.1) with $\rho_2(f) < 1$. Then using Theorem B we have

$$a_1 f(z) + a_2 f(z+c) = \frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} + \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right) = \frac{1}{\sqrt{2}}\left(A_1 e^{ih(z)} + A_2 e^{-ih(z)}\right), \tag{4.1}$$

$$b_1 f(z) + b_2 f(z+c) = \frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}} - \frac{\sin(h(z))}{\sqrt{1-\alpha}}\right) = \frac{1}{\sqrt{2}}\left(A_2 e^{ih(z)} + A_1 e^{-ih(z)}\right), \tag{4.2}$$

where $h(z)$ is a non-constant entire function. From (4.1) and (4.2) we have

$$f(z) = \frac{1}{\sqrt{2}D_1}[(b_2A_1 - a_2A_2)e^{ih(z)} + (b_2A_2 - a_2A_1)e^{-ih(z)}], \tag{4.3}$$

$$f(z+c) = -\frac{1}{\sqrt{2}D_1}[(b_1A_1 - a_1A_2)e^{ih(z)} + (b_1A_2 - a_1A_1)e^{-ih(z)}]. \tag{4.4}$$

We have from (4.3)

$$T(r, f(z)) = 2T(r, e^{ih(z)}) + S(r, f). \tag{4.5}$$

Since $\rho_2(f) < 1$, using Lemma 3.4 from (4.5) we obtain $\rho_2(f) = \rho(h) < 1$, i.e., $h(z)$ is a non-constant entire function of order less than one.

Using (4.3) and (4.4) we have

$$(b_2A_1 - a_2A_2)e^{ih(z+c)} + (b_2A_2 - a_2A_1)e^{-ih(z+c)} = (a_1A_2 - b_1A_1)e^{ih(z)} + (a_1A_1 - b_1A_2)e^{-ih(z)}. \tag{4.6}$$

Now we claim that all four terms namely $b_2A_1 - a_2A_2$, $b_2A_2 - a_2A_1$, $a_1A_2 - b_1A_1$, $a_1A_1 - b_1A_2$ in (4.6) are non-zero. Now we discuss the following cases:

Case 1: Without any loss of generality let us assume that $a_1A_1 - b_1A_2 = 0$. Since $a_1A_1 - b_1A_2 = 0$ then from (4.6) clearly we have $a_1A_2 - b_1A_1 \neq 0$, otherwise we will have $A_1^2 = A_2^2$, which contradicts our assumption $\alpha^2 \neq 0, 1$. Under the condition $a_1A_1 - b_1A_2 = 0$ we also have $b_2A_2 - a_2A_1 \neq 0$, otherwise from this two expression we have $D_1 = a_1b_2 - a_2b_1 = 0$, which contradicts our assumption.

SUBCASE 1.1: If $b_2A_1 - a_2A_2 = 0$, from (4.6) we will have

$$(b_2A_2 - a_2A_1)e^{-ih(z+c)-ih(z)} = (a_1A_2 - b_1A_1). \tag{4.7}$$

Then (4.7) shows that $-i(h(z+c)+h(z))$ must be a constant. Now we claim that $h(z)$ can not be a non-constant polynomial, if so, comparing degree of $h(z)$ we have $0$ =degree$[-i(h(z+c)+h(z))]$=degree of $2h(z) \geq 1$, a contradiction. If $h(z)$ is a non-constant transcendental entire function then differentiating we have $h'(z+c) = -h'(z)$ and using Lemma 3.2 we have $1 \equiv \frac{h'(z+c)}{h'(z)} \to -1$, which is a contradiction.

SUBCASE 1.2: Let $b_2A_1 - a_2A_2 \neq 0$, $b_2A_2 - a_2A_1 \neq 0$, $a_1A_2 - b_1A_1 \neq 0$, $a_1A_1 - b_1A_2 = 0$. Then from (4.6) we have

$$\begin{aligned}(b_2A_1 - a_2A_2)e^{ih(z+c)-ih(z)} + (b_2A_2 - a_2A_1)e^{-ih(z+c)-ih(z)} \\ = (a_1A_2 - b_1A_1).\end{aligned} \tag{4.8}$$

Now using Nevanlinna Second Fundamental theorem from (4.8) we have

$$\begin{aligned}T\left(r, e^{-ih(z+c)-ih(z)}\right) &\leq N\left(r, \frac{1}{e^{-ih(z+c)-ih(z)}}\right) + N\left(r, \frac{1}{e^{-ih(z+c)-ih(z)} - \beta}\right) \\ &\quad + N\left(r, e^{-ih(z+c)-ih(z)}\right) + S\left(r, e^{-ih(z+c)-ih(z)}\right) \\ &\leq N\left(r, \frac{1}{\frac{b_2A_1-a_2A_2}{b_2A_2-a_2A_1}e^{ih(z+c)-ih(z)}}\right) \\ &\quad + S\left(r, e^{-ih(z+c)-ih(z)}\right) = o\left(T\left(r, e^{-ih(z+c)-ih(z)}\right)\right),\end{aligned}$$

where $\beta = \frac{a_1A_2-b_1A_1}{b_2A_2-a_2A_1} \neq 0$, which shows that $-i(h(z+c)+h(z))$ is a constant.

Then from (4.8) we must have $i(h(z+c)-h(z))$ must be a non constant. Then we have

$$(b_2A_1 - a_2A_2)e^{ih(z+c)-ih(z)} + (b_2A_2 - a_2A_1)K_1 = (a_1A_2 - b_1A_1),$$

where $K_1 = e^{-ih(z+c)-ih(z)}$. Considering order of growth in both sides we get a contradiction. Hence $a_1A_1 - b_1A_2 \neq 0$.

CASE 2: Using similar arguments as done in Case 1 we can show that $b_2A_1 - a_2A_2 \neq 0$, $b_2A_2 - a_2A_1 \neq 0$, $a_1A_2 - b_1A_1 \neq 0$.

Since $h(z)$ is a non-constant entire function and $b_2A_1 - a_2A_2 \neq 0$, $b_2A_2 - a_2A_1 \neq 0$, $a_1A_2 - b_1A_1 \neq 0$, $a_1A_1 - b_1A_2 \neq 0$ then using Lemma 3.5 from (4.6) we have either $ih(z+c)+ih(z)$ or $-ih(z+c)+ih(z)$ is a constant. If $ih(z+c)+ih(z)$ is a constant then using similar arguments as done in Subcase 1.1 we clearly have a contradiction. Hence $-ih(z+c)+ih(z)$ must be a constant. Differentiating we have $h'(z+c)=h'(z)$, i.e., $h'(z)$ is a periodic entire function. We have $\rho(h)=\rho(h')$. Since after (4.5) we have already deducted $\rho(h)<1$, using Lemma 3.3 we derive that $h'$ is constant in $\mathbb{C}$, say $h'(z)=a$, $a\neq 0$, i.e., $h(z)=az+b$, $b$ is a constant.

Now using (4.3) and (4.4) we have

$$f(z)=\frac{1}{\sqrt{2}D}[(b_2A_1-a_2A_2)e^{i(az+b)}+(b_2A_2-a_2A_1)e^{-i(az+b)}],$$

$$e^{iac}=\frac{a_1A_2-b_1A_1}{b_2A_1-a_2A_2},\quad e^{-iac}=\frac{a_1A_1-b_1A_2}{b_2A_2-a_2A_1}.$$

This proves the conclusion of Theorem 2.1. □

Proof of Theorem 2.2. Let $f(z)$ be a non-constant transcendental entire solution of (2.2) with $\rho_2(f)<1$. Using Theorem B we have

$$a_0f^{(k)}(z)+a_2f(z+c)=\frac{1}{\sqrt{2}}[A_1e^{ih(z)}+A_2e^{-ih(z)}], \tag{4.9}$$

$$b_0f^{(k)}(z)+b_2f(z+c)=\frac{1}{\sqrt{2}}[A_2e^{ih(z)}+A_1e^{-ih(z)}], \tag{4.10}$$

where $h(z)$ is a non-constant entire function. Then from (4.9) and (4.10) we have

$$f^{(k)}(z)=\frac{1}{\sqrt{2}D_0}[(b_2A_1-a_2A_2)e^{ih(z)}+(b_2A_2-a_2A_1)e^{-ih(z)}], \tag{4.11}$$

$$f(z+c)=-\frac{1}{\sqrt{2}D_0}[(b_0A_1-a_0A_2)e^{ih(z)}+(b_0A_2-a_0A_1)e^{-ih(z)}]. \tag{4.12}$$

Differentiating (4.12) $k$-times we get

$$\begin{aligned} f^{(k)}(z+c)&\\ =\frac{1}{\sqrt{2}D_0}&[(a_0A_2-b_0A_1)M(z)e^{ih(z)}+(a_0A_1-b_0A_2)N(z)e^{-ih(z)}], \end{aligned} \tag{4.13}$$

where $M(z)=ih^{(k)}(z)+\cdots+i^k(h')^k$, $N(z)=-ih^{(k)}(z)+\cdots+(-i)^k(h')^k$.

Using (4.11) and (4.13) we have

$$(4.14)\quad (b_2A_1 - a_2A_2)e^{ih(z+c)} + (b_2A_2 - a_2A_1)e^{-ih(z+c)} = (a_0A_2 - b_0A_1)M(z)e^{ih(z)} + (a_0A_1 - b_0A_2)N(z)e^{-ih(z)}.$$

Now we claim that both $b_2A_1 - a_2A_2 \neq 0$ and $b_2A_2 - a_2A_1 \neq 0$.

Without any loss of generality let $b_2A_2 - a_2A_1 = 0$, then from (4.14) we have $b_2A_1 - a_2A_2 \neq 0$ and $a_0A_1 - b_0A_2 \neq 0$. Otherwise $b_2A_2 - a_2A_1 = 0$ and $b_2A_1 - a_2A_2 = 0$ implies $A_1^2 = A_2^2$, which contradicts the assumption that $\alpha^2 \neq 0, 1$; $b_2A_2 - a_2A_1 = 0$ and $a_0A_1 - b_0A_2 = 0$ contradicts our assumption $D_0 = a_0b_2 - b_0a_2 \neq 0$.

Then (4.14) reduces to

$$(4.15)\quad (b_2A_1 - a_2A_2)e^{ih(z+c)} + (b_0A_1 - a_0A_2)M(z)e^{ih(z)} = (a_0A_1 - b_0A_2)N(z)e^{-ih(z)}$$

Let us discuss the following possibilities regarding the equation (4.15):

Case i): Let $M(z) \equiv 0$ and $N(z) \equiv 0$ then from (4.15) we have

$$(b_2A_1 - a_2A_2)e^{ih(z+c)} \equiv 0,$$

since $b_2A_1 - a_2A_2 \neq 0$, $ih(z+c)$ must be a constant i.e., $h(z)$ is a constant, which is a contradiction.

Case ii): Let $M(z) \equiv 0$ and $N(z) \not\equiv 0$ then from (4.15) we have

$$(4.16)\qquad (b_2A_1 - a_2A_2)e^{i(h(z+c)+h(z))} \equiv (a_0A_1 - b_0A_2)N(z).$$

Then (4.16) implies that $i(h(z+c) + h(z))$ must be a constant. Then using similar arguments as done in Subcase 1.1 of Theorem 2.1 we must have a contradiction.

Case iii): Let $M(z) \not\equiv 0$ and $N(z) \equiv 0$ then from (4.15) we have

$$(b_2A_1 - a_2A_2)e^{ih(z+c)} + (b_0A_1 - a_0A_2)M(z)e^{ih(z)} \equiv 0.$$

If $(b_0A_1 - a_0A_2) = 0$, then we get a contradiction. If $(b_0A_1 - a_0A_2) \neq 0$, then we get $i(h(z+c) - h(z))$ is a constant. Then differentiating we have $h'(z+c) = h'(z)$. As $h'(z)$ is a periodic entire function with period $c$ and we know $\rho(h) < 1$, in view of Lemma 3.3 we have $h'(z)$ is a constant, say $a(\neq 0)$. Then we have $h(z) = az + b$, $b \in \mathbb{C}$ is a constant. Then we have $M(z) = i^k a^k$, and $N(z) = (-i)^k(a)^k$, which clearly contradicts our assumption under Case iii).

Case iv): Let $M(z)\not\equiv 0$ and $N(z)\not\equiv 0$. If $b_0A_1-a_0A_2=0$ in (4.15), then using similar arguments as done in the proof of Subcase 1.1 in Theorem 2.1 we get a contradiction. If $b_0A_1-a_0A_2\neq 0$, then using Nevanlinna Second Fundamental theorem from (4.15) we have

$$\begin{aligned}T\left(r,e^{i(h(z+c)+h(z))}\right)&\leq N\left(r,\frac{1}{e^{i(h(z+c)+h(z))}}\right)+N\left(r,\frac{1}{e^{i(h(z+c)+h(z))}-\delta}\right)\\&\quad+N\left(r,e^{i(h(z+c)+h(z))}\right)+S\left(r,e^{i(h(z+c)+h(z))}\right)\\&\leq N\left(r,\frac{1}{\frac{(b_0A_1-a_0A_2)M(z)}{b_2A_1-a_2A_2}e^{2i(h(z))}}\right)\\&\quad+S\left(r,e^{i(h(z+c)+h(z))}\right)=o\left(T\left(r,e^{i(h(z+c)+h(z))}\right)\right)\end{aligned}$$

where $\delta=\frac{(a_0A_1-b_0A_2)N(z)}{b_2A_1-a_2A_2}$, which shows that $i(h(z+c)+h(z))$ is a constant. Then form (4.15) we must have

$$(b_2A_1-a_2A_2)K_2+(b_0A_1-a_0A_2)M(z)e^{2ih(z)}=(a_0A_1-b_0A_2)N(z),$$

where $K_2=e^{i(h(z+c)+h(z))}$. Considering order of growth in both sides we clearly get a contradiction. Hence we get $b_2A_2-a_2A_1\neq 0$.

Similarly we can show that $b_2A_1-a_2A_2\neq 0$.

Now we discuss the following cases regarding the equation (4.14) when both $b_2A_2-a_2A_1\neq 0$ and $b_2A_1-a_2A_2\neq 0$:

Case 1: $(a_0A_2-b_0A_1)M(z)\equiv 0$, $(a_0A_1-b_0A_2)N(z)\equiv 0$. Then from (4.14) we get a contradiction about the fact that $h(z)$ is a non-constant entire function.

Case 2: $(a_0A_2-b_0A_1)M(z)\equiv 0$, $(a_0A_1-b_0A_2)N(z)\not\equiv 0$. Then from (4.14) we have

$$\begin{aligned}(4.17)\quad (b_2A_1-a_2A_2)e^{ih(z+c)}&+(b_2A_2-a_2A_1)e^{-ih(z+c)}\\&=(a_0A_1-b_0A_2)N(z)e^{-ih(z)}.\end{aligned}$$

Then using Lemma 3.5 we get either $i(h(z+c)+h(z))$ or, $i(-h(z+c)+h(z))$ is a constant. Now if $i(h(z+c)+h(z))$ is a constant, then using similar arguments as done in Subcase 1.1 of Theorem 2.1 we get a contradiction. If $i(-h(z+c)+h(z))$ is a constant, but $i(h(z+c)+h(z))$ is not a constant, then we have from (4.17)

$$(b_2A_1-a_2A_2)e^{i(h(z+c)+h(z))}+(b_2A_2-a_2A_1)K_3=(a_0A_1-b_0A_2)N(z),$$

where $K_3 = e^{i(h(z)-h(z+c))}$. Now comparing the order of growth in both sides we clearly get a contradiction.

Case 3: $(a_0A_2 - b_0A_1)M(z) \not\equiv 0$, $(a_0A_1 - b_0A_2)N(z) \equiv 0$. Then using similar arguments as done in Case 2 we get a contradiction.

Case 4: $(a_0A_2 - b_0A_1)M(z) \not\equiv 0$, $(a_0A_1 - b_0A_2)N(z) \not\equiv 0$. Then from (4.14) we have either $i(h(z) + h(z + c))$ or $i(h(z) - h(z + c))$ is a constant. If $i(h(z) + h(z + c))$ is a constant, then we get a contradiction. Then we must have $i(h(z) - h(z + c))$ is a constant. Then using similar analysis as done in Case iii) we have $h(z) = az + b$, $a \neq 0$, $b \in \mathbb{C}$ are constants. Then from (4.12) we have

$$f^{(k)}(z) = \frac{1}{\sqrt{2}D_0}\left[(b_2A_1 - a_2A_2)e^{i(az+b)} + (b_2A_2 - a_2A_1)e^{-i(az+b)}\right]. \tag{4.18}$$

Integrating $k$-times from (4.18) we get

$$f(z) = \frac{1}{\sqrt{2}D_0}\left[\frac{(b_2A_1 - a_2A_2)}{i^k a^k}e^{i(az+b)} + \frac{(b_2A_2 - a_2A_1)}{(-i)^k a^k}e^{-i(az+b)}\right] + S(z), \tag{4.19}$$

where $S(z)$ is a polynomial in $\mathbb{C}$ of degree $k - 1$. Since $S(z)$ is an arbitrary polynomial, using equation (2.2) we get $S(z) \equiv 0$.

From (4.19) and (4.12) we have

$$e^{iac} = \frac{i^k a^k (a_0A_2 - b_0A_1)}{b_2A_1 - a_2A_2}, \quad e^{-iac} = \frac{(-i)^k a^k (a_0A_1 - b_0A_2)}{b_2A_2 - a_2A_1}. \qquad \square$$

Proof of Theorem 2.4. Let $f(z)$ be a transcendental entire solution of (2.4) with $\rho_2(f) < 1$. Then using Lemma 3.1 we have

$$a_0 f^{(k)}(z) + a_2 f(z + c) = \frac{e^{h(z)} + e^{-h(z)}}{2}, \tag{4.20}$$

$$b_0 f^{(k)}(z) + b_2 f(z + c) = \frac{e^{h(z)} - e^{-h(z)}}{2i}, \tag{4.21}$$

where $h(z)$ is a non-constant entire function. Then from (4.20) and (4.21) we have

$$f^{(k)}(z) = \frac{(b_2 + ia_2)e^{h(z)} + (b_2 - ia_2)e^{-h(z)}}{2D_0}, \tag{4.22}$$

$$f(z + c) = \frac{(b_0 + ia_0)e^{h(z)} + (b_0 - ia_0)e^{-h(z)}}{-2D_0}. \tag{4.23}$$

Differentiating (4.23) $k$-times we get

$$(4.24)\ f^{(k)}(z+c) = \frac{(b_0+ia_0)M_1(z)e^{h(z)} + (b_0-ia_0)N_1(z)e^{-h(z)}}{-2D_0},$$

where $M_1(z) = [h^{(k)}(z) + \cdots + (h'(z))^k]$ and $N_1(z) = [-h^{(k)}(z) + \cdots + (-1)^k(h')^k]$. Using (4.23) we have

$$T(r, f(z+c)) \quad = \quad T(r, e^{2h(z)}) + S(r, f).$$

Therefore, $T(r, f(z+c)) = T(r, f(z)) + S(r, f)$ and $\rho_2(f) < 1$. Then clearly we have $\rho(h) = \rho_2(f) < 1$.

Now using (4.22) and (4.24) we have

$$(4.25)\quad (b_2+ia_2)e^{h(z+c)} + (b_2-ia_2)e^{-h(z+c)} \\ = -(b_0+ia_0)M_1(z)e^{h(z)} - (b_0-ia_0)N_1(z)e^{-h(z)}.$$

Clearly we have $b_2 + ia_2 \neq 0$ and $b_2 - ia_2 \neq 0$.

Now we discuss the following cases:

CASE 1: Let both $(b_0+ia_0)M_1(z) \equiv 0$ and $(b_0-ia_0)N_1(z) \equiv 0$. Then from (4.25) we have $h(z)$ is a constant entire function, which is a contradiction.

CASE 2: Let $(b_0 + ia_0)M_1(z) \equiv 0$ and $(b_0 - ia_0)N_1(z) \not\equiv 0$. Then from (4.25) we have

$$(b_2+ia_2)e^{h(z+c)} + (b_2-ia_2)e^{-h(z+c)} = -(b_0-ia_0)N_1(z)e^{-h(z)}.$$

Using Lemma 3.5 we have either $h(z+c)+h(z)$ or $-h(z+c)+h(z)$ is a constant. Then using same arguments as done in the proof of case 2 of Theorem 2.2 we get a contradiction.

CASE 3: Let $(b_0 + ia_0)M_1(z) \not\equiv 0$ and $(b_0 - ia_0)N_1(z) \equiv 0$. Then using same arguments as done in the proof of Case 2 of Theorem 2.2 we get a contradiction.

CASE 4: Let $(b_0+ia_0)M_1(z) \not\equiv 0$ and $(b_0-ia_0)N_1(z) \not\equiv 0$. Then from (4.25) we have either $h(z+c)+h(z)$ or $-h(z+c)+h(z)$ is a constant. If $h(z+c)+h(z)$ is a constant then using similar arguments as done in the proof of Subcase 1.1 of Theorem 2.1 we must have a contradiction. If $-h(z+c)+h(z)$ is a constant then using same arguments as done in the proof of Case iii) of Theorem 2.2 we must have $h(z) = az + b$, where $a \neq 0$ and $b$ are two constants in $\mathbb{C}$.

From (4.22), integrating $f^{(k)}(z)$ $k$-times we get

$$f(z) = \frac{1}{2D_0}\left[\frac{b_2+ia_2}{a^k}e^{az+b} + \frac{b_2-ia_2}{(-a)^k}e^{-az+b}\right] + S_1(z),$$

where $S_1(z)$ is a polynomial of degree $k-1$. Since $S_1(z)$ is arbitrary we take $S_1(z) \equiv 0$. Then from (4.22) and (4.24) we have

$$e^{iac} = \frac{-a^k(b_0 + ia_0)}{b_2 + ia_2}, \quad e^{-iac} = \frac{(-1)^{k+1}a^k(b_0 - ia_0)}{b_2 - ia_2}. \qquad \square$$

PROOF OF THEOREM 2.3. We omit the proof of Theorem 2.3 as the proof is similar to that of Theorem 2.4. $\square$

## 5. Two open questions

1. What will be the possible form of transcendental entire solution of hyper-order strictly less than one of the equations (2.1)–(2.4) when the right hand side is replaced by $e^{g(z)}$, where $g(z)$ is a non-constant polynomial in $\mathbb{C}$?
2. Is it possible to extend the theorems obtained in this paper to $\mathbb{C}^n$?

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