Annales Mathematicae Silesianae (2023),
DOI: 10.2478/amsil-2023-0020

ON DOUBLED AND QUADRUPLED FIBONACCI TYPE SEQUENCES

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Abstract. In this paper we study a family of doubled and quadrupled Fibonacci type sequences obtained by distance generalization of Fibonacci sequence. In particular we obtain doubled Fibonacci sequence, doubled and quadrupled Padovan sequence and quadrupled Narayana's sequence. We give a binomial direct formula for these sequences using graph methods, and also we derive a number of identities. Moreover, we study matrix generators of these sequences and determine connections with the Pascal's triangle.

1. Introduction

Fibonacci numbers F_n are terms of the sequence defined by the recurrence $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence is perhaps the most famous sequence, it appeared in the book Liber Abaci of Leonardo de Pisa in 1202. This sequence occurs in different fields of science, for example in some areas of algebra [24, 10, 22], graph theory [2, 3, 4, 5, 16, 27], computer algorithms [1, 11, 26] and many other areas of mathematics. The Fibonacci sequence has applications also

Received: 20.01.2023. Accepted: 07.11.2023.
(2020) Mathematics Subject Classification: 11B39, 11B37, 11C20.
Key words and phrases: Fibonacci numbers, Padovan's numbers, generalized Fibonacci numbers, Narayana's numbers, generating function, Pascal's triangle.

in other fields such as nature, art, architecture, music, finance, etc, see for example [15].

The Fibonacci sequence has many generalizations given in different directions. Some authors have generalized the Fibonacci sequence by preserving the recurrence relation and altering initial conditions of the sequence [12, 13] while others have generalized the Fibonacci sequence by preserving initial conditions and altering the recurrence equation [9, 17, 21]. Among different generalizations of the Fibonacci sequence generalizations in the distance sense play an important role.

Let $k \geq 1$, $d \geq 1$, $n \geq 0$ be integers and $d \neq k$. Distance Fibonacci numbers are defined recursively by

$$F_{d,k}(n) = F_{d,k}(n-d) + F_{d,k}(n-k) \quad \text{for } n \geq \max\{d,k\} \tag{1}$$

with initial conditions $F_{d,k}(n)$ for $n \in \{0, 1, \ldots, \max\{d,k\} - 1\}$. Numbers $F_{d,k}(n)$ are also named as distance (k,d)–Fibonacci numbers. Note that the equation (1) describes in fact a family of sequences, where each choice of d and k gives a distinct sequence.

We list only some of them which will be used in the future considerations.
$F_{1,2}(n) = F_n$ – Fibonacci numbers with $F_0 = 0, F_1 = 1$.
$F_{2,3}(n) = Pv(n)$ – Padovan's numbers with $Pv(0) = Pv(1) = Pv(2) = 1$.
$F_{1,3}(n) = N_n$ – Narayana's numbers with $N_0 = 0, N_1 = N_2 = 1$.

For an arbitrary $k \geq 3$ and fixed integer d we obtain well known generalizations which were introduced quite recently.
$F_{1,k}(n)$ – distance Fibonacci numbers (M. Kwaśnik, I. Włoch [16]).
$F_{2,k}(n) = F^{(1)}_{2,k}(n)$ – $(2,k)$-distance Fibonacci numbers (I. Włoch et al. [29]).
$F_{3,k}(n) = F_3(k,n)$ – $(3,k)$-distance Fibonacci numbers (E. Özkan et al. [20]).

For other generalizations of the Fibonacci sequence see for example [2, 3, 4, 23, 8, 30, 29, 18, 19, 25, 28, 14, 5].

Motivated by results obtained for $d \in \{1,2,3\}$ in this paper we consider the distance Fibonacci sequence $(F_{4,k}(n))$. For convenience based on notation used in [20] we will write $F_4(k,n)$ instead of $F_{4,k}(n)$.

Let $k \geq 1$, $n \geq 0$ be integers. By $(4,k)$-distance Fibonacci numbers, we mean generalized Fibonacci numbers defined recursively by the following relation

$$F_4(k,n) = F_4(k,n-4) + F_4(k,n-k) \quad \text{for } n \geq \max\{4,k\}, \tag{2}$$

with initial conditions $F_4(k,n) = 1$, for $n \in \{0, 1, 2, 3, \ldots, \max\{3, k-1\}\}$.

Although, recurrence (2) does not directly generalize Fibonacci like sequences we can observe that the family of sequences $F_4(k,n)$ for a special values of k includes double or quadruple Fibonacci type sequences. Note following observations, where we indicate sequences indexed in OEIS [23].

$F_4(1, n+1)$ – A003269 (with truncated first element).
$F_4(2, n) = F_{\lfloor \frac{n}{2} \rfloor + 1}$ – A103609 (with truncated first element).
$F_4(3, n)$ – A079398 (with truncated first element).
$F_4(4, n) = 2^{\lfloor \frac{n}{4} \rfloor}$ – A200675.
$F_4(5, n)$ – A103372.
$F_4(6, n) = Pv_{\lfloor \frac{n}{2} \rfloor}$ not indexed in OEIS.
$F_4(7, n)$ not indexed in OEIS.
$F_4(8, n) = F_{\lfloor \frac{n}{4} \rfloor + 1}$ – not indexed in OEIS.
$F_4(10, n)$ – A005686.
$F_4(12, n) = N_{\lfloor \frac{n}{4} \rfloor + 1}$ Narayana's sequence.
$F_4(16, n)$ – A003269.

For illustration nineteen initial elements of these sequences are presented in the Table 1.

Table 1. Numbers $F_4(k, n)$ for $k = \{1, 2, \ldots, 8\}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$F_4(1, n)$	1	1	1	1	2	3	4	5	7	10	14	19	26	36	50	69	95	131	181
$F_4(2, n)$	1	1	1	1	2	2	3	3	5	5	8	8	13	13	21	21	34	34	55
$F_4(3, n)$	1	1	1	1	2	2	2	3	4	4	5	7	8	9	12	15	17	21	27
$F_4(4, n)$	1	1	1	1	2	2	2	2	4	4	4	4	8	8	8	8	16	16	16
$F_4(5, n)$	1	1	1	1	1	2	2	2	2	3	4	4	4	5	7	8	8	9	12
$F_4(6, n)$	1	1	1	1	1	1	2	2	2	2	3	3	4	4	5	5	7	7	9
$F_4(7, n)$	1	1	1	1	1	1	1	2	2	2	2	3	3	3	4	5	5	5	7
$F_4(8, n)$	1	1	1	1	1	1	1	1	2	2	2	2	3	3	3	3	5	5	5

It is known that tiling defined by the Fibonacci numbers covers a plane. In [29, 20] it was shown a tiling covering of a plane defined by doubled and tripled Fibonacci sequence, respectively. We present a tiling covering of a plane by quadrupled Fibonacci sequence $F_4(8, n)$, see Figure 1.

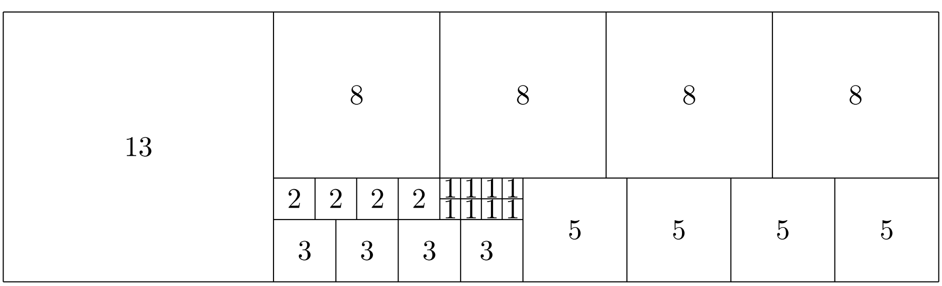

Figure 1. A tiling interpretation of the quadrupled Fibonacci sequence $F_4(8, n)$

Similar to the classical Fibonacci numbers, $(4, k)$-distance Fibonacci numbers can be extended to negative integers. Such extension is very useful for studying properties of sequences and we will use it in future considerations.

If $k \in \{1,2,3\}$, then $F_4(k,0) = F_4(k,1) = F_4(k,2) = F_4(k,3) = 1$ and

$$F_4(1,-n) = F_4(1,-n+4) - F_4(1,-n+3),$$
$$F_4(2,-n) = F_4(2,-n+4) - F_4(2,-n+2),$$
$$F_4(3,-n) = F_4(3,-n+4) - F_4(3,-n+1),$$
$$F_4(4,-n) = \frac{1}{2}F_4(4,-n+4).$$

Let $k > 4$ be integer and $F_4(k,n) = 1$ for $n \in \{0,1,\ldots,k-1\}$. Then

$$F_4(k,-n) = F_4(k,-n+k) - F_4(k,-n+(k-4)).$$

Table 2 includes the first few elements of $F_4(k,-n)$ for special k and non positive n.

Table 2. Numbers $F_4(k,n)$ for non positive n and $k = \{1,2,\ldots,8\}$

n	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
$F_4(1,n)$	6	-4	2	-3	3	-1	1	-2	1	0	1	-1	0	0	1	0	0	0	1
$F_4(2,n)$	-21	-21	13	13	-8	-8	5	5	-3	-3	2	2	-1	-1	1	1	0	0	1
$F_4(3,n)$	76	-55	40	-29	21	-15	11	-8	6	-4	3	-2	2	-1	1	0	1	0	1
$F_4(4,n)$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$F_4(5,n)$	-3	3	-1	0	1	-2	1	0	0	1	-1	0	0	0	1	0	0	0	1
$F_4(6,n)$	0	0	-2	-2	1	1	1	1	-1	-1	0	0	1	1	0	0	0	0	1
$F_4(7,n)$	-3	-3	-2	1	2	2	1	-1	-1	-1	0	1	1	1	0	0	0	0	1
$F_4(8,n)$	-3	-3	2	2	2	2	-1	-1	-1	-1	1	1	1	1	0	0	0	0	1

Linear recurrence equation with constant coefficients is typically used in conjunction with generating function which is a powerful technique for study linear homogenous recurrence relation. For the sequence $\{F_4(k,n)\}$ the generating function also can be determined.

Theorem 1. *Let $n \geq 0$, $k \geq 1$ be integers. The generating function of $\{F_4(k,n)\}$ has the following form*

$$g(x) = \frac{1+t}{1-x^4-x^k} \quad \textit{where } t = \begin{cases} 0 & \textit{for } k=1, \\ x & \textit{for } k=2, \\ x+x^2 & \textit{for } k=3, \\ x+x^2+x^3 & \textit{for } k\geq 4. \end{cases}$$

Proof. Let $g(x)=\sum_{n=0}^{\infty}F_4(k,n)x^n$. Using the recurrence (2) we have

$$g(x)-x^4g(x)-x^kg(x)=1+t,\quad \text{where } t=\begin{cases}0 & \text{for } k=1,\\ x & \text{for } k=2,\\ x+x^2 & \text{for } k=3,\\ x+x^2+x^3 & \text{for } k\geq 4.\end{cases}$$

Hence $g(x)=\frac{1+t}{1-x^4-x^k}$. □

2. Graph interpretation of $F_4(k,n)$

The recurrence (2) cannot be solved for an arbitrary k, so it is important to give a direct formula for $F_4(k,n)$ using other methods. In this section we give the direct binomial formula for $F_4(k,n)$ by graph methods. We use the standard terminology of graph theory and for concepts not defined here, see [6].

Let P_n, $n\geq 1$, be a path with the vertex set $V(P_n)=\{v_1,\dots,v_n\}$ with the vertex numbering in the natural fashion. Moreover by P_0 we put the empty graph.

Let $k\geq 1$ be an integer and $\mathcal{Y}_k=\{P_t;t\in\{4,k\}\}$ be a family of vertex disjoint subgraphs of P_n such that $V(P_n)\setminus\bigcup_{P_t\in Y_k}V(P_t)\subseteq R_k$, where $R_1=\emptyset$, $R_2=\{v_n\}$, $R_3=\{v_n,v_{n-1}\}$ and $R_k=\{v_n,v_{n-1},v_{n-2}\}$ for $k\geq 4$. We say that the family $\mathcal{Y}_k$ is a $\{P_4,P_k\}$-covering of P_n with the rest. If $V(P_n)\setminus\bigcup_{P_t\in\mathcal{Y}_k}V(P_t)=\emptyset$, then we have $\{P_4,P_k\}$-covering of P_n. For P_0 we mean that the empty set is the unique $\{P_4,P_k\}$-covering of P_0.

For example let consider a $\{P_4,P_1\}$-covering of the path P_5. We can see that there exist two coverings P_4P_1 and P_1P_4, both coverings are without the rest.

We give a graph interpretation of numbers $F_4(k,n)$ using $\{P_4,P_k\}$-covering with the rest of P_n.

Denote by $\alpha(k,n)$ the total number of $\{P_4,P_k\}$-covering with the rest of P_n.

Theorem 2. *Let $k\geq 1$, $n\geq 0$ be integers. Then $\alpha(k,n)=F_4(k,n)$.*

Proof. (By induction on n.) Let k, n be as in the statement of the theorem. Denote by $\alpha_4(k,n)$ the number of all $\{P_4,P_k\}$-covering with the rest of P_n such that $v_1\in V(P_4)$ and by $\alpha_k(k,n)$ the number of all $\{P_4,P_k\}$-covering with the rest of P_n such that $v_1\in V(P_k)$.

If $n = 0$, then the empty set is the unique $\{P_4, P_k\}$-covering with the rest of P_0 so $F_4(k, 0) = 1$. If $0 < n < \min\{4, k\}$, then $V(P_n) = R_k$, so $F_4(k, n) = 1$. If $\min\{4, k\} \le n < \max\{4, k\}$, then there is the unique $\{P_4, P_k\}$-covering with the rest of P_n realized by either P_4 or P_k, so $F_4(k, n) = 1$.

Let $n \ge \max\{4, k\}$ and suppose that $\alpha(k, n) = F_4(k, n)$ for an arbitrary n. We will show that $\alpha(k, n+1) = F_4(k, n+1)$. The $\{P_4, P_k\}$-coverings of P_{n+1} we can divide into two cases, either $v_1 \in V(P_4)$ or $v_1 \in V(P_k)$. Clearly $\alpha(k, n+1) = \alpha_4(k, n+1) + \alpha_k(k, n+1)$. Moreover $\alpha_4(k, n+1) = \alpha_4(k, n-3)$ and $\alpha_k(k, n+1) = \alpha_k(k, n+1-k)$. Then $\alpha(k, n+1) = \alpha_4(k, n-3) + \alpha_k(k, n+1-k) = F_4(k, n-3) + F_4(k, n+1-k) = F_4(k, n+1)$, which ends the proof. $\square$

Using the above graph interpretation we give the direct formula for $F_4(k, n)$.

THEOREM 3. *Let $k \ge 1$, $n \ge 0$ be integers. Then*

$$F_4(k, n+t) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{i + \lfloor \frac{n-ik}{4} \rfloor}{i} \quad \textit{where } t = \begin{cases} 3 & \textit{for } k = 1, \\ 2 & \textit{for } k = 2, \\ 1 & \textit{for } k = 3, \\ 0 & \textit{for } k \ge 4. \end{cases}$$

PROOF. If $n \le k - 1$, then $\lfloor \frac{n}{k} \rfloor = 0$ and

$$F_4(k, n) = \sum_{i=0}^{0} \binom{i + \lfloor \frac{n-ik}{4} \rfloor}{i} = \binom{0 + \lfloor \frac{n}{4} \rfloor}{0} = 1.$$

Assume that $n \ge k$. By Theorem 2, the number $F_4(k, n)$ is equal to the number of $\{P_4, P_k\}$-covering with the rest of P_n. Each $\{P_4, P_k\}$-covering with the rest of P_n consists of i monochromatic paths P_k and j monochromatic paths P_4, where $0 \le i \le \lfloor \frac{n}{k} \rfloor$, $0 \le j \le \lfloor \frac{n}{k} \rfloor$. Moreover, for a fixed i we have $j = \lfloor \frac{n-ik}{4} \rfloor$ and the number of $\{P_4, P_k\}$-covering with the rest is equal to $\binom{i+j}{i} = \binom{i + \lfloor \frac{n-ik}{4} \rfloor}{i}$. Thus $F_4(k, n) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{i + \lfloor \frac{n-ik}{4} \rfloor}{i}$. $\square$

3. Identities

In this section we give a number of identities of sums of $F_4(k, n)$. First we prove a result which will be useful in proof of the next theorem.

LEMMA 4. *Let $k \geq 1$, $n \geq 0$ be integers. Then*
(i) $F_4(4k, 4n) = F_4(4k, 4n+1) = F_4(4k, 4n+2) = F_4(4k, 4n+3)$,
(ii) $F_4(2k, 2n) = F_4(2k, 2n+1)$.

PROOF. (i) (By induction on n.) From the definition of $F_4(k,n)$ we have that $F_4(4k,0) = \cdots = F_4(4k, 4k-1) = 1$. If $n = 4k$, then by the formula (2) we have

$$F_4(4k,4k) = F_4(4k,4k+1) = F_4(4k,4k+2) = F_4(4k,4k+3) = 2.$$

Let $n \geq 4k+1$. Assume that $F_4(4k,4i) = F_4(4k,4i+1) = F_4(4k,4i+2) = F_4(4k,4i+3)$ for all $i \leq n$. We will prove the equality for $n+1$ of the form

$$\begin{aligned} F_4(4k, 4(n+1)) &= F_4(4k, 4(n+1)+1) \\ &= F_4(4k, 4(n+1)+2) = F_4(4k, 4(n+1)+3). \end{aligned} \tag{3}$$

Applying definition (2) for all numbers given in (3) we have that

$$\begin{aligned} F_4(4k, 4(n+1)) &= F_4(4k, 4n) + F_4(4k, 4n+4-4k) \\ &= F_4(4k, 4n) + F_4(4k, 4(n+1-k)), \\ F_4(4k, 4(n+1)+1) &= F_4(4k, 4n+1) + F_4(4k, 4n+5-4k) \\ &= F_4(4k, 4n+1) + F_4(4k, 4(n+1-k)+1), \\ F_4(4k, 4(n+1)+2) &= F_4(4k, 4n+2) + F_4(4k, 4n+6-4k) \\ &= F_4(4k, 4n+2) + F_4(4k, 4(n+1-k)+2), \\ F_4(4k, 4(n+1)+3) &= F_4(4k, 4n+3) + F_4(4k, 4n+7-4k) \\ &= F_4(4k, 4n+3) + F_4(4k, 4(n+1-k)+3). \end{aligned}$$

Then by the induction's hypothesis we have following equalities

$$F_4(4k,4n) = F_4(4k,4n+1) = F_4(4k,4n+2) = F_4(4k,4n+3),$$

$$\begin{aligned} F_4(4k, 4(n+1-k)) &= F_4(4k, 4(n+1-k)+1) = F_4(4k, 4(n+1-k)+2) \\ &= F_4(4k, 4(n+1-k)+3). \end{aligned}$$

Consequently the equalities (3) immediately follows.

In the same way we can prove (ii). $\square$

THEOREM 5. *Let $k \geq 1$, $n \geq 0$ be integers. Then*

(i) $\sum_{i=0}^{n} F_4(1,i) = F_4(1,n+4) - 1$,

(ii) $\sum_{i=0}^{n} F_4(2,i) = F_4(2,n+4) + F_4(2,n+3) - 2$,

(iii) $\sum_{i=0}^{n} F_4(3,i) = F_4(3,n+4) + F_4(3,n+3) + F_4(3,n+2) - 3$,

(iv) $\sum_{i=0}^{n} F_4(k,i) = F_4(k,n+k) + F_4(k,n+k-1) + F_4(k,n+k-2) + F_4(k,n+k-3) - 4$, *for* $k \geq 4$,

(v) $\sum_{i=0}^{n} F_4(1,2i) = F_4(1,2n+1) + F_4(1,2n-1)$,

(vi) $\sum_{i=0}^{n} F_4(2,2i) = F_4(2,2n+5) - 1$,

(vii) $\sum_{i=0}^{n} F_4(3,2i) = F_4(3,2n+5) + F_4(3,2n+3) - F_4(3,2n+2) - 1$,

(viii) $\sum_{i=0}^{n} F_4(k,2i) = F_4(k,2n+k+1) + F_4(k,2n+k-1) - 2$, *for even* $k \geq 4$,

(ix) $\sum_{i=0}^{n} F_4(k,2i) = F_4(k,2n+k) + F_4(k,2n+k-2) - 2$, *for odd* $k \geq 5$,

(x) $\sum_{i=0}^{n} F_4(1,2i+1) = F_4(1,2n+2) + F_4(1,2n) - 1$,

(xi) $\sum_{i=0}^{n} F_4(2,2i+1) = F_4(2,2n+4) - 1$,

(xii) $\sum_{i=0}^{n} F_4(3,2i+1) = F_4(3,2n+2) + F_4(3,2n+1) + F_4(3,2n) - 2$,

(xiii) $\sum_{i=0}^{n} F_4(4,2i+1) = F_4(4,2n+4) + F_4(4,2n+2) - 2$,

(xiv) $\sum_{i=0}^{n} F_4(k,2i+1) = F_4(k,2n+k+1) + F_4(k,2n+k-1) - 2$, *for* $k \geq 5$,

(xv) $\sum_{i=0}^{n} F_4(3,3i) = F_4(3,3n+4) - 1$,

(xvi) $\sum_{i=0}^{n} F_4(3,3i+1) = F_4(3,3n+5) - 1$,

(xvii) $\sum_{i=0}^{n} F_4(k,4i) = F_4(k,4n+k) - 1$, *for* $k \geq 4$,

(xviii) $\sum_{i=0}^{n} F_4(k,4i+1) = F_4(k,4n+k+1) - 1$, *for* $k \geq 4$.

PROOF. (i) (By telescoping method.) From the recurrence (2) we have $F_4(1,n) = F_4(1,n+4) - F_4(1,n+3)$. So $\sum_{i=0}^{n} F_4(1,i) = F_4(1,4) - F_4(1,3) + F_4(1,5) - F_4(1,4) + F_4(1,6) - F_4(1,5) + \cdots + F_4(1,n+4) - F_4(1,n+3) = F_4(1,n+4) - F_4(1,3) = F_4(1,n+4) - 1$.

(viii) (By induction on n.) Let $k \geq 4$ be even. $F_4(k,0) = 1 = 2 + 1 - 2 = F_4(k,k+1) + F_4(k,k-1) - 2$. Assume that the formula (viii) holds for n. We will prove it for $n+1$. From Lemma 4 we have $F_4(k,2n+2) = F_4(k,2n+3)$. By the induction hypothesis we have $\sum_{i=0}^{n+1} F_4(k,2i) = F_4(k,2n+k+1) + F_4(k,2n+k-1) - 2 + F_4(k,2n+2) = F_4(k,2n+k+1) + F_4(k,2n+k-1) - 2 + F_4(k,2n+3) = F_4(k,2n+k+3) + F_4(k,2n+k+1) - 2$.

(xvi) (By induction on n.) If $n = 0$, then $F_4(3,1) = 1 = 2 - 1 = F_4(3,5) - 1$. Assume that the formula (xvi) holds for n. We will prove it for $n+1$. By the induction hypothesis we have $\sum_{i=0}^{n+1} F_4(3,3i+1) = F_4(3,3n+5) - 1 + F_4(3,3n+4) = F_4(3,3n+8) - 1 = F_4(3,3(n+1)+5) - 1$.

(xvii) (By induction on n.) Let $k \geq 4$ be even. If $n = 0$, then $F_4(k,0) = 1 = 2 - 1 = F_4(k,k) - 1$. Assume that the formula (xvii) holds for n. We will prove it for $n+1$. By the induction hypothesis we have $\sum_{i=0}^{n+1} F_4(k,4i) = F_4(k,4n+k) - 1 + F_4(k,4n+4) = F_4(4,4n+4+k) - 1 = F_4(4,4(n+1)+k) - 1$.

Analogously, we prove the remaining formulas. $\square$

4. Matrix generators

Let $m = \max\{4, k\}$ and $Q_k = [q_{i,j}]_{m\times m}$ be a square matrix. For a fixed $1 \le i \le m$ an element $q_{i,1}$ is equal to the coefficient at $F_4(k,i)$ of the right hand side of the formula (2). For $1 < j \le m$ and $1 \le i \le m$ we have $q_{i,j} = 1$ if $j = i+1$ and $q_{i,j} = 0$, otherwise.

The above definition gives matrices

$$Q_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \quad Q_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ldots Q_k = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

THEOREM 6. *Let $k \ge 1$, $n \ge 0$ be integers. Then*

$$\det Q_k^n = \begin{cases} (-1)^n & \text{for } k \in \{1,2,3\}, \\ (-2)^n & \text{for } k = 4, \\ (-1)^n & \text{for even } k > 4, \\ 1 & \text{for odd } k > 4. \end{cases}$$

We define a square matrix P_k of order $\max\{4, k\}$ as the matrix of initial conditions. For $k \ge 4$

$$P_k = \begin{bmatrix} F_4(k,2k-2) & F_4(k,2k-3) & \cdots & F_4(k,k) & F_4(k,k-1) \\ F_4(k,2k-3) & F_4(k,2k-4) & \cdots & F_4(k,k-1) & F_4(k,k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_4(k,k) & F_4(k,k-1) & \cdots & F_4(k,2) & F_4(k,1) \\ F_4(k,k-1) & F_4(k,k-2) & \cdots & F_4(k,1) & F_4(k,0) \end{bmatrix}.$$

We subtract the last column, which elements are ones, from the remaining columns of P_k. Using Laplace expansions we get the following results.

THEOREM 7. *Let $k \geq 1$ be an integer. Then*

$$\det P_k = \begin{cases} -1 & \text{for } k \in \{1,2,3,4\}, \\ (-1)^{\lfloor \frac{k-1}{2} \rfloor} & \text{for } k > 4. \end{cases}$$

THEOREM 8. *Let k, n be positive integers. Then*

$$(4) \quad P_k Q_k^n$$
$$= \begin{bmatrix} F_4(k,n+2k-2) & F_4(k,n+2k-4) & \cdots & F_4(k,n+k) & F_4(k,n+k-1) \\ F_4(k,n+2k-4) & F_4(k,n+2k-3) & \cdots & F_4(k,n+k-1) & F_4(k,n+k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_4(k,n+k) & F_4(k,n+k-1) & \cdots & F_4(k,n+2) & F_4(k,n+1) \\ F_4(k,n+k-1) & F_4(k,n+k-2) & \cdots & F_4(k,n+1) & F_4(k,n) \end{bmatrix}.$$

PROOF. If $n = 1$, then by (2) and simple calculations the result immediately follows. Assume the formula (4) holds for n, we will prove it for $n+1$. Since $P_k Q_k^{n+1} = (P_k Q_k^n) Q_k$, by our assumption and by the recurrence (2) we obtain

$$A_k Q_k^{n+1}$$
$$= \begin{bmatrix} F_4(k,n+2k-2) & F_4(k,n+2k-3) & \cdots & F_4(k,n+k) & F_4(k,n+k-1) \\ F_4(k,n+2k-3) & F_4(k,n+2k-4) & \cdots & F_4(k,n+k-1) & F_4(k,n+k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_4(k,n+k) & F_4(k,n+k-1) & \cdots & F_4(k,n+2) & F_4(k,n+1) \\ F_4(k,n+k-1) & F_4(k,n+k-2) & \cdots & F_4(k,n+1) & F_4(k,n) \end{bmatrix} \times$$
$$\times \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} F_4(k,n+2k-1) & F_4(k,n+2k-2) & \cdots & F_4(k,n+k+1) & F_4(k,n+k) \\ F_4(k,n+2k-2) & F_4(k,n+2k-3) & \cdots & F_4(k,n+k) & F_4(k,n+k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_4(k,n+k+1) & F_4(k,n+k) & \cdots & F_4(k,n+3) & F_4(k,n+2) \\ F_4(k,n+k) & F_4(k,n+k-1) & \cdots & F_4(k,n+2) & F_3(k,n+1) \end{bmatrix},$$

which ends the proof. □

Remark 9. Let $k \geq 1$, $n \geq 0$ be integers. Then

$$\det P_k Q_k^n = \begin{cases} (-1)^{n+1} & \text{for } k \in \{1,2,3\}, \\ (-1)^{n+1}(2)^n & \text{for } k = 4, \\ (-1)^{n+\lfloor \frac{k-1}{2} \rfloor} & \text{for even } k > 4, \\ (-1)^{\lfloor \frac{k-1}{2} \rfloor} & \text{for odd } k > 4. \end{cases}$$

5. Connections with the Pascal's triangle

To study connections of $(4,k)$-distance Fibonacci numbers with Pascal's triangle we need to consider a family of sequences given by the same recurrence as $F_4(k,n)$ with different initial conditions.

Let $k \geq 1$, $k \geq i \geq 1$, $n \geq 0$ be integers and

$$F_4^i(k,n) = F_4^i(k,n-4) + F_4^i(k,n-k) \quad \text{for } n \geq \max\{4,k\}$$

with

$$F_4^i(k,n) = \begin{cases} 1, & \text{if } n = k-i \\ 0, & \text{in otherwise} \end{cases} \qquad \text{for } n \in \{0,1,\ldots,\max\{3,k-1\}\}.$$

Note that all sequences $F_4^i(k,n)$ have the same matrix generator Q. For illustration of the family of sequences $F_4^i(k,n)$ we present a few initial elements of these sequences for $k=5$ and special values n in the Table 3.

Table 3. Distance Fibonacci numbers $F_4^i(5,n)$ and $F_4(5,n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$F_4^1(5,n)$	0	0	0	0	1	0	0	0	1	1	0	0	1	2	1	0	1	3	3
$F_4^2(5,n)$	0	0	0	1	0	0	0	1	1	0	0	1	2	1	0	1	3	3	1
$F_4^3(5,n)$	0	0	1	0	0	0	1	1	0	0	1	2	1	0	1	3	3	1	1
$F_4^4(5,n)$	0	1	0	0	0	1	1	0	0	1	2	1	0	1	3	3	1	1	4
$F_4^5(5,n)$	1	0	0	0	0	1	0	0	0	1	1	0	0	1	2	1	0	1	3
$F_4(5,n)$	1	1	1	1	1	2	2	2	2	3	4	4	4	5	7	8	8	9	12

Proving analogously as in [30] we have

Theorem 10. *Let $k \geq 4$, $n \geq 0$, $0 \leq i \leq k-1$ be integers. Then*

$$F_4(k,n) = \sum_{i=0}^{\max\{3,k-1\}} F_4^i(k,n).$$

Er studied in [7] a similar to $F_4^i(k,n)$ family of sequences. He showed that nth power of the companion matrix of the family of sequences consists of entries of these sequences in a special order. Result from [7] applied to $F_4^i(k,n)$ has the following form.

$$Q_k^n = \begin{bmatrix} F_4^1(k,n+k-1) & F_4^1(k,n+k-2) & \dots & F_4^1(k,n) \\ F_4^2(k,n+k-1) & F_4^2(k,n+k-2) & \dots & F_4^2(k,n) \\ \vdots & \vdots & \ddots & \vdots \\ F_4^k(k,n+k-1) & F_4^k(k,n+k-2) & \dots & F_4^k(k,n) \end{bmatrix}.$$

The matrix Q_k we can interpret as adjacency matrix of a special digraph D, see the Figure 2.

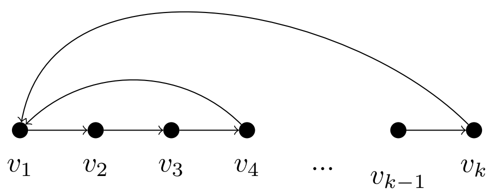

Figure 2. Digraph D for $k > 4$

It is well known that Q_k^n contains the number of all different paths of length n between corresponding vertices in the digraph D. Namely, the entry q_{ij} is equal to the number of all paths of the length n from vertex v_i to vertex v_j in the digraph D.

Using such graph interpretation of the matrix Q, we can prove analogously as in [20] the following theorem.

THEOREM 11. *Let $k \geq 4$, $n \geq 0$, $0 \leq i \leq k-1$ be integers. Then*

$$F_4^1(k,n+k-1) = \sum_{\substack{\alpha_4,\alpha_k \\ 4\alpha_4+k\alpha_k=n}} \binom{\alpha_4+\alpha_k}{\alpha_4},$$

$$F_4^j(k,n+k-1) = \sum_{\substack{\alpha_4,\alpha_k \\ 4\alpha_4+k\alpha_k=n-(4-j+1)}} \binom{\alpha_4+\alpha_k}{\alpha_4} + \sum_{\substack{\alpha_4,\alpha_k \\ 4\alpha_4+k\alpha_k=n-(k-j+1)}} \binom{\alpha_4+\alpha_k}{\alpha_4} \quad \textit{for } j=2,3,4,$$

$$F_4^j(k,n+k-1) = \sum_{\substack{\alpha_4,\alpha_k \\ 4\alpha_4+k\alpha_k=n-(k-j+1)}} \binom{\alpha_4+\alpha_k}{\alpha_4} \quad \textit{for } 4 < j \leq k.$$

Based on Theorem 10 and Theorem 11 we have

THEOREM 12. *Let $k \geq 4$, $n \geq 0$ be integers. Then*

$$F_4(k, n+k-1) = \sum_{i=1}^{3} \sum_{\substack{\alpha_4, \alpha_k \\ 4\alpha_4 + k\alpha_k = n-i}} \binom{\alpha_4 + \alpha_k}{\alpha_4} + \sum_{i=0}^{k-1} \sum_{\substack{\alpha_4, \alpha_k \\ 4\alpha_4 + k\alpha_k = n-i}} \binom{\alpha_4 + \alpha_k}{\alpha_4}.$$

From Theorem 12 we can obtain binomials whose sums are equal to numbers $F_4(k, n)$. Using these binomials we can derive new formulas for $F_4(k, n)$ numbers. For a convenience we use a graphical presentation.

For example, the number $F_4(5, 30)$ is a sum of $\binom{6}{2}$, $\binom{6}{3}$, $\binom{6}{3}$, $\binom{6}{4}$, $\binom{6}{4}$, $\binom{6}{5}$, $\binom{6}{5}$, $\binom{6}{6}$, $\binom{7}{0}$, $\binom{7}{0}$, $\binom{7}{1}$, $\binom{7}{1}$, $\binom{7}{2}$. These binomials form a geometrical pattern in the Pascal's triangle, we will call it a staircase. Corresponding entries are indicated by the blue colour, underlining entry is counted two times.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & {\color{blue}15} & {\color{blue}\underline{20}} & {\color{blue}\underline{15}} & {\color{blue}\underline{6}} & {\color{blue}1} & 0 & 0 & 0 & 0 \\ {\color{blue}\underline{1}} & {\color{blue}\underline{7}} & {\color{blue}21} & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix}.$$

We extend the staircase presented above up to infinity in both directions. Moving such infinite staircase one column to the right, we obtain next number $F_4(k, n)$. In each step of the staircase we have two binomials adjacent. Using the basic property of binomials

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

we immediately obtain a new simplest staircase on the following form.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & {\color{blue}0} & {\color{blue}0} & {\color{blue}0} & {\color{blue}0} \\ 1 & 7 & 21 & {\color{blue}35} & {\color{blue}35} & {\color{blue}21} & {\color{blue}7} & 1 & 0 & 0 & 0 \\ {\color{blue}1} & {\color{blue}8} & {\color{blue}28} & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix}.$$

Such transformations of the formula from Theorem 12 leads to

Corollary 13. *Let $k = 5$, $n \geq 0$ be integers.*

$$F_4(5, n+3) = \sum_{i=0}^{\lfloor \frac{n+2}{4} \rfloor} \binom{\lfloor \frac{n+3-i}{4} \rfloor}{i}.$$

Note that we can perform the above procedure only for sequences with the special value of k.

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